BOUNDED FUNCTIONS WITH ONE-SIDED SPECTRAL GAPS

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1. INTRODUCTION

It is a well-known theorem of Sidon that if the sequence $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ of Fourier coefficients of a bounded, measurable, 2π-periodic function f defined on the real line $\mathbb{R}$ has Hadamard lacunarity, then $\sum |\hat{f}(n)| < \infty$ (for terminology and references, see [9, vol. I, p. 247]). In particular, the gap condition implies that f is *continuous* (after correction on a set of measure zero); moreover, it is known that lacunarity hypotheses weaker than those in Sidon's theorem imply continuity (H. P. Rosenthal [3]). If we assume only that $\{\hat{f}(n)\}$ has *infinitely many* Hadamard gaps, then continuity of f is not guaranteed, but certain kinds of discontinuous behavior are ruled out. For instance, f cannot have a jump discontinuity; this is a consequence of well-known facts about conjugate Fourier series (it is not difficult to deduce it from Theorem 8.13 in Chapter 2 of [9]; I am grateful to Professor Zygmund, who supplied me with this reference).

In results of the type just described, a "gap" in the sequence of Fourier coefficients means the vanishing of both the sine and cosine coefficients, for a certain block of indices; that is, in terms of the sequence $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$, a gap is understood to be *symmetric* about $n = 0$. The main point of this paper (Corollary to Theorem 2) is that *one-sided* gaps, that is, sufficiently long blocks of consecutive zeros in the sequence $\{\hat{f}(n)\}_{n=-\infty}^{-1}$, are incompatible with jump discontinuities. More generally, one-sided gaps force a kind of matching behavior, in a sense of averages, on the values of a function in left- and right-hand neighborhoods of each point. Results of the latter kind do not seem to be explicitly known, even for symmetric gaps; at any rate, we do not know of any studies along these lines.

Observe that no *one-sided* gap condition can force so strong a regularity as continuity upon a bounded function: even the most drastic conceivable one-sided gap condition, namely that $\hat{f}(n) = 0$ for all $n < 0$, means only that f is the radial boundary function of a bounded analytic function, which needn't be continuous. But such a function cannot have a jump discontinuity, by virtue of a classical theorem of Pringsheim and Lindelöf; generalizations of this, involving matching average behavior, were noted in [5]. (For other generalizations of the no-jump theorem, see [7], [8].)

The present paper can be viewed as a sequel to [5], and it overlaps that paper slightly; however, here we employ a variant of the method in [5] that enables us to handle functions having one-sided gaps, not merely boundary values in H^{∞}. Because the generalization does not complicate matters, we formulate our results for functions on $\mathbb{R}^n$ (as in [5]). Specialization to $n = 1$ and 2π-periodic functions yields results on traditional Fourier series.

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2. FUNCTIONS WHOSE SPECTRUM LIES IN A CONE

By $L^\infty = L^\infty(\mathbb{R}^n)$ we denote as usual the bounded measurable functions on Euclidean n-space $\mathbb{R}^n$. We employ customary vector notation; in particular, t and u denote points of $\mathbb{R}^n$, and dt the (Haar) measure in $\mathbb{R}^n$. By $B(t^0, a)$ we denote the open ball of radius a centered at t^0; by $|E|$ we denote the measure of E.

Before formulating our first theorem, which does not mention gaps explicitly, we discuss certain properties that a subspace S of L^∞ may have:

(a) The constant function 1 is in S.

(b) S is weak*-closed in L^∞.

(c) S is *translation-invariant;* that is, $f \in S \Rightarrow T_u f \in S$ for all $u \in \mathbb{R}^n$, where $(T_u f)(t) = f(t - u)$.

(d) S is *dilation-invariant;* that is, $f \in S \Rightarrow D_a f \in S$ for all $a > 0$, where $(D_a f)(t) = f(at)$.

(e) "Quasi-analyticity": an element of S that vanishes on a nonempty open cone is identically zero.

An example of a subspace S having these properties is the set of Fatou boundary functions of functions bounded and holomorphic in a Cartesian product of n half-planes (see [5]; a detailed discussion of the analogue for the polydisc is in [4]). This space shall be denoted by $H^\infty(\mathbb{R}^n)$. If for each closed set Q in the dual n-space $\hat{\mathbb{R}}^n$ we denote by $L^\infty(\mathbb{R}^n, Q)$ the set of $f \in L^\infty$ whose (distributional) Fourier transforms are supported in Q, then $H^\infty(\mathbb{R}^n)$ is easily identified with $L^\infty(\mathbb{R}^n, Q^+)$, where Q^+ denotes the first "quadrant" in $\hat{\mathbb{R}}^n$, that is, the set of points with nonnegative coordinates. Observe that the space $S = L^\infty(\mathbb{R}^n, Q)$ satisfies (b) and (c) for every closed set Q; moreover, it satisfies (a) if the origin lies in Q, and (d) if Q is a cone with vertex at the origin. The condition (e) is more delicate; it is certainly satisfied if Q is what we shall call a *minor cone*, that is, a closed convex cone such that (assuming its vertex to be the origin) we can find a vector $u \in \mathbb{R}^n$ whose scalar product with each nonzero vector of Q is positive (for example, a half-line in $\mathbb{R}^1$, or a sector of opening less than π in $\mathbb{R}^2$). Indeed, we can then easily find n linearly independent vectors $u^1, \cdots, u^n$ near u such that $u^j t > 0$ for all nonzero $t \in Q$ and for $j = 1, 2, \cdots, n$. This gives a nonsingular linear transformation of $\mathbb{R}^n$ that carries Q into Q^+; consequently, the set $L^\infty(\mathbb{R}^n, Q)$, being carried into a set of functions satisfying (e) by an invertible linear change of variables, itself satisfies (e).

THEOREM 1. *Let* S *denote any subspace of* L^∞ *satisfying* (a), (b), (c), (d), *and* (e). *Let* K *and* K' *denote open cones in* $\mathbb{R}^n$ *with common vertex* t^0. *Suppose, for some* $f \in S$ *and some complex number* c, *that*

$$\lim_{\substack{t \to t^0 \\ t \in K}} f(t) = c .$$

Then $\lim_{a \to 0+} |E(a)|^{-1} \int_{E(a)} f(t)\,dt = c$, *where* $E(a) = K' \cap B(t^0, a)$.

Remark. Thus, if $f \in S$ tends to c as $t \to t^0$ from inside *some* cone with vertex at t^0, it tends to c "on the average" as $t \to t^0$ from inside any cone with vertex at t^0. In particular, f cannot tend to distinct limits from inside two cones with common vertex. For the case where $n = 1$, $S = H^\infty$, and K and K' are the positive and negative halflines, we recover the Pringsheim-Lindelöf theorem.

Proof. We may assume that $c = 0$, since by (a) the function $f - c$ is in S. Also, since by (c) the function $g(t) = f(t + t_0)$ is in S, we may assume that t^0 is the origin. Assuming these normalizations, we prove first that

$$(1) \qquad \lim_{a \to 0+} \int f(at)k(t)\,dt = 0 \qquad \text{for all } k \in L^1(\mathbb{R}^n).$$

Indeed, if (1) does not hold, then there exist a function $h \in L^1$, a positive sequence $\{a_i\}$ tending to 0, and a complex number $b \neq 0$, such that

$$(2) \qquad \lim_{i \to \infty} \int f(a_i t)h(t)\,dt = b.$$

Since the dilates $\{D_a f\}$ constitute a bounded subset of L^∞, we may further assume, by passing to a subsequence, that the functions $f(a_i t)$ converge, in the weak* topology of L^∞, to some function ϕ. This ϕ is again in S, by virtue of (d) and (b); and

$$(3) \qquad \lim_{i \to \infty} \int f(a_i t)k(t)\,dt = \int \phi(t)k(t)\,dt \qquad \text{for all } k \in L^1.$$

Now, let k in (3) be any element of $L^1(\mathbb{R}^n)$ that is supported in K. Since $f(a_i t) \to 0$, for each $t \in K$, the left side of (3) equals 0, by the dominated-convergence theorem. Therefore, since $\int \phi k\,dt$ vanishes for all k supported in K, the function ϕ vanishes a.e. on K, and so it vanishes identically, by (e). Thus, the right side of (3) may be replaced by 0, and taking $k = h$, we have a contradiction of (2). Hence (1) is established.

The conclusion of the theorem now follows at once if for k in (1) we take the characteristic function of $K' \cap B(0, 1)$, and if we observe that, with this choice of k,

$$\int f(at)k(t)\,dt = \int f(t)a^{-n}k(t/a)\,dt = a^{-n}\int_{E(a)} f(t)\,dt.$$

Since $|E(a)|$ is a constant multiple of a^n, the proof is complete.

Remarks. (i) Evidently, in the hypothesis that $f(t) \to c$ as $t \to t^0$ from within K, we may allow an exceptional set of measure 0. Elsewhere in the paper we shall take corrections on sets of measure zero for granted, wherever they are appropriate.

(ii) The essence of the preceding theorem is the observation that if $\{f_n\}$ is a bounded sequence in S, and $\{f_n \mid E\}$ converges to 0 in the weak* topology of $L^\infty(E)$, where E is any measurable subset of $\mathbb{R}^n$ whose complement is too small to support a nonnull function in S, then $\{f_n\}$ converges to 0 in the weak* topology of $L^\infty(\mathbb{R}^n)$. In the case at hand, the "sequence" was the family of dilates $\{D_a f\}_{a>0}$, and E was K; the hypothesis $f(t) \to 0$ as $t \to 0$ $(t \in K)$ is simply a convenient way of assuring that the restrictions $D_a f \mid K$ tend to 0 weak* in $L^\infty(K)$. It is worth pointing out that for $n = 1$ and $K = \mathbb{R}^+$, the hypothesis that $f(t) \to 0$ as $t \to 0$ through positive values can be weakened to read $\lim_{a \to 0+} a^{-1} \int_0^a f(t) = 0$. Indeed, this condition implies that

(4) $$\lim_{a \to 0+} \int f(at)\,k(t)\,dt = 0$$

when k is k_0, the characteristic function of $[0, 1]$. Therefore (4) holds also for all dilates $\{D_a k_0\}_{a>0}$, as well as for all $k \in L^1$ that lie in the annihilator of S. But this annihilator, together with the dilates of k_0, spans L^1, as is easily shown by means of duality and (e) (see [5]); therefore $D_a f \to 0$ (weak*), and the proof proceeds as before. Variants with k_0 replaced by other averaging kernels are also possible.

(iii) Variants of the theorem could obviously be formulated wherein $\{D_a\}$ is replaced by a more general family of affine mappings, or where a different special choice of k is made in (1), and so forth. We leave these possibilities to the reader. We also note that when S is the space H^∞, we could take for K any cone of positive measure (the assumption that K has *interior* points being unnecessary), since an element of $H^\infty(\mathbb{R}^n)$ that vanishes on a set of positive measure on $\mathbb{R}^n$ vanishes identically.

3. FUNCTIONS WITH ONE-SIDED GAPS

Theorem 1 does not yet yield gap results of the kind alluded to in the title and introduction of the paper, because it is applicable only to classes $L^\infty(\mathbb{R}^n, Q)$ for which Q is a cone, and a minor one at that. Theorem 2 is a genuine gap theorem, demanding of f only that there be occasional large holes in its spectrum; the local behavior thereby forced upon f is similar to that in Theorem 1, but rather weaker.

LEMMA. *Let* J *be a minor cone in* $\mathbb{R}^n$ *with vertex at the origin, that is, a closed convex cone such that for some* $u \in \mathbb{R}^n$ *all the scalar products* ut *with* $0 \neq t \in J$ *are positive. If* $\phi \in L^\infty(\mathbb{R}^n)$ *is supported in* J *and the support of* $\hat{\phi}$ *is not all of* $\hat{\mathbb{R}}^n$, *then* $\phi = 0$.

Remark. In principle, this lemma is well known, and it is valid even when ϕ is a tempered distribution; the case $n = 1$ seems to have been formulated first by B. F. Logan, Jr. (see [6, beginning p. 150]). For convenience, we sketch the proof.

Proof of the lemma. By means of a nonsingular linear transformation of $\mathbb{R}^n$, we carry J into the first quadrant Q^+ (see the remarks preceding Theorem 1), and since the spectrum is mapped covariantly by the transformation, we may assume without loss of generality that $J \subset Q^+$. Let now g be a rapidly decreasing function in $C^\infty(\mathbb{R}^n)$ such that $g(0) = 1$ and $\hat{g}$ is supported in the unit ball. Then the function $\psi_a = (D_a g)\phi$ is in $L^1(\mathbb{R}^n)$, and it is supported in Q^+; moreover, if a is positive and sufficiently small, then $\hat{\psi}_a(x)$ vanishes on some ball, since the spectrum of $D_a g$ lies in $B(0, a)$. Now, denoting by U the open upper half-plane (thought of as bounded by $\mathbb{R}$), and recalling that $\hat{\psi}_a(x)$ is the boundary value on the distinguished boundary $\mathbb{R}^n$ of U^n of a function holomorphic and bounded on U^n, we deduce that $\psi_a = 0$. Thus, for each sufficiently small $a > 0$, the function $g(at)\,\phi(t)$ vanishes a.e., and this implies that $\phi = 0$.

Remark. The lemma remains valid if the vertex of the cone is at some point other than the origin, since translation of the spectrum corresponds to multiplication of ϕ by an exponential.

THEOREM 2. *Suppose that* $f \in L^\infty(\mathbb{R}^n)$ *and that there exist a number* $\alpha > 0$ *and a sequence of points* $x^j \in \hat{\mathbb{R}}^n$ $(|x^j| \to \infty)$ *such that* $B(x^j, \alpha|x^j|)$ *does not meet the spectrum of* f. *Let* K *and* K' *denote open cones in* $\mathbb{R}^n$ *with common vertex.*

Suppose, moreover, that the complementary cone $\mathbb{R}^n \setminus K$ *is minor (relative to the origin* t^0*), and that*

$$\lim_{\substack{t \to t^0 \\ t \in K}} f(t) = c$$

for some complex number c. *Then*

$$\liminf_{a \to 0+} \left| |E(a)|^{-1} \int_{E(a)} (f(t) - c)\, dt \right| = 0,$$

where $E(a)$ *denotes* $K' \cap B(t^0, a)$.

Remarks. In particular, the gap hypothesis is satisfied if the spectrum of f omits some nonempty open cone with vertex at 0. For $n = 1$, the gap hypothesis is equivalent to the existence of infinitely many Hadamard gaps, that is, gaps of the type $[x, \beta x]$, where β is a fixed number greater than 1. The hypothesis concerning K is satisfied for $n = 1$ if K is a half-ray, and for $n = 2$ if K is a sector of opening greater than π. Finally, observe that the conclusion of the theorem, while weaker than that of Theorem 1, nevertheless implies that $f(t)$ cannot tend to a limiting value other than c as $t \to t^0$ from within K'.

Proof. Again without loss of generality, we may suppose that t^0 is the origin and $c = 0$. Write $a_j = |x^j|^{-1}$ and $\xi^j = a_j x^j$. It follows from the hypotheses that the spectrum of each dilate $D_{a_j} f$ fails to intersect the ball $B(\xi^j, \alpha)$. Passing to a subsequence for which $\{\xi^j\}$ converges, say to ξ ($|\xi| = 1$), we may suppose without loss of generality that the spectrum of each $D_{a_j} f$ fails to intersect $B(\xi, \alpha/2)$. By passing to a new subsequence, we may further suppose the sequence $\{D_{a_j} f\}$ to converge weak* to some $\phi \in L^\infty$ whose spectrum is disjoint from $B(\xi, \alpha/2)$.

Now, reasoning as in the proof of Theorem 1, we see that

$$\int \phi k\, dt = \lim_{j \to \infty} \int f(a_j t) k(t)\, dt = 0$$

for every $k \in L^1$ supported in K. Thus, ϕ is supported in the minor cone $\mathbb{R}^n \setminus K$, and its spectrum omits the ball $B(\xi, \alpha/2)$. By the lemma, this implies that ϕ is identically zero. Hence, for every $k \in L^1$,

$$\lim_{j \to \infty} \int f(a_j t) k(t)\, dt = 0;$$

specializing k to be the characteristic function of $K' \cap B(0, 1)$, we obtain the desired conclusion.

COROLLARY. *Suppose the function* $f \in L^\infty(\mathbb{R})$ *has period* 2π, *and the sequence of Fourier coefficients* $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ *has an infinity of one-sided Hadamard gaps; that is, suppose there exist a number* $\beta > 1$ *and an infinite sequence of* $n_j \to +\infty$ *such that* $\hat{f}(n) = 0$ *for* $n_j \le n \le \beta n_j$ *(or the corresponding condition for the negatively indexed Fourier coefficients). Then* f *cannot have distinct left- and right-hand limits at any point. Moreover, if* $\lim_{t \to u-0} f(t) = c$ *exists for some point* u, *then there is* *a sequence* $\{a_j\}$ *decreasing to* 0 *such that the mean value of* f *on the interval* $[u, u + a_j]$ *tends to* c *(indeed, we may take* $a_j = n_j^{-1}$).

4. CONCLUDING REMARKS

(i) In the theorems of this paper, the dilation parameter a could be made to approach $+\infty$, rather than 0, with only trivial changes in the proofs. (In Theorem 2, we must of course then assume that $|x^j| \to 0$.) We thus obtain a set of results connecting the behavior of f(t) within each of two cones K and K' having a common vertex, as t tends to *infinity* within the respective cones.

(ii) A cone that is not minor may indeed support a function whose spectrum omits a nontrivial open set. A simple example in $\mathbb{R}^2$ is obtained if we choose a rapidly decreasing C^∞-function g on $\mathbb{R}^1$ that vanishes for negative arguments, but does not vanish identically, and set $f(t_1, t_2) = g(t_1)\hat{g}(t_2)$. Then both f and $\hat{f}$ vanish in a half-plane. By means of a similar product, we can construct an f supported on only two of the 2^n "quadrants" of $\mathbb{R}^n$, and such that $\hat{f}$ vanishes on a half-space.

Added December 6, 1971. For n = 1, the lemma in Section 3 is contained in much more powerful results of N. Levinson [1, Chapter V, especially Theorem XXVII]. In its dual version (phrased as an approximation theorem in L^1), the lemma (case n = 1) was recently rediscovered by D. J. Newman [2]. The dual version of the lemma in $\mathbb{R}^n$ is an approximation theorem similar to Newman's for minor cones.

REFERENCES

1. N. Levinson, *Gap and density theorems.* Amer. Math. Soc. Colloquium Publications, vol. 26. Amer. Math. Soc., New York, 1940.

2. D. J. Newman, *Translates are always dense on the half line.* Proc. Amer. Math. Soc. 21 (1969), 511-512.

3. H. P. Rosenthal, *On trigonometric series associated with weak* closed subspaces of continuous functions.* J. Math. Mech. 17 (1967), 485-490.

4. W. Rudin, *Function theory in polydiscs.* Benjamin, New York, 1969.

5. H. S. Shapiro, *Boundary values of bounded holomorphic functions of several variables.* Bull. Amer. Math. Soc. 77 (1971), 111-116.

6. ———, *Topics in approximation theory.* Lecture Notes in Mathematics, 187. Springer-Verlag, New York, 1971.

7. ———, *Boundary values of bounded holomorphic functions.* Studia Math. (to appear).

8. H. S. Shapiro and A. L. Shields, *A generalized no-jump theorem for boundary values of bounded analytic functions* (preliminary title; paper in preparation).

9. A. Zygmund, *Trigonometric series: Vols. I, II.* Cambridge University Press, London and New York, 1968.

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