BEHAVIOR OF NORMAL MEROMORPHIC FUNCTIONS ON THE MAXIMAL IDEAL SPACE OF H^∞

Leon Brown and P. M. Gauthier

Many theorems about bounded holomorphic functions hold also in the larger class of normal meromorphic functions. We recall that bounded holomorphic functions can be extended continuously to the maximal ideal space M of H^∞. The main purpose of this paper is to point out that a (meromorphic) function is normal if and only if it admits a continuous extension to the set G of "regular points" of M. In fact, it turns out that if f is meromorphic, then such extensions are actually meromorphic on G. This result is sharp; we present an example of a normal meromorphic function f that cannot be extended continuously to any nonregular point. An examination of this function f yields a new proof that the nonregular points are rare in the sense that they constitute a closed, nowhere dense set [11, p. 102].

K. Stroyan has pointed out to us that the extendibility mentioned above can be established by means of the nonstandard characterization of the Gleason parts of M obtained by M. F. Behrens (unpublished).

1. PRELIMINARIES

We shall consider functions that are defined in the unit disc D with the non-Euclidean hyperbolic metric ρ, and that take their values on the Riemann sphere Ω endowed with the chordal metric χ. The hyperbolic distance $\rho(z, z')$ and the pseudohyperbolic distance $\psi(z, z')$ are defined by

$$\psi(z, z') = \left|\frac{z - z'}{1 - \bar{z}' z}\right| = \tanh[\rho(z, z')].$$

LEMMA 1 (Pick; see [9, p. 239]). *Suppose* f *is holomorphic and bounded by* 1 *in* D. *Then*

$$\rho(f(z), f(z')) \le \rho(z, z'),$$

for all $z, z' \in D$.

For subsets S and T of D, we define the three pseudometrics

a) $H_\rho(S, T) = \inf\{\varepsilon: S \subset \{z: \rho(z, T) < \varepsilon\},\ T \subset \{z: \rho(z, S) < \varepsilon\}\}$,

b) $\sigma(S, T) = \inf\limits_r H_\rho(S \cap \{|z| > r\},\ T \cap \{|z| > r\})$,

c) $\lambda(S, T) = \inf\limits_r \rho(S \cap \{|z| > r\},\ T \cap \{|z| > r\})$;

the first is called the non-Euclidean Hausdorff pseudometric.

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A function f in D is said to be *normal* if the family $\{f \circ L\}$, where L ranges over the conformal mappings of D onto itself, is a normal family in the sense of Montel. For a brief discussion of normal meromorphic functions, see [7, p. 86]. P. Lappan [15, Theorem 1, p. 155] has given the following characterization of normal functions.

LEMMA 2. *A function is normal if and only if it is uniformly* (ρ, χ)*-continuous.*

Another way of stating Lemma 2 is as follows.

LEMMA 3. *A function* f *is not normal if and only if there exist two sequences* $\{z_n\}$, $\{z_n'\}$ *such that* $\rho(z_n, z_n') \to 0$ *and* $\chi(f(z_n), f(z_n'))$ *is bounded away from zero.*

Let H^∞ denote the algebra of holomorphic functions bounded in D, and let M be the maximal ideal space of H^∞. For a discussion of the properties of M, see [10, Chapter 10], [11], [8], and [12]. M is a compact Hausdorff space, and by Carleson's corona theorem [5] it contains D as a dense open subset. Furthermore, each $f \in H^\infty$ can be extended to a continuous function $\hat{f}$ on M, and each pair of points in M can be separated by one of the functions $\hat{f}$. Let $\beta = M \setminus D$ denote the ideal boundary of D, and if $S \subset D$, write $\beta(S) = \bar{S} \setminus D$, where $\bar{S}$ denotes the closure of S in M. The set $\beta(S)$ is the (closed) subset that S generates on the boundary β.

If $m \in \beta$ and f is meromorphic on D, we define the *cluster set* C(f, m) of f at m to be the set of all values w on the Riemann sphere Ω for which there exists a net $\{z_\lambda\}$ ($z_\lambda \in D$, $z_\lambda \to m$) such that $\{f(z_\lambda)\}$ converges to w (for convenience, our notation for nets omits all reference to the directed set). It is easily verified that

$$C(f, m) = \bigcap \overline{f(V \cap D)},$$

where the intersection is over all neighborhoods V of m. Similarly, the *range* of f at m is defined as the set of values $w \in \Omega$ for which there is a net $\{z_\lambda\}$ in D converging to m with $f(z_\lambda) = w$ for each λ. Thus, the range is the set of values assumed infinitely often by f in each neighborhood of m, and

$$R(f, m) = \bigcap f(V \cap D),$$

where the intersection is over all neighborhoods V of m.

We recall that M is contained in the unit ball of the dual space of H^∞. Two points $m_1, m_2 \in M$ are said to be in the same *Gleason part* [4] if $\|m_1 - m_2\| < 2$. Gleason has shown that this is an equivalence relation, and we denote by P(m) the Gleason part of a point $m \in M$. Furthermore, if $S \subset D$, we write

$$\mathscr{P}(S) = \bigcup \{P(m): m \in \beta(S)\}$$

for the set of Gleason parts generated by S.

LEMMA 4 (see [2, p. 128]). *Let* $m_1, m_2 \in M$. *Then* $m_2 \in P(m_1)$ *if and only if*

$$\sup\{|\hat{f}(m_2)|: f \in H^\infty, \|f\| \leq 1, \hat{f}(m_1) = 0\} < 1.$$

Each Gleason part P(m) is either a singleton or an analytic disc. We call m a *regular point* if P(m) is an analytic disc, and we denote by G the set of all regular points in M.

An interpolating sequence is a Blaschke sequence $\{z_n\}$ in D such that

$$\prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \bar{z}_n z_k} \right| \geq \delta > 0 \qquad (n = 1, 2, \cdots).$$

We give a partial statement of a theorem of K. Hoffman [11, p. 75].

LEMMA 5. *Let* $m \in M$. *The following conditions are equivalent.*

a) *The point* m *is regular.*

b) $P(m)$ *contains at least two points.*

c) *The point* m *lies in the closure of an interpolating sequence.*

2. PRINCIPAL RESULTS

We begin this section with a series of geometric theorems.

LEMMA 6. *If* $\{z_\lambda\}$ *is a net converging to* $m \in M$, *and if* $\{\xi_\lambda\}$ *is a net such that* $\rho(z_\lambda, \xi_\lambda) \to 0$, *then* $\{\xi_\lambda\}$ *also converges to* m.

Proof. If $\{\xi_\lambda\}$ does not converge to m, then it has a subnet (which we continue to denote by $\{\xi_\lambda\}$) that converges to a point m_0 different from m. If $f \in H^\infty$, then f is normal, and by Lemma 2, f is uniformly (ρ, χ)-continuous. Thus $\chi(f(z_\lambda), f(\xi_\lambda)) \to 0$, and therefore $\hat{f}(m) = \hat{f}(m_0)$. Since this holds for every $f \in H^\infty$, $m = m_0$, and we arrive at a contradiction.

THEOREM 1. *Let* $S, T \subset D$. *Then* $\beta(S) = \beta(T)$ *if and only if* $\sigma(S, T) = 0$.

Proof. Suppose $\sigma(S, T) = 0$ and $m \in \beta(S)$. Let $\{x_\lambda\}$ be any net in S that converges to m. Choose $y_\lambda \in T$ so that

$$\rho(x_\lambda, y_\lambda) < 2\rho(x_\lambda, T).$$

Since $|x_\lambda| \to 1$ and $\sigma(S, T) = 0$, it follows that $\rho(x_\lambda, y_\lambda) \to 0$. By Lemma 6, $\{y_\lambda\}$ converges to m, and we see that $\beta(S) \subset \beta(T)$. Similarly, the opposite inclusion holds, and therefore $\beta(S) = \beta(T)$.

Conversely, suppose $\sigma(S, T) > 0$. Then we may choose an interpolating sequence $\{z_n\}$ $(z_n \in S)$ such that $\rho(z_n, T) \geq \delta > 0$ for all n. Let B be the Blaschke product associated with $\{z_n\}$. It follows from [6, p. 796] and [13, p. 532] that B is bounded away from zero on T; thus $\hat{B}(m) \neq 0$ for each $m \in \beta(T)$. Since $\{z_n\} \subset S$, there is a point $m \in \beta(S)$ such that $\hat{B}(m) = 0$, and thus $\beta(S) \neq \beta(T)$; this completes the proof.

THEOREM 2. *The three conditions*

$$\mathscr{P}(S) = \mathscr{P}(T), \qquad \sigma(S, T) < \infty, \qquad H_\rho(S, T) < \infty$$

are equivalent.

Proof. It is sufficient to show that $\mathscr{P}(S) = \mathscr{P}(T)$ if and only if $H_\rho(S, T) < \infty$.

Suppose $H_\rho(S, T) < M < \infty$. We choose $m \in \beta(S)$ and $x_\lambda \in S$ so that $x_\lambda \to m$. For each λ, choose $y_\lambda \in T$ so that $\rho(x_\lambda, y_\lambda) < M$. By taking subnets if necessary, we may assume $y_\lambda \to m_0$. We now show that m and m_0 are in the same Gleason part. Indeed, if $f(m) = 0$ and $\|f\| \leq 1$, then, by Pick's Lemma,

$$\rho(f(x_\lambda), f(y_\lambda)) \leq \rho(x_\lambda, y_\lambda) < M,$$

and this implies that $\rho(0, f(m_0)) \leq M$. Thus $|f(m_0)| \leq K < 1$, and by Lemma 4, m and m_0 are in the same part. Thus $\mathscr{P}(S) \subset \mathscr{P}(T)$, and by symmetry, $\mathscr{P}(S) = \mathscr{P}(T)$.

Conversely, let $H_\rho(S, T) = \infty$. Then there exists an interpolating sequence $\{z_n\}$, say in S, such that $\rho(z_n, T) \to \infty$. In other words, $\{z_n\}$ satisfies the condition

$$\psi(T, z_n) \to 1 \quad \text{as } n \to \infty. \tag{1}$$

Moreover, we may assume that $\{z_n\}$ satisfies the condition

$$\lim_{k \to \infty} \prod_{\substack{n=1 \\ n \neq k}}^{\infty} \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| = 1 . \tag{2}$$

Let B be the Blaschke product associated with the sequence $\{z_n\}$. A. Kerr-Lawson has shown [13, p. 533] that (2) implies that for each $\delta_0 < 1$, there exists an $\varepsilon > 0$ such that $|B(z)| \geq \delta_0$ whenever $\psi(z, z_n) \geq \varepsilon$ for all n. From (1) it follows that $|\hat{B}(m)| = 1$ for all $m \in \beta(T)$. On the other hand, if $m_0 \in \beta(\{z_n\})$, then $\hat{B}(m_0) = 0$. Hence, by Lemma 4, no m in $\beta(T)$ is in the same Gleason part as m_0. Since $m_0 \in \beta(S)$, the theorem follows.

The following is an immediate consequence of Theorem 2.

COROLLARY. *If* S *and* T *are two subsets of* D *such that* $H_\rho(S, T) < \infty$, *then*

$$\beta(S) \setminus G = \beta(T) \setminus G .$$

We present one more geometric theorem.

THEOREM 3. *If* S *and* T *are subsets of* D, *then* $G \cap \beta(S) \cap \beta(T) \neq \emptyset$ *if and only if* $\lambda(S, T) = 0$.

Proof. Suppose $\lambda(S, T) = 0$. We choose the points $z_n^1 \in S$ and $z_n^2 \in T$ such that $\rho(z_n^1, z_n^2) < 1/n$ and $\{z_n^1\}$ is an interpolating sequence. Let m_1 be in $\beta(\{z_n^1\})$. We choose a subnet of $\{z_n^1\}$, say $\{z_{n(\lambda)}^1\}$, that converges to m_1.

Since $n(\lambda) \to \infty$, $\rho(z_{n(\lambda)}^1, z_{n(\lambda)}^2) \to 0$, and by Lemma 6, $\{z_{n(\lambda)}^2\}$ converges to m. Moreover, by Lemma 5, m is in G, and therefore $G \cap \beta(S) \cap \beta(T) \neq \emptyset$.

The converse is simply a rephrasing of the statement that Hoffman's condition (1) implies his condition (4) (see [11, p. 75]).

We are now in a position to state our main results.

THEOREM 4. *A function* f *is normal in* D *if and only if* f *admits a (spherically) continuous extension to the set* G *of regular points of* M.

Proof. First we shall show that if $m \in G$, then C(f, m) is a singleton. Suppose, to obtain a contradiction, that ω_1 and ω_2 are in C(f, m) and $\varepsilon = \chi(\omega_1, \omega_2) > 0$. For each neighborhood V of m, we choose two points z_v^1 and z_v^2 in $D \cap V$ such that

$$\chi(f(z_v^j), \omega_j) < \varepsilon/3 \qquad (j = 1, 2).$$

Let $S = \{z_v^1\}$ and $T = \{z_v^2\}$. Then $m \in \beta(S) \cap \beta(T) \cap G$, and Theorem 3 implies that $\lambda(S, T) = 0$. By the uniform continuity of f, we can choose $z_1 \in S$ and $z_2 \in T$ so that $\rho(z_1, z_2) < \delta$, where δ is chosen so small that $\chi(f(z_1), f(z_2)) < \varepsilon/3$. We now arrive at a contradiction:

$$\varepsilon = \chi(\omega_1, \omega_2) \leq \chi(\omega_1, f(z_1)) + \chi(f(z_1), f(z_2)) + \chi(f(z_2), \omega_2) < \varepsilon .$$

Thus $C(f, m)$ is a singleton for $m \in G$, and we set $\hat{f}(m) = C(f, m)$.

If $\hat{f}$ is not continuous at m, then for some $\varepsilon > 0$, each relative neighborhood $V \cap G$ of m contains a point m_V such that $\chi(\hat{f}(m_V), \hat{f}(m)) \geq \varepsilon$. We can choose $z_V \in V \cap D$ so that $\chi(f(z_V), \hat{f}(m_V)) < \varepsilon/2$. The net $\{z_V\}$ converges to m, but $\chi(f(z_V), \hat{f}(m)) \geq \varepsilon/2$, and this is a contradiction.

Conversely, if f is a nonnormal function, then by Lemma 3 there exist two sequences $\{z_n^1\}$ and $\{z_n^2\}$ and a positive ε such that $\rho(z_n^1, z_n^2) \to 0$ but $\chi(f(z_n^1), f(z_n^2)) \geq \varepsilon$ $(n = 1, 2, \cdots)$. Clearly, we may assume that $\{z_n^1\}$ is an interpolating sequence, and by Lemma 5 it follows that $\beta(\{z_n^1\}) \subset G$. Since $\rho(z_n^1, z_n^2) \to 0$, we see that $\sigma(\{z_n^1\}, \{z_n^2\}) = 0$, and Theorem 1 implies that

$$\beta(\{z_n^1\}) = \beta(\{z_n^2\}) \subset G .$$

Suppose $m \in \beta(\{z_n^1\})$, and let $\{z_{n(\lambda)}^1\}$ be a subnet converging to m. By Lemma 6, $\{z_{n(\lambda)}^2\}$ also converges to m, and since $\chi(f(z_{n(\lambda)}^1), f(z_{n(\lambda)}^2)) \geq \varepsilon$, the cluster set $C(f, m)$ is not a singleton; this completes the proof.

By Theorem 4, every normal meromorphic function has a continuous extension to a dense open subset of the boundary β. We recall that for each $m \in G$, the Gleason part $P(m)$ is nontrivial; indeed it is an analytic disc. We shall show that on such parts the extension $\hat{f}$ is actually meromorphic (or identically infinite).

THEOREM 5. *If* f *is a normal meromorphic* (*holomorphic*) *function in* D *and* $\hat{f}$ *is the extension of* f *to the set* G *of regular points of* M, *then on each nontrivial Gleason part,* $\hat{f}$ *is either meromorphic* (*holomorphic*) *or identically infinite.*

Proof. Let $m \in G$. We recall that the analytic structure on $P(m)$ is obtained as follows [11, p. 75]. If $\alpha \in D$ converges to m, then

$$L_\alpha(z) = \frac{z + \alpha}{1 + \bar{\alpha} z}$$

converges pointwise to L_m, a one-to-one mapping of D onto $P(m)$. We shall show that $\hat{f} \circ L_m$ is a meromorphic (holomorphic) function. Fix $z_0 \in D$, and suppose first that $\hat{f} \circ L_m(z_0)$ is finite. We may assume that α lies in D and in some neighborhood of m for which $f \circ L_\alpha$ is uniformly bounded. We claim that $f \circ L_\alpha$ is uniformly bounded in some neighborhood of z_0. If it is not, there exist sequences $\{z_n\}$ and $\{\alpha_n\}$ such that $z_n \to z_0$ and

$$f \circ L_{\alpha_n}(z_n) \to \infty . \tag{3}$$

But $\{f \circ L_{\alpha_n}\}$ is a normal family of functions (f being normal); consequently one of its subsequences converges uniformly on compact subsets, either to a function g meromorphic on the unit disc, or to ∞. From (3) it follows that $g(z_0)$ is infinite; but since the family $\{f \circ L_\alpha\}$ is uniformly bounded at z_0, we have a contradiction. The family $\{f \circ L_\alpha\}$ converges to $\hat{f} \circ L_m$ pointwise, and since $\{f \circ L_\alpha\}$ is uniformly bounded in a neighborhood of z_0, we see that $\hat{f} \circ L_m$ is holomorphic in a neighborhood of z_0.

If $\hat{f} \circ L_m(z_0) = \infty$, we consider the family $\{1/f \circ L_\alpha\}$. The family $\{f \circ L_\alpha\}$ is normal, and hence spherically equicontinuous. Since the spherical metric is invariant with respect to the taking of reciprocals, the family $\{1/f \circ L_\alpha\}$ of reciprocals is equicontinuous and thus normal. Now $1/\hat{f} \circ L_m(z_0)$ is zero, and the argument for the finite case shows that $1/f \circ L_m$ is holomorphic in a neighborhood of z_0. Thus, for each point $z \in D$, $\hat{f} \circ L_m$ is either meromorphic (holomorphic) at z or identically infinite in a neighborhood of z. Hence $\hat{f} \circ L_m$ is either meromorphic (holomorphic) on D or identically infinite. The proof is complete.

3. EXAMPLES

We exhibit a normal meromorphic function f, such that for each $m \in M \setminus G$, $C(f, m) = R(f, m) = \Omega$.

Let f be a Schwarz triangle function [3, Part 7, pp. 173-194] whose initial triangle is strictly interior to the unit disc. It is well known that f is a normal meromorphic function (for example, f is easily seen to be uniformly (ρ, χ)-continuous). Let a be any value on the Riemann sphere Ω, and let $\{z_n\}$ consist of the a-points of f. Since each triangle has the same finite ρ-diameter, there exists an $\varepsilon > 0$ such that an ε-neighborhood (in the metric ρ) of $\{z_n\}$ covers the disc. By a result of K. Hoffman [12, Corollary, p. 84], or by the Corollary to Theorem 2,

$$\beta(\{z_n\}) \supset M \setminus G .$$

Therefore a is in $R(f, m)$ for each $m \in M \setminus G$. Thus this example has the required property.

From the example above we obtain anew Hoffman's result [11, p. 89] that $M \setminus G$ is closed, so that it must be a nowhere dense subset of the boundary β. For we have shown that

$$\beta(f^{-1}(a)) \cap \beta(f^{-1}(b)) \supset M \setminus G .$$

But Theorem 3 implies the opposite inclusion. Thus $M \setminus G$ is closed, and it follows easily from Lemma 5 that $\beta \cap G$ is dense.

Consider now the class of holomorphic triangle functions (for example, the elliptic-modular functions). These functions are known to be normal; see [1, p. 5] and [14, Theorem 16, p. 54]. At least one of the cusps of the initial triangle lies on the circumference of D; this implies that each triangle is of infinite hyperbolic diameter.

Let A denote the set of values omitted by f (A consists of ∞ and at most two finite points). We show that if $m \in M$, then

$$C(f, m) \setminus A = R(f, m). \tag{4}$$

Clearly, $R(f, m)$ is contained in $C(f, m) \setminus A$. Suppose $\omega \in C(f, m) \setminus A$, and let $z_\lambda \to m$ with $f(z_\lambda) \to \omega$. Let Q be a fundamental quadrilateral for f, and for each z_λ, let $\xi_\lambda \in Q$ be equivalent to z_λ (so that $f(z_\lambda) = f(\xi_\lambda)$). Since ω does not correspond to a cusp of Q, and since $f(\xi_\lambda) \to \omega$, the sequence $\{\xi_\lambda\}$ converges to a value ξ in Q with $f(\xi) = \omega$. For each λ, we choose a point a_λ equivalent to ξ such that $\rho(a_\lambda, z_\lambda) = \rho(\xi, \xi_\lambda)$. Since $\rho(a_\lambda, z_\lambda) \to 0$, it follows from Lemma 6 that $a_\lambda \to m$, and thus $\omega \in R(f, m)$.

We note that if a and b are not in A, then $H_\rho\{f^{-1}(a), f^{-1}(b)\} < \infty$; by the Corollary to Theorem 2, this implies that if $m \in M \setminus G$ and $a \in C(f, m) \setminus A$, then $b \in R(f, m)$. Thus, for all $m \in M \setminus G$, either $C(f, m)$ is a subset of A or $R(f, m) = \Omega \setminus A$ and $C(f, m) = \Omega$.

We do not know whether the latter case occurs, but we strongly suspect that it is actually typical; that is, we conjecture that for $m \in M \setminus G$, the cluster set is always total. If this is true, it implies that Theorem 4 is sharp also for holomorphic functions.

REFERENCES

1. F. Bagemihl and W. Seidel, *Sequential and continuous limits of meromorphic functions*. Ann. Acad. Sci. Fenn. Ser. AI No. 280 (1960), 17 pp.
2. A. Browder, *Introduction to function algebras*. Benjamin, New York, 1969.
3. C. Carathéodory, *Theory of functions of a complex variable*. Vol. 2. Translated by F. Steinhardt. Chelsea Publ. Co., New York, 1954.
4. G. T. Cargo, *Normal functions, the Montel property, and interpolation in* H^∞. Michigan Math. J. 10 (1963), 141-146.
5. L. Carleson, *Interpolation by bounded analytic functions and the corona problem*. Ann. of Math. (2) 76 (1962), 547-559.
6. J. A. Cima and P. Colwell, *Blaschke quotients and normality*. Proc. Amer. Math. Soc. 19 (1968), 796-798.
7. E. F. Collingwood and A. J. Lohwater, *The theory of cluster sets*. Cambridge Univ. Press, Cambridge, 1966.
8. A. Gleason, *Function algebras*. Seminars on Analytic Functions, Vol. II. Princeton, 1957, pp. 213-216.
9. E. Hille, *Analytic function theory*. Vol. II. Ginn and Company, Boston, 1959.
10. K. Hoffman, *Banach spaces of analytic functions*. Prentice-Hall, Englewood Cliffs, N. J., 1962.
11. ———, *Bounded analytic functions and Gleason parts*. Ann. of Math. (2) 86 (1967), 74-111.
12. A. Kerr-Lawson, *A filter description of the homomorphisms of* H^∞. Canad. J. Math. 17 (1965), 734-757.
13. ———, *Some lemmas on interpolating Blaschke products and a correction*. Canad. J. Math. 21 (1969), 531-534.
14. P. Lappan, *Some sequential properties of normal and non-normal functions with applications to automorphic functions*. Comment. Math. Univ. St. Paul. 12 (1964), 41-57.
15. ———, *Some results on harmonic normal functions*. Math. Z. 90 (1965), 155-159.

Wayne State University
Detroit, Michigan 48202
and
Université de Montréal
Montréal, Canada