HOLOMORPHIC IDEMPOTENTS AND COMMON FIXED POINTS ON THE 2-DISK

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1. INTRODUCTION

Several authors have studied the question whether two functions that map a set into itself and commute under composition must have a common fixed point. In [4], J. P. Huneke shows that continuous, commuting functions on the unit interval need not have a common fixed point. H. H. Glover and Huneke [5] have discussed the general problem of spaces without the common-fixed-point property for commuting selfmaps. In [7], A. L. Shields showed that if $\mathscr{F}$ is a commuting family of functions holomorphic in the unit disk Δ in $\mathbb{C}$, continuous in $\bar{\Delta}$, and mapping $\bar{\Delta}$ into $\bar{\Delta}$, then the elements of $\mathscr{F}$ have a common fixed point.

In this note, we prove an analogous result for the 2-disk. To do this, we first obtain a characterization of the holomorphic idempotents of the 2-disk into itself.

We wish to thank Henry Glover for his continued interest in this material.

2. HOLOMORPHIC IDEMPOTENTS ON Δ^2

The *2-disk* is the set $\Delta \times \Delta = \Delta^2$ in $\mathbb{C} \times \mathbb{C}$. For each pair (z_1, z_2) in $\mathbb{C} \times \mathbb{C}$, let $\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}$. By a *disk in* $\mathbb{C} \times \mathbb{C}$ we shall mean a set of the form $\{(\rho_1 z, \rho_2 z): z \in \Delta\}$, where $\|(\rho_1, \rho_2)\| \neq 0$. We shall need the following form of Schwarz's lemma in Δ^2.

LEMMA. *If* $F: \Delta^2 \to \Delta$ *is holomorphic, with* $F(0, 0) = 0$ *and* $|F| \leq M$, *then* $|F(z_1, z_2)| \leq M\|(z_1, z_2)\|$.

Moreover, if there exists a pair (z_1^*, z_2^*) *in* $\Delta^2 - \{(0, 0)\}$ *such that* $|F(z_1^*, z_2^*)| = M\|(z_1^*, z_2^*)\|$, *then, with the notation* $\rho_i \|(z_1^*, z_2^*)\| = z_i^*$ $(i = 1, 2)$, F *is linear on the disk* $\{(\rho_1 z, \rho_2 z): z \in \Delta\}$.

Proof. Writing each pair (z_1, z_2) in Δ^2 as (zw_1, zw_2), where $\|(w_1, w_2)\| = 1$ and $|z| = \|(z_1, z_2)\|$, we see, by applying Schwarz's lemma to the function $G(z) = F(zw_1, zw_2)$, that for $\|(z_1, z_2)\| = r$,

$$|F(z_1, z_2)| \leq \max_{|z|=r} |F(zw_1, zw_2)| \leq r \max_{|z|<1} |F(zw_1, zw_2)| \leq Mr.$$

Now, if (z_1^*, z_2^*) is a point such that $\|(z_1^*, z_2^*)\| = r$ $(0 < r < 1)$ and $|F(z_1^*, z_2^*)| = Mr$, then, setting $\rho_i = z_i^*/r$ for $i = 1, 2$ and applying Schwarz's lemma to the function $G(z) = F(\rho_1 z, \rho_2 z)$, we see that $G(z) = \eta z$, where $|\eta| = 1$. From the double power series for F, we find that there are constants A_1 and A_2 such that $F(\rho_1 z, \rho_2 z) = A_1 \rho_1 z + A_2 \rho_2 z$; this yields the result.

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Let us now consider a holomorphic idempotent F mapping Δ^2 into Δ^2. If $F(0, 0) \neq (0, 0)$, then, since the relation $FF = F$ implies that $F(\Delta^2)$ is the fixed-point set for F, there exists a pair $L = (L_1, L_2)$ of Möbius transformations, each a holomorphic bijection from Δ to Δ, such that $F^* = L^{-1}FL$ is an idempotent from Δ^2 to Δ^2 with $F^*(0, 0) = (0, 0)$.

THEOREM 1. *The holomorphic idempotents mapping* Δ^2 *into* Δ^2 *are of the form* LFL^{-1}, *where* L *is a holomorphic bijection of* Δ^2 *onto* Δ^2, *and where* F *has one of the following forms:*

(i) F *is the constant zero mapping,*

(ii) F *is the identity mapping,*

(iii) $F(\Delta^2) = \{(z, h(z)): z \in \Delta\}$, [*or* $\{(h(z), z): z \in \Delta\}$], *where* h *is a holomorphic function mapping* Δ *into* Δ *with* $h(0) = 0$.

Proof. We shall say that F has a (complex) one-dimensional range when case (iii) of the theorem occurs.

Let $F = (F_1, F_2)$, where F_1 and F_2 are the holomorphic coordinate functions mapping Δ^2 into Δ. We assume $F(0, 0) = (0, 0)$. If F is not the constant zero mapping, then there are two possibilities for the range of F.

Case A. $F(\Delta^2) \subset \{(\rho_1 z, \rho_2 z): z \in \Delta\}$ for some pair (ρ_1, ρ_2). Since we have excluded the case where F is the constant zero mapping, we can assume that not both F_1 and F_2 are zero mappings from Δ^2 to Δ. If $\|(\rho_1, \rho_2)\| < 1$, then the iterates of F converge to the zero mapping, which is excluded since F is idempotent. Hence, we see that $\|(\rho_1, \rho_2)\| = 1$ and F has a one-dimensional range. [The idempotent $F = (F_1, F_2)$, where $F_1(z_1, z_2) = F_2(z_1, z_2) = (z_1 + z_2)/2 + (z_1 - z_2)^2/4$, is an illustration of this case where the coordinate functions are not linear.]

Case B. $F(\Delta^2)$ is not contained in a disk. Since $F(w) = w$ for each w in $F(\Delta^2)$, either the equation $F_2(z_1, z_2) = z_2$ holds on an infinite, connected set that is the intersection of a disk in Δ^2 with the set in Δ^2 where $|z_2| > |z_1|$, or else the corresponding statement holds for F_1. The lemma implies that if $F_2(z_1, z_2) = z_2$ on such a set, then F_2 is linear on an infinite number of disks. Then, as a consequence of the Weierstrass preparation theorem [6, page 9], F_2 is linear on all of Δ^2. Therefore, there is no loss of generality in assuming that F_2 is linear on Δ^2. Let

$$F_2(z_1, z_2) = A_1 z_1 + A_2 z_2 \text{ on } \Delta^2. \tag{2.1}$$

The idempotency of F implies that

$$A_1 F_1(z_1, z_2) = A_1(1 - A_2)z_1 + A_2(1 - A_2)z_2. \tag{2.2}$$

Case 1. $A_1 = 0$. Because A_2 is 0 or 1, we see that F_2 is either the constant zero mapping or the identity mapping in the second coordinate. If F_2 is the zero mapping, then the function $F_1^*(z) = F_1(z, 0)$ is a holomorphic idempotent on Δ, and consequently it is either the zero function or the identity function. The idempotency of F implies that F_1 is identically zero if F_1^* is the zero function. F is then the zero mapping. If F_1^* is the identity function, then $F_1(z_1, z_2) = z_1$ for all (z_1, z_2) in Δ^2, as we can see from the double power series expansion of F_1 and the condition that $|F_1| < 1$. Thus, $F(z_1, z_2) = (z_1, 0)$, and F has a one-dimensional range.

If $F_2(z_1, z_2) = z_2$ on Δ^2, then for each z_2, the function $F_1(\cdot, z_2)$ is an idempotent from Δ to Δ. Either $F_1(z_1, z_2) = z_1$ for a set of values z_2 dense in Δ, and hence, for all z_2, or $F_1(z_1, z_2)$ is independent of z_1 on such a set and is therefore a function of z_2 alone. Thus, either the idempotent F is the identity mapping or it has a one-dimensional range.

Case 2. $A_1 \neq 0$. Both F_1 and F_2 are linear on Δ^2 and are given by equations (2.1) and (2.2). Since both F_1 and F_2 have modulus less than 1, we can show, by a judicious choice of points in Δ^2, that $1 - |A_2| \geq |A_1| \geq |1 - A_2|$. It follows that A_2 is real and nonnegative and that $A_1 = \eta(1 - A_2)$ for some η with $|\eta| = 1$. Thus F has a one-dimensional range. We can reduce this case further by using the holomorphic bijection of Δ^2 onto Δ^2 given by $L(z_1, z_2) = (z_1, \eta z_2)$. Then $F^* = L^{-1} F L$ is an idempotent, with $F^*(z_1, z_2) = (pz_1 + qz_2, pz_1 + qz_2)$, where p and q are nonnegative real numbers such that $p + q = 1$. The proof of the theorem is complete.

We can extend the characterization to idempotents that are holomorphic on Δ^2 and continuous on $\overline{\Delta}^2$. For such an idempotent, we have the possibility that $F(\Delta^2)$ contains a boundary point of Δ^2. Then one of the coordinate functions of F maps a point of Δ^2 to the boundary of Δ. By the maximum principle, that coordinate function is constant. If $F(z_1, z_2) = (c, F_2(z_1, z_2))$ for all (z_1, z_2) in Δ^2, where $|c| = 1$, then $F_2(c, \cdot)$ is a holomorphic idempotent on Δ, and is therefore either a constant or the identity function. The idempotency of F implies that F_2 is either a constant on Δ^2 or is the identity function on $\{(c, z): z \in \Delta\}$.

THEOREM 2. *The idempotent mappings of* $\overline{\Delta}^2$ *onto* $\overline{\Delta}^2$ *that are holomorphic on* Δ^2 *and continuous on* $\overline{\Delta}^2$ *are of one of the following types:*

(i) F *maps* Δ^2 *into* Δ^2 *and is characterized by Theorem* 1,

(ii) $F(z_1, z_2) = (c_1, c_2)$, *for all* (z_1, z_2) *in* $\overline{\Delta}^2$, *and* $\|(c_1, c_2)\| = 1$,

(iii) $F(z_1, z_2) = (c, F_2(z_1, z_2))$ [*or* (F_1, c)], *for all* (z_1, z_2) *in* $\overline{\Delta}^2$ *with* $|c| = 1$, *and* $F_2(c, z_2) = z_2$ *for* $z_2 \in \Delta$ [*or* $F_1(z_1, c) = z_1$ *for* $z_1 \in \Delta$].

3. COMMON FIXED POINTS

We recall that if G is a bounded, connected, open subset of $\mathbb{C} \times \mathbb{C}$ and H(G) is the set of holomorphic mappings of G into G, then, with the operation of composition of mappings and the topology of uniform convergence on compact subsets, H(G) is a topological semigroup. If f is in H(G) and $\Gamma(f)$, the closure of the iterates of f in the topology of uniform convergence on compact subsets of G, is a subset of H(G), then $\Gamma(f)$ is a compact topological semigroup, and consequently it contains exactly one idempotent [2, page 100]. As usual, A(G) denotes the mappings in H(G) that have a continuous extension to $\overline{G}$. We shall need the following result of A. Denjoy [1] and J. Wolff [8], [9].

THEOREM (Denjoy and Wolff). *If* f *is a holomorphic function mapping* Δ *into* Δ *that is not a Möbius transformation with a single fixed point in* Δ, *then the iterates of* f *converge uniformly on compact subsets of* Δ *to a constant* z_0 $(|z_0| \leq 1)$.

We can now prove the main result.

THEOREM 3. *If* f *and* g *are commuting, continuous mappings of the closed* 2-*disk, and if they are holomorphic on the open* 2-*disk, then they have a common fixed point.*

Proof. Case I. If there is a mapping $F = (F_1, F_2)$ in $\Gamma(f)$ that is not in $H(\Delta^2)$, then F must map some element of Δ^2 onto the boundary. Without loss of generality, we can assume that there is a constant η of modulus 1 such that $F_1(z_1, z_2) = \eta$

for some pair (z_1, z_2) in Δ^2. By the maximum principle, $F_1 \equiv \eta$ on Δ^2. If the coordinate function F_2 is also a constant function, then F is a constant mapping, and f and g have a common fixed point, since each commutes with F.

If F_2 is not a constant function, then, since f and g commute with F, the coordinate functions f_1 of $f = (f_1, f_2)$ and g_1 of $g = (g_1, g_2)$ are constant on the set $\{(\eta, z): z \in F_2(\Delta^2)\}$, and hence they must be constant on $\{(\eta, z): z \in \Delta\}$. Let $f^* = f_2(\eta, \cdot)$ and $g^* = g_2(\eta, \cdot)$. The functions f^* and g^* commute on $\bar{\Delta}$ and are holomorphic on Δ, since functions in $A(\Delta^2)$ are holomorphic on the "undistinguished" boundary of Δ^2 [6, page 3]. Applying the result of Shields to f^* and g^*, we conclude that if τ is a common fixed point for f^* and g^*, then (η, τ) is a common fixed point for f and g.

In what follows, we can assume that neither of the commuting functions f and g maps points of Δ^2 to the boundary of Δ^2, since we have already discussed the case where $\Gamma(f) \not\subset H(\Delta^2)$ (or, by symmetrical argument, where $\Gamma(g) \not\subset H(\Delta^2)$).

Case II. If $\Gamma(f)$ is a subset of $H(\Delta^2)$, then $\Gamma(f)$ is a compact semigroup. Let F be the holomorphic idempotent in $\Gamma(f)$. If L is a holomorphic bijection of Δ^2 onto Δ^2, then the transformation $h \to L^{-1} h L$ preserves commutativity and the common-fixed-point property between pairs of mappings. Therefore, we may assume that $F(0, 0) = (0, 0)$, and then, from Theorem 1, we conclude that either F is the zero mapping or the identity mapping, or F has one-dimensional range.

If F is the zero mapping, then $(0, 0)$ is the common fixed point for f and g.

If F is the identity mapping, then $\Gamma(f)$ is a group, and f has a holomorphic inverse. The holomorphic bijections of Δ^2 onto Δ^2 are mappings such that

$$(z_1, z_2) \to (L_1(z_1), L_2(z_2)) \quad \text{or} \quad (z_1, z_2) \to (L_2(z_2), L_1(z_1)),$$

where L_1 and L_2 are Möbius transformations of Δ onto Δ [2, page 312]. If M is a Möbius transformation of Δ onto Δ that is not the identity and does not have exactly one fixed point in Δ, then the iterates of M converge uniformly on compact subsets of Δ to a fixed point of M on the boundary of Δ [7, page 705].

We shall consider separately the mappings $f = (L_1, L_2)$ and $f = (L_2, L_1)$.

If $f = (L_1, L_2)$, then, since the identity mapping is in $\Gamma(f)$, neither L_1 nor L_2 can be a Möbius transformation with a fixed point on the boundary of Δ. If both L_1 and L_2 have a single fixed point in Δ, then f has a single fixed point in Δ^2, and it is a common fixed point with g. If f is the identity mapping, then each fixed point of g is a common fixed point with f. Finally, with no loss of generality, we can assume that L_1 is the identity on Δ and that L_2 has a single fixed point w_0 in Δ. Then, taking z_0 as a fixed point of the function $g_1^* = g_1(\cdot, w_0)$, we see that (z_0, w_0) is a common fixed point of f and g.

If $f = (L_2, L_1)$, then $f^2 = (L_2 L_1, L_1 L_2)$. Since $\Gamma(f^2) \subset \Gamma(f) \subset H(\Delta^2)$, neither $L_2 L_1$ nor $L_1 L_2$ can be a Möbius transformation with a fixed point on the boundary of Δ^2. If f^2 has a single fixed point in Δ^2, then it is the only fixed point of f and the common fixed point for f and g. Finally, if either $L_2 L_1$ or $L_1 L_2$ is the identity mapping on Δ, then they both are. Then, if z_0 is a fixed point of the function $g_1^* = g_1(\cdot, L_1(\cdot))$, $(z_0, L_1(z_0))$ is a common fixed point for f and g.

The final possibility for the idempotent F is that the range of F is one-dimensional. There is no loss of generality in assuming that $F(\Delta^2) = \{(z, h(z)): z \in \Delta\}$, where h is a holomorphic function mapping Δ into Δ with $h(0) = 0$.

If the function $f_1^* = f_1(\cdot, h(\cdot))$ has a single fixed point z_0, then the commutativity of f, g, and F implies that $(z_0, h(z_0))$ is a common fixed point for f and g.

If f_1^* does not have a unique fixed point in Δ, then consider the mapping $f^* = f(\cdot, h(\cdot))$ from Δ to Δ^2. With $f^* = (f_1^*, f_2^*)$, we see from the commutativity of F and f that $(f_1^*)^n$ is the first-coordinate function of $(f^*)^n$. By the theorem of Denjoy and Wolff, $(f_1^*)^n$ converges uniformly on compact subsets of Δ to a point z_0 in $\overline{\Delta}$. However, z_0 must lie in Δ, since $\Gamma(f)$ contains no functions that map points of Δ^2 to the boundary of Δ^2. Therefore $(z_0, h(z_0))$ is in Δ^2 and $(z_0, h(z_0))$ is a fixed point of f. It follows that $(f^*)^n$ converges uniformly to $(z_0, h(z_0))$, in every compact subset of Δ. The commutativity of f and g implies that $g(f^*)^n$ converges to both $g(z_0, h(z_0))$ and $(z_0, h(z_0))$. The point $(z_0, h(z_0))$ is a common fixed point for f and g. This completes the proof of the theorem.

4. COMMUTING FAMILIES

In [7], Shields considered families of commuting, continuous functions on the closed disk. He showed that if $\mathscr{F}$ is such a family, then there exists a common fixed point for the family, provided that the range of each function contains points of Δ and that the intersection $\mathscr{F} \cap A(\Delta^2)$ contains a function different from the identity. For the 2-disk, the corresponding result fails, in a somewhat trivial manner. To see this, we take g and h to be continuous, commuting functions on $\overline{\Delta}$ that fail to have a common fixed point (the existence of such follows from Huneke's example). Let

$$G(z_1, z_2) = (g(z_1), 0), \qquad H(z_1, z_2) = (h(z_1), 0), \qquad F(z_1, z_2) = (z_1, \gamma(z_2)),$$

where γ is holomorphic and $\gamma(0) = 0$. Then $\{G, H, F\}$ is a commuting family, with F holomorphic, and without a common fixed point.

However, with minor modifications in the proof of Theorem 3, we can prove the following result for commuting families of functions.

THEOREM 4. *Let $\mathscr{F}$ be a family of continuous, commuting mappings of $\overline{\Delta}^2$ onto $\overline{\Delta}^2$ such that the range of each mapping in $\mathscr{F}$ contains points of Δ^2. Then there exists a common fixed point for $\mathscr{F}$ provided one of the following conditions is satisfied:*

(i) *All but one of the mappings are holomorphic on Δ^2.*

(ii) *There exists a holomorphic mapping in $\mathscr{F}$ such that neither of its coordinate functions is the identity when restricted to any disk in Δ^2.*

REFERENCES

1. A. Denjoy, *Sur l'itération des fonctions analytiques.* C. R. Acad. Sci. Paris Sér. A-B 182 (1926), 255-257.

2. B. A. Fuks, *Special chapters in the theory of analytic functions of several complex variables.* Translated from the Russian by A. Jeffrey and N. Mugibayashi. Translations of Mathematical Monographs, Vol. 14. Amer. Math. Soc., Providence, R.I., 1965.

3. E. Hewitt and K. A. Ross, *Abstract harmonic analysis.* Vol. I. Academic Press, New York, 1963.

4. J. P. Huneke, *On common fixed points of commuting continuous functions on an interval.* Trans. Amer. Math. Soc. 139 (1969), 371-381.

5. J. P. Huneke and H. H. Glover, *Some spaces that do not have the common fixed point property.* Proc. Amer. Math. Soc. 29 (1971), 190-196.

6. W. Rudin, *Function theory in polydiscs.* Benjamin, New York, 1969.

7. A. L. Shields, *On fixed points of commuting analytic functions.* Proc. Amer. Math. Soc. 15 (1964), 703-706.

8. J. Wolff, *Sur l'itération des fonctions holomorphes dans une région, et dont les valeurs appartiennent à cette région.* C. R. Acad. Sci. Paris Sér. A-B 182 (1926), 42-43.

9. ———, *Sur l'itération des fonctions bornées.* C. R. Acad. Sci. Paris Sér. A-B 182 (1926), 200-201.

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