DERIVATIVES OF SINGULAR INNER FUNCTIONS

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Let U denote the open unit disc $\{z: |z| < 1\}$, and let T denote the unit circle $\{z: |z| = 1\}$. For $0 < p < \infty$, the Hardy class H^p consists of all functions f analytic in U for which

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta$$

is finite. An analytic function f is said to be of bounded characteristic ($f \in N$) in case

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta$$

is finite, and to be in class B^p $(0 < p < 1)$ if

$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f(re^{i\theta})| \, (1 - r)^{1/p-2} \, dr \, d\theta$$

is finite. It is well known that $H^p \subset N$ [4, p. 16], and that $H^p \subset B^p$ for $0 < p < 1$ [5, p. 415].

A singular inner function is a function of the form

$$S(z; \mu) = \exp\left(- \int \frac{e^{it} + z}{e^{it} - z} \, d\mu(e^{it}) \right),$$

where μ is a positive measure on T, singular with respect to Lebesgue measure on T (see Chapter 5 of [6] for details). Recently, much attention has been given to the factorization and boundary properties of functions with derivatives in H^p (see [1], [2], and [3], for instance). In [2], J. G. Caughran and A. L. Shields have raised the problem of finding conditions on the singular measure μ sufficient to insure that $S'(z; \mu) \in H^p$ for some $p > 0$. Is it possible that $S'(z; \mu) \in H^{1/2}$? Does there exist a singular inner function $S(z; \mu)$ such that $S'(z; \mu) \in H^p$ and the distribution function of μ is continuous? Theorems 1 and 4 of this paper give conditions on μ sufficient to insure that $S'(z; \mu)$ belongs to H^p or N, and they answer the latter question in the affirmative. Theorem 2 shows that in case $S'(z; \mu) \in H^{1/2}$, the support $\sigma(S)$ of μ must be perfect and may not be a Carleson set. Recall that a Carleson set is a closed subset of T that has measure zero and whose complement is the union of open arcs of lengths ε_n, where $\sum \varepsilon_n \log 1/\varepsilon_n < \infty$. Finally, we use Theorem 4 to give an example of a singular inner function whose derivative is in H^p $(p < 1/4)$ and whose support is a perfect non-Carleson set.

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THEOREM 1. *Let* $S(z; \mu)$ *be a singular inner function with support* $\sigma(S)$, *and suppose the complement of* $\sigma(S)$ *in* T *is the union of arcs of lengths* ε_n.

(1) *If* $\sigma(S)$ *is a Carleson set, then* $S'(z; \mu) \in N$.

(2) *If* $0 < k < 1$ *and* $\{\varepsilon_n\} \in \ell^k$, *then* $S'(z; \mu) \in H^p$, *where* $p = (1 - k)/2$.

Proof. Let $f(z) = -\int \frac{2e^{it}}{(e^{it} - z)^2} d\mu(e^{it})$. Then, since $S'(z; \mu) = S(z; \mu) f(z)$, $S' \in N$ if and only if $f \in N$, and if $f \in H^p$, then $S'(z; \mu) \in H^p$. Let

$$d(\theta) = \operatorname{dist}(e^{i\theta}, \sigma(S)).$$

For $r \geq 1/2$ and $e^{it} \in \sigma(S)$,

$$|e^{it} - re^{i\theta}|^2 \geq \frac{1}{2} d(\theta)^2,$$

since $1 + r^2 - 2r\cos(\theta - t) \geq \frac{1}{2}(2 - 2\cos(\theta - t))$. Consequently, $|f(re^{i\theta})| < \frac{c}{d(\theta)^2}$ for some constant c independent of r and θ. It is easily verified that

$$\int_0^{2\pi} \log^+ \frac{1}{d(\theta)^2} d\theta < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \varepsilon_n \log \frac{1}{\varepsilon_n} < \infty.$$

Also,

$$\int_0^{2\pi} \frac{1}{d(\theta)^{2p}} d\theta < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} \varepsilon_n^{1-2p} < \infty \text{ and } 0 < p < 1/2.$$

Thus, if $\{\varepsilon_n\} \in \ell^k$, then $S'(z; \mu) \in H^{(1-k)/2}$, and the proof is complete.

Remark. Minor modifications of the argument above show that in case $\sigma(S)$ is a Carleson set, $S^{(n)}(z; \mu) \in N$ for all n. In case $\{\varepsilon_n\} \in \ell^k$, $S^{(n)}(z; \mu) \in H^p$, where $p = \frac{1-k}{n+1}$.

COROLLARY. *Let* f *be analytic in* U, *with* $f' \in H^1$. *Then all derivatives of the singular inner factor of* f *are functions of bounded characteristic.*

Proof. It is shown in [3, Theorem 1] that $\sigma(S)$ is a Carleson set.

THEOREM 2. *Suppose* $S(z; \mu)$ *is a singular inner function with* $S'(z; \mu) \in H^{1/2}$. *Then* $\sigma(S)$ *is not a Carleson set and is perfect.*

Proof. If $\sigma(S)$ is a Carleson set, then, by a result of Caughran [1, see the remark following Theorem 1], $S'/S \in H^{1/2}$. But the function $f(z) = S'(z; \mu)/S(z; \mu)$ has the anti-derivative

$$F(z) = -\int \frac{e^{it} + z}{e^{it} - z} d\mu(e^{it}).$$

By a result of Hardy and Littlewood [5, p. 415, Theorem 33], $F' \in H^p$ $(0 < p < 1)$ implies $F \in H^{(1-p)/p}$. Consequently, $F(z) \in H^1$. Since μ is singular, $\lim_{r \to 1} F(re^{i\theta})$ is pure imaginary almost everywhere. This is a contradiction, since

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|e^{it} - z|^2} F(e^{it})\, dt$$

whenever $F \in H^1$ (see [5, pp. 33-34]).

For the second part, write $\mu = \mu_1 + \mu_2$, where μ_1 is the point mass measure at an isolated point $e^{i\theta_0}$ of $\sigma(S)$ and $\mu_2 = \mu \mid \sigma(S) - \{e^{i\theta_0}\}$. Then

$$S(z; \mu) = S(z; \mu_1) S(z; \mu_2),$$

and consequently

$$S'(z; \mu) = S(z; \mu_1) S'(z, \mu_2) + S'(z; \mu_1) S(z; \mu_2) .$$

The first term is bounded in a small neighborhood of $e^{i\theta_0}$, and $|S(z; \mu_2)| = 1$ in this neighborhood. If $S'(z; \mu) \in H^{1/2}$, then the function

$$|S'(z; \mu_1)| = \frac{c}{|e^{i\theta_0} - z|^2} \quad \text{a.e.}$$

would be in $L^{1/2}$, a contradiction. Thus, $\sigma(S)$ must be perfect.

It follows from Theorems 1 and 2 that if $\sigma(S)$ satisfies the condition $\{\varepsilon_n\} \in \ell^k$ for all $k > 0$, then $S'(z; \mu) \in H^p$ $(p < 1/2)$, but $S'(z; \mu) \notin H^{1/2}$. Cantor sets satisfying this condition are easily constructed (use intervals of length proportional to n^{-n}, for example). In case $\sigma(S)$ is the classical middle-third Cantor set translated to T, then Theorem 1 gives the relation $S'(z; \mu) \in H^p$ for $p < \frac{1 - (\log 2)/(\log 3)}{2} = 0.1845\ldots$

The following theorem shows that no restrictions on μ are needed in order that $S'(z; \mu) \in B^p$ $(p < 1/2)$:

THEOREM 3. *$S'(z; \mu) \in B^p$ for all $p < 1/2$.*

Proof. $|S'(re^{i\theta}; \mu)| \leq \int \frac{2}{|e^{it} - re^{i\theta}|^2} d\mu(e^{it}) \leq \frac{2}{1 - r^2} \int_T \frac{1 - r^2}{|e^{it} - z|^2} d\mu(e^{it})$.
Since the function defined by the integral is positive and harmonic, it follows that $\int_0^{2\pi} |S'(re^{i\theta}; \mu)|\, d\theta \leq c/(1 - r)$, where c is a constant independent of r. The result now follows, since $\int_0^1 (1 - r)^{1/p-3} dr$ is finite if and only if $p < 1/2$.

We conjecture that it is impossible for $S'(z; \mu)$ to belong to $B^{1/2}$, and since $H^{1/2} \subset B^{1/2}$, it is impossible that $S'(z; \mu) \in H^{1/2}$.

The following theorem gives a criterion that makes use not only of $\sigma(S)$, but of μ as well.

THEOREM 4. *Let μ be a singular Borel measure on T, and define $\alpha(t) = \mu(\{e^{i\theta}: 0 \leq \theta \leq t\})$. Suppose $\alpha(t)$ has constant value c_n on the complementary interval of length ε_n, and that $\{c_n \varepsilon_n\} \in \ell^k$ for some $k < 1/4$. Then $S'(z; \mu) \in H^p$ for all $p < 1/4$. Moreover, $S'(z; \mu)$ can not belong to the class $H^{1/2}$.*

Proof. As in Theorem 1, it is sufficient to prove that $f(z) \in H^p$, where

$$f(z) = -\int \frac{2e^{it}}{(e^{it} - z)^2} d\mu(e^{it}) .$$

Fix z with $|z| < 1$, and let $g(t) = \frac{e^{it}}{(e^{it} - z)^2}$. Then, by partial integration,

$$f(z) = \frac{-2\mu(T)}{(1 - z)^2} + 2 \int_0^{2\pi} \alpha(t) g'(t) dt .$$

Suppose $T - (\sigma(S) \cup \{1\}) = \bigcup_{n=1}^{\infty} \{e^{it}: a_n < t < b_n\}$, and $\alpha(t) = c_n$ on (a_n, b_n). Since

$$\int_{a_n}^{b_n} \alpha(t) g'(t) dt = c_n (e^{ib_n} - e^{ia_n}) \frac{z^2 - e^{i(a_n+b_n)}}{(e^{ia_n} - z)^2 (e^{ib_n} - z)^2} ,$$

it follows that

$$|f(z)| \le \frac{M_1}{|1 - z|^2} + M_2 \sum_{n=1}^{\infty} c_n \varepsilon_n \frac{1}{|e^{ib_n} - z|^2 |e^{ia_n} - z|^2} ,$$

where M_1 and M_2 are constants independent of z.

Now $\{c_n \varepsilon_n\} \in \ell^p$ for all $p \ge k$, since $(c_n \varepsilon_n)^k \ge (c_n \varepsilon_n)^p$ for $c_n \varepsilon_n < 1$. One can now easily complete the proof by using the Cauchy-Schwarz inequality together with the elementary facts that

(1) if $0 < p < 1$ and $x, y \ge 0$, then $(x + y)^p \le x^p + y^p$,

(2) if $|\beta| = 1$, then $\int_0^{2\pi} \frac{1}{|re^{it} - \beta|^{4p}} dt \le \int_0^{2\pi} \frac{1}{|e^{it} - \beta|^{4p}} dt = A_{4p}$, where A_{4p} denotes a constant independent of β and finite for $p < 1/4$.

For the second part, if $S' \in H^{1/2}$, then $f(e^{it}) \in L^{1/2}$, since $|S| = 1$ a.e. on T. Since $f \in H^p$ for $p < 1/4$, this is sufficient to imply that $f \in H^{1/2}$. A contradiction follows exactly as in Theorem 2.

Example. The following is an example of a singular inner function S for which $\sigma(S)$ is perfect and is not a Carleson set, while $S' \in H^p$ for $p < 1/4$. On $[0, 2\pi]$, let $\{(a_n, b_n)\}$ be a sequence of intervals converging to 0 with $b_{n+1} < a_n$ and $b_n - a_n$ proportional to $1/n(\log n)^2$. Then

$$\sum_{n=2}^{\infty} (b_n - a_n) \log \frac{1}{(b_n - a_n)} = \infty .$$

On $[0, 2\pi]$, construct a perfect set E of measure zero by removing intervals I_n of lengths δ_n satisfying the condition $\sum_{n=1}^{\infty} \delta_n^k < \infty$ for $k > 1/5$. For $n > 1$, let E_n be the copy of this set in $[b_n, a_{n-1}]$. On $[a_n, b_n]$, define $\alpha(t) = n^{-5}$. On $[b_1, 2\pi]$, define $\alpha(t) = 1$. Finally, on $[b_n, a_{n-1}]$, define $\alpha(t) = \beta_n(t)$, where $\beta_n(t)$ is a continuous, singular, nondecreasing function such that $\beta_n(b_n) = n^{-5}$, $\beta_n(a_{n-1}) = (n-1)^{-5}$, and the support of the corresponding measure is E_n. If μ is the resulting measure on T and $S(z; \mu)$ is the corresponding inner function, then $\sigma(S)$ is perfect and is not a Carleson set, and

$$\sum_{n=1}^{\infty} (c_n \varepsilon_n)^k < \left(\sum_{n=1}^{\infty} \delta_n^k \right) \left(\sum_{n=1}^{\infty} n^{-5k} \right) + \sum_{n=2}^{\infty} n^{-6k} (\log n)^{-2k} + (2\pi - a_1)^k ;$$

the right-hand side is finite for $k > 1/5$. Hence, Theorem 4 implies that $S'(z; \mu) \in H^p$ for $p < 1/4$ and also that $S'(z; \mu)$ is not in the class $H^{1/2}$.

Two unpublished results of Caughran and Shields should be mentioned. First, when $\sigma(S)$ is a Carleson set, the derivatives $S^{(n)}(z; \mu)$ are in the class N^+ (see [4, pages 25-28]). It follows that $S^{(n)} \in H^p$ if and only if the radial limit function $S^{(n)}(e^{it}) \in L^p$. Second, if S is a singular function, then S'/S is not in the class $B^{1/2}$. Note that the proof of Theorem 3 shows that S'/S is in B^p, for all $p < 1/2$.

REFERENCES

1. J. G. Caughran, *Factorization of analytic functions with* H^p *derivative.* Duke Math. J. 36 (1969), 153-158.

2. J. G. Caughran and A. L. Shields, *Singular inner factors of analytic functions.* Michigan Math. J. 16 (1969), 409-410.

3. D. J. Caveny and W. P. Novinger, *Boundary zeros of functions with derivative in* H^p. Proc. Amer. Math. Soc. 25 (1970), 776-780.

4. P. L. Duren, *Theory of* H^p *spaces.* Academic Press, New York, 1970.

5. G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals. II.* Math. Z. 34 (1931), 403-439.

6. K. Hoffman, *Banach spaces of analytic functions.* Prentice-Hall, Englewood Cliffs, N.J., 1962.

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