THE DISTANCE TO VERTICAL ASYMPTOTES FOR SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS

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A function $y(t)$ is said to have a vertical asymptote at a if either

$$\lim_{t \to a-} |y(t)| = \infty \quad \text{or} \quad \lim_{t \to a+} |y(t)| = \infty .$$

Solutions of nonlinear differential equations of the form

$$(1) \qquad y'' = p(t) f(y)$$

may have vertical asymptotes; for example, $\tan(t - \alpha)$ is the solution of

$$(2) \qquad y'' = 2y(1 + y^2)$$

satisfying the conditions $y(\alpha) = 0$ and $y'(\alpha) = 1$. This solution is defined to the right of α up to $\alpha + \pi/2$, where it has a vertical asymptote. Similarly, $\sec(t - \alpha)$ satisfies the equation

$$y'' = 2y^3 - y$$

and has a vertical asymptote at $\alpha + \pi/2$.

If for the solution $y(t; \alpha) = y(t)$ of (1) satisfying the conditions $y(\alpha) = a$ and $y'(\alpha) = b$ we denote by $t(\alpha)$ the location of the vertical asymptote of y to the right of α ($t(\alpha) = \infty$ if y is defined on $[\alpha, \infty)$), then it is meaningful to discuss the asymptotic behavior of $t(\alpha) - \alpha$ as $\alpha \to \infty$. This is the analogue of the question of the asymptotic distribution of zeros for oscillatory solutions of differential equations. Theorem 1 below answers this question under certain assumptions on p and f.

Theorems 2 and 3 give implicit lower bounds on the distance to vertical asymptotes of solutions of certain equations of the form (1). S. B. Eliason [1] has obtained such lower bounds under restrictions on f different from ours.

1. ASYMPTOTIC BEHAVIOR OF $t(\alpha) - \alpha$

Concerning (1), we assume *that* f *is continuous on* $(-\infty, \infty)$, *that* p *is positive and continuously differentiable on* $[0, \infty)$, *and that* $p'(t)p(t)^{-3/2} \to 0$ *as* $t \to \infty$. We shall deal only with the case where $y(\alpha) > 0$ and $y'(\alpha) \geq 0$; similar statements and proofs apply to the case where $y(\alpha) \leq 0$ and $y'(\alpha) \leq 0$, also to the problem of the distance to the vertical asymptote to the left of α. We consider then the solution $y(t) = y(t; \alpha)$ (assumed unique) of (1) satisfying the conditions

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$$y(\alpha;\ \alpha) = a \geq 0, \tag{3}$$
$$y'(\alpha;\ \alpha) = b \geq 0 \qquad (a + b > 0).$$

If $a > 0$, *we require that* f *satisfy the condition* $xf(x) > 0$ *for* $x \geq a$; *if* $a = 0$ *we require that* $xf(x) > 0$ *for* $x > 0$.

The following lemma is a slight strengthening of a lemma in [2]. We shall use the notation

$$M(p;\ t,\ s) = \sup_{u \in [t,s]} p(u), \qquad m(p;\ t,\ s) = \inf_{u \in [t,s]} p(u).$$

LEMMA. *Suppose* $p \in C^1([0,\ \infty))$, $p > 0$ *on* $[0,\ \infty)$, *and for some* $\gamma > 0$, $|p'(t)|\, p(t)^{-\gamma} \to 0$ *as* $t \to \infty$. *Then, corresponding to each pair of positive numbers* ε *and* K, *there exists an* N *such that*

$$\left|\frac{M(p;\ t,\ s)}{m(p;\ t,\ s)} - 1\right| < \varepsilon$$

whenever $t \geq s \geq N$ *and* $|t - s| \leq K[m(p;\ t,\ s)]^{1-\gamma}$.

Proof. Choose an N such that $t \geq N$ implies $|p'(t)|\, p(t)^{-\gamma} \leq \dfrac{\varepsilon}{2K(1+\varepsilon)^{\gamma+1}}$. Let $s \geq N$, and define

$$g(t,\ s) = \frac{M(p;\ t,\ s)}{m(p;\ t,\ s)}.$$

Suppose $|g(t,\ s) - 1| \geq \varepsilon$ for some $t,\ s \geq N$ satisfying the condition $|t - s| \leq K[m(p;\ t,\ s)]^{1-\gamma}$. Since $g(s,\ s) = 1$, there exists a t^* such that $|g(t^*,\ s) - 1| = \varepsilon$ and $|g(z,\ s) - 1| < \varepsilon$ for all z between s and t^*. Now $g(t,\ s)$ is not differentiable with respect to t, but it is continuous with right and left t-derivatives $D_+ g$ and $D_- g$, and

$$|D_{\pm} g(t,\ s)| \leq \frac{|p'(t)|\, M(p;\ t,\ s)}{[m(p;\ t,\ s)]^2}.$$

Hence

$$\varepsilon = |g(t^*,\ s) - 1| \leq \sup_{t \in [t^*, s]} |D_{\pm} g(t,\ s)| \cdot |t^* - s|$$
$$\leq \frac{M(p';\ t^*,\ s)\, M(p;\ t^*,\ s)}{[m(p;\ t^*,\ s)]^2}\, |t^* - s| \leq \frac{[M(p;\ t^*,\ s)]^{\gamma+1}}{[m(p;\ t^*,\ s)]^2}\, \frac{\varepsilon}{2K(1+\varepsilon)^{\gamma+1}}\, |t^* - s|$$
$$\leq \left[\frac{M(p;\ t^*,\ s)}{m(p;\ t^*,\ s)}\right]^{\gamma+1} \frac{\varepsilon}{2(1+\varepsilon)^{\gamma+1}} = g(t^*,\ s)^{\gamma+1}\, \frac{\varepsilon}{2(1+\varepsilon)^{\gamma+1}},$$

so that $g(t^*,\ s)^{\gamma+1} \geq 2(1+\varepsilon)^{\gamma+1}$. But the condition $|g(t^*,\ s) - 1| = \varepsilon$ implies that $g(t^*,\ s) \leq 1 + \varepsilon$; this contradiction establishes the lemma.

THEOREM 1. *Let* $y(t;\ \alpha)$ *be the solution of* (1) *satisfying* (3), *and let* f *satisfy the condition*

$$\int_a^\infty \frac{dx}{\sqrt{\int_a^x f(u)\,du}} < \infty.$$

Then, for all sufficiently large α, $y(t;\alpha)$ *has a vertical asymptote at* $t(\alpha) \in (\alpha, \infty)$, *and as* $\alpha \to \infty$,

$$t(\alpha) - \alpha \sim \int_a^\infty \frac{dx}{\sqrt{b^2 + 2\,p(\alpha) \int_a^x f(u)\,du}} \tag{4}$$

in the sense that the ratio approaches unity.

Proof. For the moment, we postpone showing that y has a vertical asymptote to the right of α, and we assume the existence of $t(\alpha) < \infty$. It is clear from (1) that $y > 0$ and $y' > 0$ on $(\alpha, t(\alpha))$, and that $\lim_{t \to t(\alpha)-} y(t) = +\infty$. Multiplication of (1) by y' and integration from α to $s \in (\alpha, t(\alpha))$ lead to the relation

$$y'(s)^2 = y'(\alpha)^2 + 2 \int_\alpha^s p(t)\, f(y(t))\, y'(t)\, dt.$$

For convenience, set $m_\alpha = m(p;\alpha, t(\alpha))$, $M_\alpha = M(p;\alpha, t(\alpha))$; then

$$y'(\alpha)^2 + 2m_\alpha \int_{y(\alpha)}^{y(s)} f(u)\,du \le y'(s)^2 \le y'(\alpha)^2 + 2M_\alpha \int_{y(\alpha)}^{y(s)} f(u)\,du\,,$$

whence

$$\sqrt{2m_\alpha}\,(t(\alpha) - \alpha) \le \int_\alpha^{t(\alpha)} \frac{y'(s)\,ds}{\sqrt{\frac{y'(\alpha)^2}{2m_\alpha} + \int_{y(\alpha)}^{y(s)} f(u)\,du}}\,, \tag{5}$$

$$\int_\alpha^{t(\alpha)} \frac{y'(s)\,ds}{\sqrt{\frac{y'(\alpha)^2}{2M_\alpha} + \int_{y(\alpha)}^{y(s)} f(u)\,du}} \le \sqrt{2M_\alpha}\,(t(\alpha) - \alpha)\,. \tag{6}$$

Thus

$$\sqrt{\frac{m_\alpha}{p(\alpha)}} \le \frac{1}{t(\alpha) - \alpha} \int_a^\infty \frac{dx}{\sqrt{b^2\left(\frac{p(\alpha)}{m_\alpha}\right) + 2\,p(\alpha) \int_a^x f(u)\,du}}\,, \tag{7}$$

$$\frac{1}{t(\alpha) - \alpha} \int_a^\infty \frac{dx}{\sqrt{b^2\left(\frac{p(\alpha)}{M_\alpha}\right) + 2\,p(\alpha) \int_a^x f(u)\,du}} \le \sqrt{\frac{M_\alpha}{p(\alpha)}}\,. \tag{8}$$

Below, we show that

$$\left|\frac{p(t)}{p(s)} - 1\right| \to 0 \tag{9}$$

uniformly for $t, s \in [\alpha, t(\alpha)]$ as $\alpha \to \infty$, and hence that

$$\frac{m_\alpha}{p(\alpha)} \to 1, \quad \frac{M_\alpha}{p(\alpha)} \to 1$$

as $\alpha \to \infty$. The result (4) then obviously follows from (7) and (8) if $b = 0$. If $b > 0$, it will be enough to show that the ratios of the integrals in (4) and (7) and in (4) and (8) approach 1 as $\alpha \to \infty$, for then (4) follows readily. The integrals in (7) and (8) are bounded above and below by the (convergent) integrals

$$\int_a^\infty \frac{dx}{\sqrt{b^2 \pm \varepsilon + 2\,p(\alpha)\cdot\int_a^x f(u)\,du}}$$

for all large α, since we are assuming that (9) holds and that ε/b is small. Thus it suffices to compare two integrals of the forms

$$\int_a^\infty \frac{dx}{\sqrt{Q(x) \pm \varepsilon}}, \quad \int_a^\infty \frac{dx}{\sqrt{Q(x)}}, \tag{10}$$

where $Q(x) \geq b^2 > 0$. But

$$\left|\int_a^\infty \frac{dx}{\sqrt{Q(x) \pm \varepsilon}} - \int_a^\infty \frac{dx}{\sqrt{Q(x)}}\right| \leq \varepsilon \int_a^\infty \frac{dx}{\sqrt{Q(x)(Q(x) \pm \varepsilon)}\,(\sqrt{Q(x)} + \sqrt{Q(x) \pm \varepsilon})}$$

$$\leq \frac{\varepsilon}{b^2} \int_a^\infty \frac{dx}{\sqrt{Q(x)}},$$

and it follows that the ratio of the two integrals in (10) approaches 1 as $\alpha \to \infty$.

We turn now to the proof of (9). By the lemma (with $\gamma = 3/2$), it is enough to produce a constant K such that $|t(\alpha) - \alpha| \leq K m_\alpha^{-1/2}$. But from (5) we see that

$$t(\alpha) - \alpha \leq \frac{1}{\sqrt{2m_\alpha}} \int_a^\infty \frac{dx}{\sqrt{\int_a^x f(u)\,du}}, \tag{11}$$

and this provides the desired estimate.

It remains only to show that a vertical asymptote to the right of α exists for all large α. Assume that $y(t; \alpha)$ has no vertical asymptote in $[\alpha, t]$; then, by the method that led to (5) and (11), we can show that

$$(t - \alpha)[m(p; \alpha, t)]^{1/2} \leq 2^{-1/2} \int_a^{y(t)} \frac{dx}{\sqrt{\int_a^x f(u)\, du}} \leq 2^{-1/2} \int_a^{\infty} \frac{dx}{\sqrt{\int_a^x f(u)\, du}} \equiv K(a).$$

Hence $y(t; \alpha)$ can be continued to the right of α no further than this inequality holds. It is thus enough to show that for each large α there exists a sequence $\{t_n(\alpha)\}$ with $t_n(\alpha) \to \infty$ such that

$$(t_n(\alpha) - \alpha)[m(p; \alpha, t_n(\alpha))]^{1/2} \to \infty . \tag{12}$$

This will clearly be the case unless

$$[m(p; \alpha, t)]^{1/2} \to 0 \quad \text{as } t \to \infty \tag{13}$$

for all large α; therefore we assume that (13) holds. If (12) fails, then $(t - \alpha)[m(p; \alpha, t)]^{1/2} \leq K$ for some K, all large α, and all $t \geq \alpha$. Applying the lemma with $\varepsilon = 1/2$, we are forced to conclude that

$$\left| \frac{M(p; \alpha, t)}{m(p; \alpha, t)} - 1 \right| < \frac{1}{2}$$

for sufficiently large α and $t > \alpha$. But this is clearly a contradiction to (13). Thus (12) must hold, and it follows that a vertical asymptote to the right of α exists for all large α. This concludes the proof of the theorem.

Remark. The hypotheses of the theorem cannot be weakened much. Indeed, if the solution $y(t; \alpha)$ satisfying the condition $y(\alpha, \alpha) = a > 0$ and $y'(\alpha, \alpha) = 0$ always has a vertical asymptote to the right of α, then (7) implies the existence of the integral

$$\int_a^{\infty} \frac{dx}{\sqrt{\int_a^x f(u)\, du}} .$$

The differential equation

$$y'' = 2t^{-6} y^3$$

fails to satisfy the condition $p'(t)/p^{3/2}(t) \to 0$ as $t \to \infty$, and it has the solution $y = t^2$ with no vertical tangent. Finally, the problem

$$y'' = 2y^3, \quad y(1) = 1, \quad y'(1) = -1$$

has the solution $y = 1/t$ without a vertical tangent to the right of 1, showing the need for the restriction $y(\alpha)\, y'(\alpha) \geq 0$.

COROLLARY. *If* $p(t) \leq Q$ *for* $0 \leq t < \infty$ *and*

$$\int_a^{\infty} \frac{dx}{\sqrt{b^2 + 2Q \int_a^x f(u)\, du}} = +\infty ,$$

then $y(t;\alpha)$ *has no vertical asymptote to the right of* α.

Proof. If y is defined on (α, t) but has a vertical asymptote at t, then, by the method that led to (6), we can show that

$$\int_a^\infty \frac{dx}{\sqrt{b^2 + 2Q\int_a^x f(u)\,du}} \le t - \alpha,$$

since $M_\alpha \le Q$. But this is obviously impossible for $t < \infty$.

2. LOWER BOUNDS FOR $t(\alpha) - \alpha$

This section contains lower bounds on $t(\alpha) - \alpha$. First we drop the assumption $p \ge 0$.

THEOREM 2. *Suppose* p *is continuous on* $[\alpha, \bar{t}]$; *let* $f > 0$ *be differentiable, with* $f' \ge 0$ *on* $[M, \infty)$; *and suppose* y *is a solution of* (1) *on* $[\alpha, \bar{t})$ *satisfying the condition* $y(t) \ge M > 0$ *for* $t \in [\alpha, \bar{t})$. *If* $\lim_{t\to\bar{t}-} y(t) = +\infty$, *then an implicit lower bound on* $\bar{t} - \alpha$ *is given by*

$$\int_{y(\alpha)}^\infty \frac{du}{f(u)} \le \int_\alpha^{\bar{t}} (\bar{t} - s)p(s)\,ds + \frac{y'(\alpha)}{f(y(\alpha))}(\bar{t} - \alpha). \tag{14}$$

Proof. Division of (1) by $f(y(t))$ and integration from α to $t \in (\alpha, \bar{t})$ yields the formula

$$\frac{y'(t)}{f(y(t))} = -\int_\alpha^t f'(y(s))\left[\frac{y'(s)}{f(y(s))}\right]^2 dx + \frac{y'(\alpha)}{f(y(\alpha))} + \int_\alpha^t p(s)\,ds$$

after an integration by parts. Observing that the first integral on the right is nonnegative, and integrating again from α to $\bar{t}$, we obtain (14).

COROLLARY. *If, in addition to the hypotheses of the theorem,*

$$\int_{y(\alpha)}^\infty \frac{du}{f(u)} = \infty,$$

then y *has no vertical asymptote to the right of* α.

The lower bound on $(\bar{t} - \alpha)$ contained in (14) is not sharp, in contrast to that of [1]. However, the present result applies to a larger class of functions f, y than that of [1].

Theorem 2 and the results of [1] do not apply if y vanishes on $[\alpha, t(\alpha))$, which happens, for example, with the solution $\tan(t - \alpha)$ of (2). The following theorem provides a lower bound on $t(\alpha) - \alpha$ in this case, at least if $p > 0$ and $p' \ge 0$.

THEOREM 3. *Let* f *be continuous on* $[a, \infty)$, *with* $f(u) > 0$ *for* $u \ge a$ *if* $a > 0$, *and* $f(u) > 0$ *for* $u > 0$ *if* $a = 0$; *let* $p > 0$ *have a nonnegative derivative on* $[\alpha, \infty)$. *Then, if the solution* y *of* (1) *satisfying the conditions* $y(\alpha) = a \ge 0$ *and* $y'(\alpha) = b \ge 0$ *has a vertical tangent at* $t(\alpha) > \alpha$,

$$\int_a^\infty \frac{dx}{\sqrt{\frac{b^2}{p(\alpha)} + 2\int_a^x f(u)\,du}} \leq \int_\alpha^{t(\alpha)} \sqrt{p(s)}\,ds\ . \tag{15}$$

Proof. Observing that $y(t) > 0$ and $y'(t) > 0$ on $(\alpha, t(\alpha))$, we multiply (1) by $y'(t)/p(t)$ and integrate (by parts) from α to $t \in (\alpha, t(\alpha))$ to get the relation

$$\frac{y'(t)^2}{p(t)} - \frac{y'(\alpha)^2}{p(\alpha)} + \int_\alpha^t p'(s)\left[\frac{y'(s)}{p(s)}\right]^2 ds = 2\int_{y(\alpha)}^{y(t)} f(u)\,du\ .$$

Since the integral on the left side is nonnegative,

$$\frac{y'(t)}{\sqrt{\frac{b^2}{p(\alpha)} + 2\int_a^{y(t)} f(u)\,du}} \leq \sqrt{p(t)}\ .$$

Integration from α to $s \in (\alpha, t(\alpha))$ and passage to the limit as $s \to t(\alpha)-$ yields (15).

It is clear from the proof that equality holds in (15) if p is a constant; thus (15) is sharp.

REFERENCES

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