SETS OF CONSTANT DISTANCE FROM A PLANAR SET

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Let A be a compact subset of the Euclidean plane R^2. For each $\varepsilon > 0$, define

$$\partial_\varepsilon(A) = \varepsilon\text{-boundary of } A = \{x \in R^2 \colon \|x - A\| = \varepsilon\},$$

where $\|x - A\| = \inf_{a \in A} \|x - a\|$ is the distance from x to A. I shall prove that

(i) $\partial_\varepsilon(A)$ is the union of a finite collection of simple closed curves minus the union of their interiors, and therefore

(ii) each component of $\partial_\varepsilon(A)$ is locally connected, which implies that

(iii) for all but a countable number of ε, each component of $\partial_\varepsilon(A)$ is a point, a simple arc, or a simple closed curve.

The key idea for (i) works in R^n, but (ii) and (iii) require restriction to the plane.

First consider the case where ε is large compared with the diameter $\sup_{\alpha, \alpha' \in A} \|\alpha - \alpha'\|$ of A.

LEMMA 1. *Let* A *have diameter* δ, *where* $\delta < \varepsilon$, *and suppose that* A *contains the origin* 0. *Then* $\partial_\varepsilon(A)$ *is an* (n - 1)-*sphere. In fact, there exists a homeomorphism* H *of* R^n *upon itself such that*

$$\text{(i)} \quad \begin{cases} \dfrac{H(x)}{\|H(x)\|} = \dfrac{x}{\|x\|} & (x \neq 0), \\ H(0) = 0, \end{cases}$$

(ii) H *carries the unit* (n - 1)-*sphere onto* $\partial_\varepsilon(A)$,

(iii) H *carries the interior of the unit* (n - 1)-*sphere onto*

$$V_\varepsilon(A) = \{x \in A \mid \|x - A\| < \varepsilon\}.$$

Proof. For each point σ on the unit sphere S^{n-1}, let Λ_σ denote the half-line $\Lambda_\sigma = \{x \in R^n \mid x/\|x\| = \sigma\}$. If x and y are two points of Λ_σ and $\delta \leq \|x\| < \|y\|$, then $\|x - A\| < \|y - A\|$. To see this, let T_σ be the (n - 1)-hyperplane normal to Λ_σ at $\delta\sigma \in \Lambda_\sigma$. By elementary geometry, each point on the other side of T_σ from x is closer to x than to y. This includes all points of A. Now let σ be a fixed point of S^{n-1}, and consider the function $t \to \|t\sigma - A\|$ $(0 < t < \infty)$. We have just observed that $\|t\sigma - A\|$ is strictly increasing with t as long as $\delta \leq t$. For $t < \sigma$,

$$\|t\sigma - A\| \leq \|t\sigma - 0\| = \|t\sigma\| < \delta < \varepsilon,$$

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and for $t > \varepsilon + \delta$,

$$\|t\sigma - A\| > \|(\varepsilon+\delta)\sigma - A\| \geq \|(\varepsilon+\delta)\sigma - \delta\sigma\| = \varepsilon .$$

Thus the line Λ_σ intersects $\partial_\varepsilon(A)$ in precisely one point b_σ. In other words, the map $b_\sigma \to \sigma$ defines a bijection from $\partial_\varepsilon(A)$ onto S^{n-1}. It is continuous (being the restriction to $\partial_\varepsilon(A)$ of the map $x \to x/\|x\|$), and hence it is a homeomorphism, since $\partial_\varepsilon(A)$ is compact. Let $h: S^{n-1} \to \partial_\varepsilon(A)$ be its inverse. Now define $H: R^n \to R^n$ by the rule

$$\begin{cases} H(0) = 0, \\ H(t\sigma) = th(\sigma) \quad (t > 0). \end{cases}$$

THEOREM 1. *Let* A *be a compact subset of* R^n, *and let* $\varepsilon > 0$. *Then there exists a finite collection of starlike* n-*cells such that* $\partial_\varepsilon(A)$ *is the union of the boundaries of the cells minus the union of their interiors.*

Proof. Since A is compact, there exists a finite collection $A_1, A_2, \cdots, A_k$ of compact sets such that $A = \bigcup_{i=1}^k A_i$ and each A_i has diameter less than ε. The set A has the property that $\partial_\varepsilon(A) = \bigcup_i \partial_\varepsilon(A_i) - \bigcup_i V_\varepsilon(A_i)$.

By Lemma 1, $\bar{V}_\varepsilon(A_i)$ is a starlike n-cell whose interior is $V_\varepsilon(A_i)$.

COROLLARY 1. *Let* A *be a compact set in* R^2, *and let* $\varepsilon > 0$. *Then* $\partial_\varepsilon(A)$ *is the union of a finite collection of simple closed curves minus the union of their interiors.*

Definitions. A *regular curve* is a metrizable, compact, connected space such that each point has arbitrarily small neighborhoods whose boundaries are finite sets. A set is *nondegenerate* if it contains more than one point. A *simple arc* is a homeomorphic copy of a rectilinear interval. A point is of *order* 2 in a space if it has arbitrarily small neighborhoods whose boundaries have exactly two points.

LEMMA 2. *A Hausdorff space* X *that is the union of finitely many simple arcs is a regular curve.*

Proof. The proof will be by induction on n. Because the theorem is trivial for n = 1, we consider the induction step $n - 1 \Rightarrow n$. Let C be the set of all points of X at which X is not regular. By [3, p. 98], C is empty or contains a nondegenerate continuum $\tilde{C}$. Suppose $c \in \tilde{C}$. Let $1 \leq j \leq n$. If $c \notin \alpha_j$, then by the induction hypothesis c is a regular point of $\sum_{i \neq j} \alpha_i$. But since $X - \alpha_j$ is open in X, c must be a regular point of X. Therefore $c \in \alpha_j$ for each j. Thus $\tilde{C} \subset \bigcap_{i=1}^n \alpha_i$. But this implies that $\tilde{C}$ is an arc common to all α_i, and this in turn implies that $\tilde{C}$ contains points of order 2 in X, a contradiction. Hence C is empty and X is regular.

THEOREM 2. *Each component of* $\partial_\varepsilon(A)$ *is locally connected.*

Proof. According to Corollary 1 and Lemma 2, $\partial_\varepsilon(A)$ is a subset of a regular curve. By [3, p. 99], every subcontinuum of a regular curve is locally connected.

THEOREM 3. *For all but a countable number of* ε, *each component of* $\partial_\varepsilon(A)$ *is a point, a simple arc, or a simple closed curve.*

Proof. Recall that a *triod* is a homeomorph of the cone on three points. By [1], it is impossible to embed the union of an uncountable collection of pairwise disjoint triods in the plane. Hence, for all but a countable number of ε, $\partial_\varepsilon(A)$ contains no triod. Now, by Theorem 75 of [2, p. 218], the only atriodic, locally connected, metrizable continua are the point, the simple arc, and the simple closed curve.

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