TWO EXAMPLES IN SURFACE AREA THEORY

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1. INTRODUCTION

By a k-surface in R^k, we mean the class of Fréchet-equivalent, continuous mappings $f: X \to R^k$ from a compact topological k-cell X in R^k $(k \geq 2)$. We investigate representation problems for such k-surfaces of finite Lebesgue k-area. In particular, we examine the following two notions.

Absolutely continuous mappings. A continuous mapping $f: X \to R^k$ $(X \subset R^k)$ is said to be *absolutely continuous* (briefly, AC) if there exists a Lebesgue-integrable function ϕ on int X such that $L(f, G) = \int_G \phi(x)\,dx$ for every open subset G of X. Here, L(f, G) denotes the Lebesgue k-area of the restriction of f to G.

Differentiably absolutely continuous mappings. A continuous mapping $f: X \to R^k$ $(X \subset R^k)$ is said to be *differentiably absolutely continuous* (DAC) if it is AC and possesses a weak total differential a.e. in int X. (See [7].)

Equivalent definitions of absolute continuity have been used in [2] for $k = 2$, in [1] for $k > 2$, and in [7] for $k \geq 2$. If f is AC, then we may take $\phi = |J|$, where J is the generalized Jacobian of f. If f is DAC, then we may take $\phi = |j|$, where j is the ordinary Jacobian of f.

By means of two examples of three-dimensional Fréchet surfaces of finite Lebesgue 3-area, we show that

(1) finiteness of 3-area of a Fréchet surface does not imply the existence of an absolutely continuous representation, and

(2) there exists a Fréchet surface of finite 3-area with an absolutely continuous representation but no differentiabily absolutely continuous representation.

We use the surface discussed in [6] in the first example, and a surface of the type discussed in [3] in the second example.

It is known that for two-dimensional Fréchet surfaces, such examples never exist. (See [2].)

2. PRELIMINARIES

For use in the examples below, we recall the construction of some multiplicity functions and k-areas associated with Lebesgue k-area.

O(y, f, I) denotes the usual topological index of a point y in R^k with respect to the restriction of f to a polyhedral region I contained in X. Corresponding to each subset A of X, we define the *essential multiplicity*

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$$N(y, f, A) = \sup \sum |O(y, f, I)| ,$$

where the supremum is taken over all finite collections $D = [I]$ of nonoverlapping polyhedral regions, and where the sum ranges over all I in D that are contained in A. It is well known that

$$L(f, A) = \int_{R^k} N(y, f, A)\, dy .$$

Associated with the essential multiplicity N is the *stable multiplicity* S(y, f, A), which counts the number (possibly ∞) of essential components of $f^{-1}(y)$ contained in int A (see [3] or [7]). We set

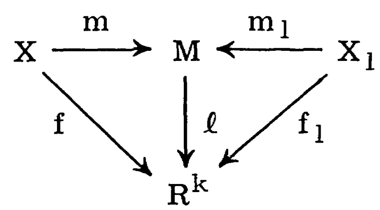
$$S(f, A) = \int_{R^k} S(y, f, A)\, dy .$$

We also define

$$L^*(f, A) = \sup \sum L(f, q) ,$$

where the supremum is taken over all finite collections $Q = [q]$ of nonoverlapping compact topological k-cells q, and where the sum ranges over all q in Q that are contained in A.

If $f_1 \colon X_1 \to R^k$ is a continuous mapping, Fréchet-equivalent to f, and if the diagram



represents simultaneous monotone-light factorizations of f and f_1 with common middle space M and light factor ℓ, then

$$L(f, m^{-1}(G)) = L(f_1, m_1^{-1}(G))$$

for every open subset G of M, and corresponding statements hold for the functionals S and L^*. This was proved in [4] for L and S; it holds also for L^*, since we can easily verify that L^* is lower-semicontinuous with respect to uniform convergence, L^* is invariant with respect to Lebesgue equivalence, and $L^*(f, A) = \sup L^*(f, K)$, where the supremum is taken over all compact subsets K of A.

3. THE FIRST EXAMPLE

It is convenient to describe points of R^3 in terms of the usual cylindrical coordinates (r, θ, z). By $\mathscr{L}^3$ we denote the 3-dimensional Lebesgue measure.

Let

$$X = \{(r, \theta, z): 0 \leq r \leq 3,\ 0 \leq \theta < 2\pi,\ 0 \leq z \leq 1\},$$

$$B = \{(r, \theta, z): r = 0,\ 0 \leq z \leq 1\},$$

$$A = \{(r, \theta, z): 1 < r < 3,\ 0 \leq \theta < 2\pi,\ 0 < z < 1\},$$

$$Y = \{(r, \theta, z): 0 \leq r \leq 1,\ 0 \leq \theta < 2\pi,\ 0 \leq z \leq 1\}.$$

Define $g: A \to Y$ by

$$g(r, \theta, z) = \begin{cases} ((2 - r)^2, \theta, z) & \text{if } 1 < r \leq 2, \\ ((2 - r)^2, -\theta, z) & \text{if } 2 \leq r < 3, \end{cases}$$

and let h be a homeomorphism of Y onto itself such that $\mathscr{L}^3[h(B)] > 0$. Define $f: A \to Y$ by the formula $f = h \circ g$. Straightforward computations show that

$$N(y, f, A) = \begin{cases} 2 & \text{if } y \in \operatorname{int} Y, \\ 0 & \text{otherwise}. \end{cases}$$

But $N(y, f, q) = 0$ for every y in $h(B)$ and every compact topological 3-cell q contained in A, and it follows that $L^*(f, A) < L(f, A) < \infty$.

Now extend f to a continuous mapping defined on all of X such that $L(f, X) < \infty$, let

$$\begin{array}{ccc} X & \xrightarrow{\ m\ } & M \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \ell} \\ & & R^k \end{array}$$

represent the monotone-light factorization of f, and observe that if $G = m(A)$, then G is open in M and $A = m^{-1}(G)$. Since clearly $L^*(f_1, U) = L(f_1, U)$ for every AC mapping $f_1: X_1 \to R^k$ and every open subset U of X_1, the inequality $L^*(f, A) < L(f, A)$ implies that f is not Fréchet-equivalent to any AC mapping.

4. THE SECOND EXAMPLE

Let X and B be defined as in the preceding example, and let

$$g(r, \theta, z) = (r, 2\theta, z) \qquad \text{for } (r, \theta, z) \in X.$$

Let h be a homeomorphism of X onto itself such that

(i) $\mathscr{L}^3[h(B)] > 0$,

(ii) h satisfies Lusin's condition (N) on $X - B$, and

(iii) h^{-1} satisfies Lusin's condition (N) on $X - h(B)$.

Define the mapping $f: X \to X$ by the composition $f = h \circ g$. Straightforward computations yield the relations

$$N(y, f, X) = \begin{cases} 2 & \text{if } y \in \operatorname{int} X, \\ 0 & \text{otherwise}, \end{cases}$$

$$S(y, f, X) = \begin{cases} 2 & \text{if } y \in \operatorname{int} X - h(B), \\ 1 & \text{if } y \in \operatorname{int} X \cap h(B), \\ 0 & \text{otherwise}, \end{cases}$$

so that $S(f, X) < L(f, X) < \infty$. With the help of [7, p. 356], we can verify, however, that $S(f_1, X_1) = L(f_1, X_1)$ for every DAC mapping $f_1: X_1 \to R^k$. Thus f is not Fréchet-equivalent to any DAC mapping.

On the other hand, if $f_1: X \to X$ is defined by $f_1 = f \circ h^{-1}$, then f is Lebesgue-equivalent and therefore Fréchet-equivalent to f_1. Moreover, f_1 is AC, since it satisfies Lusin's condition (N) on X (see [7, p. 255]).

REFERENCES

1. J. C. Breckenridge, *Cesari-Weierstrass surface integrals and lower* k-*area.* Ann. Scuola Norm. Sup. Pisa 25 (1971), 423-446.

2. L. Cesari, *Surface area.* Princeton University Press, Princeton, N.J., 1956.

3. H. Federer, *Essential multiplicity and Lebesgue area.* Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 611-616.

4. ———, *Measure and area.* Bull. Amer. Math. Soc. 58 (1952), 306-378.

5. T. Nishiura, *The Geöcze* k-*area and flat mappings.* Rend. Circ. Mat. Palermo (2) 11 (1962), 106-125.

6. ———, *Interval functions related to Geöcze* k-*area.* Applicable Anal. (to appear).

7. T. Rado and P. V. Reichelderfer, *Continuous transformations in analysis.* Grundlehren, Bd. LXXV. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955.

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