MEROMORPHIC FUNCTIONS OF BOUNDED BOUNDARY ROTATION

James Noonan

1. INTRODUCTION

For $k \geq 2$, let Λ_k denote the class of functions f, given by

$$f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots, \tag{1.1}$$

that are analytic in $U = \{z: |z| < 1\}$ except for a simple pole at $z = 0$ and have an integral representation of the form

$$f'(z) = -\frac{1}{z^2} \exp\left\{\int_0^{2\pi} \log(1 - ze^{-it})\, dm(t)\right\}, \tag{1.2}$$

where m is a real-valued function of bounded variation on $[0, 2\pi]$ satisfying the conditions

$$\int_0^{2\pi} dm(t) = 2, \qquad \int_0^{2\pi} |dm(t)| \leq k, \qquad \int_0^{2\pi} e^{-it}\, dm(t) = 0. \tag{1.3}$$

Simple calculations show that the third of conditions (1.3) guarantees that the singularity of f at $z = 0$ is a simple pole with no logarithmic term.

The class Λ_k was introduced by J. Pfaltzgraff and B. Pinchuk in [5], where they showed that $f \in \Lambda_k$ if and only if f maps the unit disc onto a domain containing infinity, with boundary rotation at most $k\pi$ (for a definition of this concept, see [4]). Hence the union of the classes Λ_k is called the family of meromorphic functions of bounded boundary rotation.

Let V_k denote the class of functions g, given by

$$g(z) = z + b_2 z^2 + \cdots,$$

that are analytic in U, satisfy the condition $g'(z) \neq 0$ in U, and map U onto a domain with boundary rotation at most $k\pi$. V. Paatero [4] showed that $g \in V_k$ if and only if

$$g'(z) = \exp\left\{\int_0^{2\pi} \log(1 - ze^{-it})^{-1}\, dm(t)\right\}, \tag{1.4}$$

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where m is a real-valued function of bounded variation on $[0, 2\pi]$ and satisfies the first two of the conditions in (1.3).

Let Σ^* denote the class of functions of the form (1.1) that are univalent in U and map U onto a domain whose complement is starlike with respect to the origin. It is known (see [5, p. 5]) that $\phi \in \Sigma^*$ if and only if there exists a nondecreasing function m on $[0, 2\pi]$ with $\int_0^{2\pi} dm(t) = 2$ such that

$$\phi(z) = \frac{1}{z} \exp\left\{ \int_0^{2\pi} \log(1 - ze^{-it})\,dm(t) \right\}. \tag{1.5}$$

Finally, following Ch. Pommerenke [6, p. 267], we denote by $K^*(\alpha)$ the class of all functions h of the form

$$h(\zeta) = \zeta + c_0 + \frac{c_1}{\zeta} + \cdots$$

that are analytic in $1 < |\zeta| < \infty$ and for which there exists a function s, starlike in $|\zeta| > 1$, such that

$$\left|\arg \frac{\zeta h'(\zeta)}{s(\zeta)}\right| \leq \frac{\pi}{2} \cdot \alpha. \tag{1.6}$$

In this paper, we derive some relations between Λ_k and the classes V_k, Σ^*, and $K^*(\alpha)$, and we use these relations to study the coefficient problem for Λ_k. We also prove a sharp distortion theorem for Λ_k.

2. RELATIONS BETWEEN Λ_k AND V_k, Σ^*, AND $K^*(\alpha)$

THEOREM 2.1. *A function* f *belongs to* Λ_k *if and only if there exists* $g \in V_k$ *of the form* $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, *with* $b_2 = 0$, *such that*

$$-\frac{1}{z^2 f'(z)} = g'(z).$$

Proof. This theorem follows directly from the representation formulas (1.2) and (1.4) and the formula

$$2b_2 = \int_0^{2\pi} e^{-it}\,dm(t).$$

Using either Theorem 3.1 of [1] in combination with our Theorem 2.1, or else the representation formula (1.5), we are able to relate Λ_k to Σ^*.

THEOREM 2.2. *A function* f *belongs to* Λ_k *if and only if there exist functions* ϕ_1 *and* ϕ_2 *in* Σ^*, *given by*

$$\phi_1(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \qquad \phi_2(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \tag{2.1}$$

such that $a_0 = \frac{k-2}{k+2} b_0$ *and*

$$f'(z) = -\frac{1}{z^2} \frac{[z\phi_1(z)]^{(k+2)/4}}{[z\phi_2(z)]^{(k-2)/4}}. \tag{2.2}$$

Proof. Combine Theorem 3.1 of [1] with our Theorem 2.1, or else use formula (1.5). The relations

$$a_0 = \frac{k-2}{k+2} b_0 \quad \text{and} \quad \int_0^{2\pi} e^{-it}\, dm(t) = 0$$

and equivalent. Note that if we use (1.5), we may find

$$\int_0^{2\pi} dm^+(t) < (k+2)/2 \quad \text{and} \quad \int_0^{2\pi} dm^-(t) < (k-2)/2,$$

where $m = m^+ - m^-$ is the canonical decomposition of m. In this case, we obtain (2.2) by noting that if $\phi \in \Sigma^*$ and $0 < a < 1$, then $\phi_1 \in \Sigma^*$, where $\phi_1(z) = z^{-1}[z\phi(z)]^a$. The remainder of the proof is straightforward, and we omit it.

We shall now relate Λ_k to $K^*(\alpha)$. If $h \in K^*(\alpha)$ and $R > 1$, then the image under h of the circle $|\zeta| = R$ is a closed curve. It is clear that (1.6) is a restriction on the increase of the argument of the tangent vector to this curve. If $f \in \Lambda_k$ and $0 < r < 1$, then the image under f of the circle $|z| = r$ is also a closed curve. Since $f \in \Lambda_k$, the total variation of the argument of the tangent vector to this curve is at most $k\pi$ (see [5, p. 5] for a proof). These facts furnish the basis for a geometric proof of the following theorem.

THEOREM 2.3. *Let* $f \in \Lambda_k$ *with* $2 \leq k \leq 4$. *For* $|\zeta| > 1$, *let* f_1 *be defined by* $f_1(\zeta) = f(1/\zeta)$. *Then* $f_1 \in K^*(k/2 - 1)$.

Proof. For $R > 1$, let $C(R)$ be the image under f_1 of the circle $|\zeta| = R$. Choose w_0 and w_1 in $C(R)$, and suppose that between w_0 and w_1 the tangent to $C(R)$ turns back on itself by $\alpha\pi$ radians. Since the total increase of the argument of the tangent vector is 2π, from w_1 to w_0 the argument of the tangent vector increases at least $2\pi + \alpha\pi$ radians. Thus the total variation of the argument of the tangent vector is at least $2\pi + 2\alpha\pi$. Thus $2\pi + 2\alpha\pi \leq k\pi$, so that $\alpha \leq k/2 - 1$. Therefore $f_1 \in K^*(k/2 - 1)$. This proof is essentially that of Theorem 2.2 in [1].

It is also possible to give an analytic proof that $f_1 \in K^*(k/2 - 1)$. Combining our Theorem 2.1 with Theorem 2.2 in [1], we see that for any r $(0 < r < 1)$ and any θ_1 and θ_2 with $\theta_1 < \theta_2$ we have the inequality

$$\int_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \right\} d\theta \leq (k/2 - 1)\pi.$$

Following the procedure of R. J. Libera and M. S. Robertson [3, pp. 170-171], we construct an $F \in \Sigma^*$ such that $\left|\arg \frac{zf'(z)}{F(z)}\right| < (k/2 - 1)\pi/2$. If we now define F_1 for $|\zeta| > 1$ by $F_1(\zeta) = -F(1/\zeta)$, we see that F_1 is starlike in $|\zeta| > 1$ and that

$$\left|\arg \frac{\zeta f_1'(\zeta)}{F_1(\zeta)}\right| < (k/2 - 1)\pi/2 .$$

Thus $f_1 \in K^*(k/2 - 1)$.

Before concluding this section, let us note that for $k > 2$, Λ_k contains nonunivalent functions [5]. However, by using the same argument as in Theorem 2.1 of [1], we see that Λ_k contains only finitely-valent functions. Specifically, if $f \in \Lambda_k$ assumes some value p times, then $p < k/2 + 1$.

3. THE COEFFICIENT PROBLEM FOR Λ_k

For $n \geq 1$, consider the problem of finding

$$\max\left\{ |a_n| : f(z) = \frac{1}{z} + \sum_{j=0}^{\infty} a_j z^j \in \Lambda_k \right\}.$$

By a normal-families argument, it is easy to see that this problem has a solution. By using the theorems of Section 2, we can solve the problem for $n = 1, 2$. These theorems also enable us to give estimates on the rate of growth of the coefficients as $n \to \infty$.

THEOREM 3.1. *Let* $f \in \Lambda_k$ *be given by* $f(z) = \frac{1}{z} + a_0 + a_1 z + \cdots$. *Then*

$$|a_1| \leq k/2 \quad \textit{and} \quad |a_2| \leq k/6 .$$

Both estimates are sharp for all k.

Proof. By Theorem 2.1, there exists $g \in V_k$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, such that $b_2 = 0$ and

$$-\frac{1}{z^2 f'(z)} = g'(z).$$

By cross-multiplying and comparing coefficients of z, we see that

$$a_1 - 3b_3 + 4b_2^2 = 0, \qquad a_2 + b_2 a_1 + 3b_3 b_2 - 2b_4 = 0 .$$

Since $2b_2 = \int_0^{2\pi} e^{-it}\, dm(t) = 0$, we have the equations

$$a_1 = 3b_3, \qquad a_2 = 2b_4 .$$

From a recursion formula of O. Lehto [2] for the coefficients of $g \in V_k$, we see that

$$b_3 = \frac{1}{6}\int_0^{2\pi} e^{-2it}\, dm(t), \qquad b_4 = \frac{1}{12}\int_0^{2\pi} e^{-3it}\, dm(t).$$

Therefore

$$|a_1| = |3b_3| \leq k/2 \quad \text{and} \quad |a_2| = |2b_4| \leq k/6 .$$

The function $f \in \Lambda_k$ given by

$$f'(z) = -\frac{1}{z^2}\frac{(1+z^2)^{(k+2)/4}}{(1-z^2)^{(k-2)/4}}$$

shows that the bound $|a_1| \leq k/2$ is sharp, and the function given by

$$f'(z) = -\frac{1}{z^2}\frac{(1+z^3)^{(k+2)/6}}{(1-z^3)^{(k-2)/6}}$$

shows that the bound $|a_2| \leq k/6$ is sharp.

We have been unable to solve the coefficient problem in Λ_k for $n \geq 3$. The major reason for this is that the constrained-optimization problems arising from the coefficient problem become much more complicated as n increases. However, Theorems 2.2 and 2.3 enable us to prove the following theorem on the rate of growth of the coefficients.

THEOREM 3.2. *Let* $f \in \Lambda_k$, $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$. *Then, for* $n \geq 1$,

$$a_n = \begin{cases} O(n^{-2}) & \text{if } k = 2, \\ o(n^{-1}) & \text{if } 2 < k < 4, \\ O(n^{k/2-3}) & \text{if } k \geq 4. \end{cases}$$

The exponents in the cases $k = 2$ *and* $k \geq 4$ *are best possible.*

Proof. Suppose first that $k = 2$. Then f is meromorphic and convex, so that $F \in \Sigma^*$, where $F(z) = zf'(z)$. By a theorem of Ch. Pommerenke [7], we then have the estimate

$$n|a_n| \leq \frac{2}{n+1},$$

which proves $a_n = O(n^{-2})$.

If $2 < k \leq 4$, it follows from Theorem 2.3 that $f_1 \in K^*(k/2 - 1)$, where $f_1(\zeta) = f(1/\zeta)$. If $k = 4$, then $f_1 \in K^*(1)$, and therefore [6, p. 267] $a_n = O(n^{-1})$. If $2 < k < 4$, then $f_1 \in K^*(\alpha)$, with $0 < \alpha < 1$, so that Theorem 4 of [6] yields the estimate $a_n = o(n^{-1})$.

We now use a method of D. Brannan [1] to study the case $k > 4$. By Theorem 2.2,

$$-z^2 f'(z) = \frac{[z\phi_1(z)]^{(k+2)/4}}{[z\phi_2(z)]^{(k-2)/4}},$$

where $\phi_1, \phi_2 \in \Sigma^*$. Since the function $1/\phi_1$ belongs to the class $\mathscr{S}^*$ of analytic starlike functions, $|z\phi_1(z)| \leq 4$. Similarly, $1/\phi_2 \in \mathscr{S}^*$, so that

$$[z\phi_2(z)]^{-1} \prec (1-z)^{-2}.$$

Therefore, if $\lambda > 0$, we have the inequality

$$\int_0^{2\pi} \frac{1}{|z\,\phi_2(z)|^{\lambda(k-2)/4}}\,d\theta \le \int_0^{2\pi} \frac{1}{|1-z|^{\lambda(k-2)/2}}\,d\theta .$$

But Pommerenke [6] has shown that

$$\int_0^{2\pi} \frac{1}{|1-z|^{\alpha}}\,d\theta \sim c(\alpha)\frac{1}{(1-r)^{\alpha-1}} \qquad (r \to 1)$$

whenever $\alpha > 1$. Letting $\lambda = 1$ and $\alpha = (k-2)/2$ (note that $k > 4$ implies $\alpha > 1$), we see that

$$\int_0^{2\pi} |z^2 f'(z)|\,d\theta \le \frac{c(k)}{(1-r)^{k/2-2}} . \tag{3.1}$$

Now

$$2\pi n a_n r^{n+1} = \int_0^{2\pi} z^2 f'(z)\, e^{-i(n+1)\theta}\,d\theta . \tag{3.2}$$

If $r = 1 - 1/n$, we combine (3.1) and (3.2) to obtain the estimate $a_n = O(n^{k/2-3})$. The function $f \in \Lambda_k$ given by

$$f'(z) = -\frac{1}{z^2}\frac{\left(1 + z^2 - 2z\dfrac{k-2}{k+2}\right)^{(k+2)/4}}{(1-z)^{(k-2)/2}}$$

shows that the exponent $k/2 - 3$ is best possible.

4. A DISTORTION THEOREM FOR Λ_k

It is easy to see that for each fixed nonzero $z \in U$, the problems of finding

$$\max\{|f'(z)|: f \in \Lambda_k\} \tag{4.1}$$

and

$$\min\{|f'(z)|: f \in \Lambda_k\} \tag{4.2}$$

have solutions. In [5, p. 40] it was shown that any solution of (4.1) or (4.2) must be of the form

$$f'(z) = -\frac{1}{z^2}\frac{\prod_{j=1}^{2}(1-\varepsilon_j z)^{p_j}}{\prod_{j=1}^{2}(1-e_j z)^{n_j}}, \tag{4.3}$$

where $|\varepsilon_j| = |e_j| = 1$, $p_j \ge 0$, $n_j \ge 0$, and

$$\sum_{j=1}^{2} p_j - \sum_{j=1}^{2} n_j = 2, \qquad \sum_{j=1}^{2} (p_j + n_j) \leq k, \qquad \sum_{j=1}^{2} p_j \varepsilon_j = \sum_{j=1}^{2} n_j e_j .$$

The authors state in [5] that they have been unable to find complete solutions to (4.1) and (4.2), but they conjecture that for each z the solution is of the form

$$f'(z) = -\frac{1}{z^2} \frac{\left(1 + 2z\frac{k-2}{k+2} + z^2\right)^{(k+2)/4}}{(1+z)^{(k-2)/2}} . \tag{4.4}$$

In our next theorem we obtain solutions both to (4.1) and (4.2). We shall show that f given by (4.4) is the solution to (4.1) for some values of z, but is never the solution to (4.2). We first need a technical lemma, which we state without proof.

LEMMA 4.1. *Let*

$$H(x) = \frac{1+x^2}{2x} - \left(\log\frac{1+x}{1-x}\right)^{-1} \qquad \textit{for } 0 < x < 1 .$$

Then $0 < H(x) < 1$, $H(x)$ *is strictly increasing, and* $\lim_{x\to 0} H(x) = 0$, $\lim_{x\to 1} H(x) = 1$.

In our next lemma, we examine functions of the form (4.3). Let $H(x)$ be defined as in Lemma 4.1.

LEMMA 4.2. *Let* $k > 2$, *and let* f' *be given by* (4.3), *where we assume* $p_1 > 0$, $p_2 > 0$, $n_1 > 0$, $n_2 > 0$, *and* $\sum_{j=1}^{2} (p_j + n_j) = k$. *Let* r_k *denote the unique solution of the equation* $H(x) = (k-2)/(k+2)$. *Then*

(i) $$|f'(re^{i\theta})| \leq \frac{1}{r^2}\left(\frac{2r}{\log\frac{1+r}{1-r}}\right)^{(k+2)/4} \frac{1}{(1-r^2)^{t_1}} \left(\frac{1+r}{1-r}\right)^{t_2} \qquad (0 < r \leq r_k),$$

where $t_1 = \dfrac{k-2}{4}$ *and* $t_2 = \dfrac{k+2}{4} H(r)$, *and*

(ii) $$|f'(re^{i\theta})| \leq \frac{1}{r^2} \frac{\left(1 + r^2 - 2r\frac{k-2}{k+2}\right)^{(k+2)/4}}{(1-r)^{(k-2)/2}} \qquad (r_k \leq r < 1) .$$

Proof. Let $\varepsilon_j = e^{i\theta_j}$ and $e_j = e^{i\phi_j}$. Then

$$\log|r^2 f'(re^{i\theta})| = S(r, p_j, \theta_j) - S(r, n_j, \phi_j),$$

where

$$S(r, x_j, y_j) = \sum_{j=1}^{2} \frac{x_j}{2} \log(1 + r^2 - 2r\cos(\theta - y_j)) .$$

Since $\sum_{j=1}^{2} p_j \varepsilon_j = \sum_{j=1}^{2} n_j e_j$, let

$$N = n_1\cos(\theta - \phi_1) + n_2\cos(\theta - \phi_2) = p_1\cos(\theta - \theta_1) + p_2\cos(\theta - \theta_2) . \tag{4.5}$$

Fix p_1, p_2, n_1, n_2, and N. Then, by considering $S(r, p_j, \theta_j)$ as a function of $\cos(\theta - \theta_1)$ and $\cos(\theta - \theta_2)$, subject to the constraint (4.5), we see that

$$S(r, p_j, \theta_j) \leq \frac{p_1 + p_2}{2} \log\left(1 + r^2 - 2r \frac{N}{p_1 + p_2}\right).$$

Thus

$$\log|r^2 f'(re^{i\theta})| \leq \frac{p_1 + p_2}{2} \log\left(1 + r^2 - 2r \frac{N}{p_1 + p_2}\right) - S(r, n_j, \phi_j).$$

Considering $S(r, n_j, \phi_j)$ as a function of $\cos(\theta - \phi_1)$ and $\cos(\theta - \phi_2)$, subject to constraint (4.5), we see by examining critical points that the minimum of $S(r, n_j, \phi_j)$ occurs when $\cos(\theta - \phi_1)$ is either 1, -1, $(N + n_2)/n_1$, or $(N - n_2)/n_1$. (The latter two are considered only if they have modulus less than 1.)

We now examine these cases individually. If $\cos(\theta - \phi_1) = 1$, then (4.5) yields the relation $\cos(\theta - \phi_2) = (N - n_1)/n_2$. Thus

$$\log|r^2 f'(re^{i\theta})| \tag{4.6}$$
$$\leq \frac{p_1 + p_2}{2} \log\left(1 + r^2 - 2r\frac{N}{p_1 + p_2}\right) - \frac{n_2}{2} \log\left(1 + r^2 - 2r \frac{N - n_1}{n_2}\right).$$

We now continue to fix n_1 and n_2, but we allow N to vary. Since we assume that $\cos(\theta - \phi_1) = 1$, we must have the relations $n_1 - n_2 \leq N \leq (k - 2)/2$. It is now easy to verify that the maximum in (4.6) occurs when $N = (k - 2)/2$ or $N = n_1 - n_2$.

We proceed similarly in all four cases, and after a long but straightforward computation we see that

$$\log|r^2 f'(re^{i\theta})| \leq \max\left\{\frac{k + 2}{4} \log\left(1 + r^2 - 2r \frac{4n - k + 2}{k + 2}\right) - n \log(1 - r) - \left(\frac{k - 2}{2} - n\right) \log(1 + r)\right\}, \tag{4.7}$$

where the maximum is taken over $0 \leq n \leq (k - 2)/2$. By differentiation, we see that the unconstrained maximum occurs when

$$n = n(\max) = \frac{k - 2}{4} + \frac{k + 2}{4} H(r).$$

But in order to be feasible, $n(\max)$ must satisfy $0 \leq n(\max) \leq (k - 2)/2$. Using Lemma 4.1, we see that $0 \leq n(\max)$ is true for all r, and that $n(\max) \leq (k - 2)/2$ is true only for $r \leq r_k$.

We also note that for $n < n(\max)$, the function in (4.7) increases with n. Thus, if $0 < r \leq r_k$, the maximum in (4.7) occurs at the feasible value $n(\max)$. If $r_k \leq r < 1$, the maximum in (4.7) occurs at $n = (k - 2)/2$. Putting these values of n into (4.7) and taking exponential functions, we arrive at the conclusion of the lemma.

The following lemma, which we state without proof, will be used in the solution of problem (4.2). The proof is similar to the proof of Lemma 4.2.

LEMMA 4.3. *Let the hypothesis of Lemma* 4.2 *hold. Then*

$$|f'(re^{i\theta})| \geq \frac{1}{r^2} \frac{(1+r)^{(k+2)/2-s}(1-r)^s}{\left(\dfrac{2r}{\log \dfrac{1+r}{1-r}}\right)^{(k-2)/2}},$$

where

$$s = \frac{k-2}{8r}\left\{-\frac{2r}{\log \dfrac{1+r}{1-r}} + 1 + r^2 + 2r\frac{k+2}{k-2}\right\}.$$

We can now solve problems (4.1) and (4.2). In particular, we see that for each problem, the extremal function depends on r.

THEOREM 4.1. *Let* $f \in \Lambda_k$, *and let* r_k, t_1, t_2, *and* s *satisfy the conditions in Lemmas* 4.2 *and* 4.3. *Then*

$$(4.8)\quad |f'(re^{i\theta})| \leq \frac{1}{r^2}\left(\frac{2r}{\log \dfrac{1+r}{1-r}}\right)^{(k+2)/4} \frac{1}{(1-r^2)^{t_1}}\left(\frac{1+r}{1-r}\right)^{t_2} \qquad \text{if } 0 < r \leq r_k,$$

$$(4.9)\quad |f'(re^{i\theta})| \leq \frac{1}{r^2}\frac{\left(1+r^2-2r\dfrac{k-2}{k+2}\right)^{(k+2)/4}}{(1-r)^{(k-2)/2}} \qquad \text{if } r_k \leq r < 1,$$

$$(4.10)\quad |f'(re^{i\theta})| \geq \frac{1}{r^2}\frac{(1+r)^{(k+2)/2-s}(1-r)^s}{\left(\dfrac{2r}{\log \dfrac{1+r}{1-r}}\right)^{(k-2)/4}} \qquad \text{for all } r \ (0 < r < 1).$$

All estimates are sharp.

Proof. We first establish (4.8) and (4.9). Let $re^{i\theta}$ be fixed. By Theorem 6.1 in [5], the extremal function for problem (4.1) is of the form (4.3). Suppose that the integrator m is related to f by formula (1.2). Let $\int_0^{2\pi} |dm(t)| = j$. If $j = k$, then (4.8) and (4.9) follow from Lemma 4.2. If $j < k$, then $r_j < r_k$, and we must consider three cases.

First, suppose $r > r_k > r_j$. By Lemma 4.2,

$$(4.11)\qquad |f'(re^{i\theta})| \leq \frac{1}{r^2}\frac{\left(1+r^2-2r\dfrac{j-2}{j+2}\right)^{(j+2)/4}}{(1-r)^{(j-2)/2}}.$$

A straightforward computation shows that the right-hand side of (4.11) increases with j, so that (4.9) is valid.

Next, suppose $r_j < r < r_k$. Let

$$n(j) = \frac{j-2}{4} + \frac{j+2}{4}H(r).$$

By Lemma 4.1, $n(j) < n(k)$. Also, since $r_j < r < r_k$, we have the inequalities

$$\frac{j-2}{2} < n(j) < n(k) \le \frac{k-2}{2}.$$

A long computation shows that the function

$$A(j, n) = \frac{j+2}{4} \log \left(1 + r^2 - 2r \frac{4n - j + 2}{j+2}\right) - n \log(1 - r) - \left(\frac{j-2}{2} - n\right) \log(1 + r)$$

increases with j, for each fixed $n \le (j-2)/2$. Thus $A(j, n) \le A(k, n)$. But in Lemma 4.2 we showed that $A(k, n)$ increases with n, if k is fixed and $n \le n(k)$. Thus $A\left(j, \frac{j-2}{2}\right) \le A(k, n(k))$. Combining this inequality with Lemma 4.2, we arrive at the conclusion of the theorem.

Finally, suppose $r < r_j < r_k$. Then

$$n(j) < \frac{j-2}{2} \quad \text{and} \quad n(j) < n(k) \le \frac{k-2}{2}.$$

As above, $A\left(j, \frac{j-2}{2}\right) \le A\left(k, \frac{k-2}{2}\right)$, and this implies the conclusion of the theorem.

A similar argument, using Lemma 4.3 in place of Lemma 4.2, shows that (4.10) holds. In order to show that all estimates are sharp, we merely re-examine Lemmas 4.2 and 4.3 and pick out the values for p_1, p_2, n_1, n_2, θ_1, θ_2, ϕ_1, and ϕ_2 at which the various maxima and minima are attained. It is then possible to construct the extremal function for each r. This completes the proof of the theorem.

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University of Maryland
College Park, Maryland 20742