THE HARDY CLASS OF A SPIRAL-LIKE FUNCTION

Lowell J. Hansen

1. INTRODUCTION

A univalent function f analytic on the open unit disk Δ is said to be *spiral-like* of order σ ($|\sigma| < \pi/2$) if it is normalized ($f(0) = 0$ and $f'(0) = 1$), and if in addition it satisfies the condition

$$\Re[e^{i\sigma} z f'(z)/f(z)] > 0 \qquad (z \in \Delta).$$

(Spiral-like functions were considered by L. Špaček [5].) *For each spiral-like function* f, *we shall determine, by studying the region* $f(\Delta)$, *the Hardy classes* H_p *to which* f *belongs*. This is the object of Theorem 1, which is stated in Section 2 and proved in Section 3. The proof will use the notion of the Hardy number of a region, which was defined and studied in [1]. Theorem 1 enables us to draw some conclusions concerning the growth of the maximum modulus and the Taylor coefficients of spiral-like functions (Section 4).

2. PRELIMINARIES

Let Ω be a region (that is, a connected nonempty open set in the finite complex plane) that contains the point $z = 0$. We shall say that Ω is *spiral-like* of order σ ($|\sigma| < \pi/2$) if, whenever $z_0 \in \Omega$, the spiral $\{z_0 \exp(te^{-i\sigma}): t \leq 0\}$ is also contained in Ω. We note that if $\sigma = 0$, then Ω is starlike with respect to the point $z = 0$.

The relationship between spiral-like functions and spiral-like regions is indicated by the following lemma.

LEMMA 1. *Let* f *be a normalized univalent function analytic on the unit disk* Δ. *Then* f *is spiral-like of order* σ *if and only if* $f(\Delta)$ *is spiral-like of order* σ.

A proof of the special case $\sigma = 0$ is given by W. Hayman in [3, pp. 14-15]. The proof of the general case is similar.

Let Ω be a region, and let I_Ω denote the identity map on Ω. We recall from [1] that the *Hardy number* of Ω is defined by the condition

(1) $$h(\Omega) = \sup\{p \geq 0: |I_\Omega|^p \text{ possesses a harmonic majorant}\}.$$

The most significant property of $h(\Omega)$ is that if f is analytic on Δ, $f(\Delta) \subseteq \Omega$, and $h(\Omega) > 0$, then f belongs to each Hardy class H_p ($0 < p < h(\Omega)$). We shall also use the facts that

(i) if $\Omega_1 \subseteq \Omega_2$, then $h(\Omega_2) \leq h(\Omega_1)$,

(ii) if $\Omega_2 = \{az + b: z \in \Omega_1\}$ ($a \neq 0$), then $h(\Omega_2) = h(\Omega_1)$.

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Let Ω be a region, and let t be a positive real number. We define $\alpha_\Omega(t)$ to be the angular Lebesgue measure of the largest arc contained in $\Omega \cap \{|z| = t\}$. If Ω is spiral-like, then the function $t \to \alpha_\Omega(t)$ is nonincreasing and nonnegative, and therefore $\lim_{t\to\infty} \alpha_\Omega(t)$ exists.

THEOREM 1. *Let* f *be spiral-like of order* σ *on the unit disk* Δ. *Let* $\Omega = f(\Delta)$ *and* $A = \lim_{t\to\infty} \alpha_\Omega(t)$. *Then* $h(\Omega) = \pi/A\cos^2\sigma$. *Consequently* $f \in H_p$ $(0 < p < \pi/A\cos^2\sigma)$. *For* $p = \pi/A\cos^2\sigma$, *the latter conclusion does not hold if* $A > 0$.

Theorem 1 is already known in the case $\sigma = 0$ (Theorem 4.1 of [1]).

3. PROOF OF THEOREM 1

Since $f \in H_\infty$ when f is bounded, we consider only the unbounded case.

It follows from Corollary 3.2 of [1] that if $\alpha_\Omega(t) \to 0$, then $h(\Omega) = \infty$. Therefore the theorem holds in the special case where $A = 0$.

Suppose that $0 < A \leq 2\pi$. Then, as in the case where Ω is starlike (Theorem 4.1 of [1]), we can find two spirals

$$\{\exp[i(\theta - A/2) + te^{-i\sigma}]: -\infty < t < +\infty\} \quad \text{and} \quad \{\exp[i(\theta + A/2) + te^{-i\sigma}]: -\infty < t < +\infty\}$$

such that the region S between them with angular opening A is contained in Ω. The region S is the image under the exponential map of the infinite strip

$$\{x + iy: |y - (\theta - x\tan\sigma)| < A/2\}.$$

Therefore the function

$$F(z) = e^{i\theta}\exp\left[\frac{A\cos\sigma}{\pi} e^{-i\sigma}\operatorname{Log}\left(\frac{1+z}{1-z}\right)\right] \tag{2}$$

maps Δ univalently onto S, where $\operatorname{Log}(w)$ denotes the principal logarithm of w. Since F is univalent, we see that

$$h(S) = \sup\{p: F \in H_p\}.$$

Since

$$|F(z)|^p = \exp\left\{\frac{pA\cos\sigma}{\pi}\left[(\cos\sigma)\log\left|\frac{1+z}{1-z}\right| + (\sin\sigma)\operatorname{Arg}\left(\frac{1+z}{1-z}\right)\right]\right\},$$

we have the inequalities

$$K^{-1}\left|\frac{1+z}{1-z}\right|^{pA\cos^2\sigma/\pi} \leq |F(z)|^p \leq K\left|\frac{1+z}{1-z}\right|^{pA\cos^2\sigma/\pi},$$

where $K = \exp[(pA\cos|\sigma|\sin|\sigma|)/2]$. We conclude that $F \in H_p$ precisely when $0 < p < \pi/A\cos^2\sigma$, since $(1+z)/(1-z) \in H_q$ for $0 < q < 1$. That is, $h(S) = \pi/A\cos^2\sigma$. Since $S \subseteq \Omega$, we have the inequality

$$h(\Omega) \leq h(S) = \pi/A\cos^2\sigma. \tag{3}$$

To obtain the reverse inequality, we construct a sequence $\{\Omega_n\}$ of regions containing Ω such that $h(\Omega_n) \to \pi/A \cos^2\sigma$. For this construction we need the following lemma.

LEMMA 2. *Let* G *be an open set in the interval* $[a, b] = \{x: a \leq x \leq b\}$ *and suppose that each component of* G *has length at most* m. *Then* G *is contained in an open subset of* [a, b] *whose components are finite in number and have length at most* m.

For the proof, arrange the components of G into a sequence $\{S_n\}$, and cover S_1 with a maximal open interval (call it T_1) of length at most m and with endpoints in $[a, b]\setminus G$. Then cover the first component of $G\setminus T_1$ with a similar interval T_2 disjoint from T_1, and so forth. Clearly, the process terminates after finitely many steps.

We apply the lemma to get a finite union $\mathscr{U}_n$ of open arcs in $\{|z| = n\}$ with the properties that

(i) $\Omega \cap \{|z| = n\} \subseteq \mathscr{U}_n$, and

(ii) each arc in $\mathscr{U}_n$ has angular measure no greater than $\alpha_\Omega(n)$.

We then let

$$\Omega_n = \{|z| < n\} \cup \left(\bigcup_{z \in \mathscr{U}_n} S_{\sigma,z} \right),$$

where $S_{\sigma,z} = \{z \exp(te^{-i\sigma}): t \geq 0\}$. The region Ω_n is spiral-like of order σ and contains Ω. For each n large enough so that Ω_n is not the entire complex plane, we fix a boundary point z_n of Ω_n and let $\Omega_n' = \{z - z_n: z \in \Omega_n\}$. Then $h(\Omega_n') = h(\Omega_n)$, and thus $h(\Omega) \geq h(\Omega_n')$. We conclude by Theorem 5.2 of [1] that

$$h(\Omega_n') \geq \frac{\pi}{\alpha_\Omega(n)} (1 + \tan^2\sigma) = \pi/[\alpha_\Omega(n) \cos^2\sigma].$$

(The theorem to which we refer states that $h(\Omega_n') \geq \pi(1 + \lambda^2)/\beta$, where, in the present case, $\lambda = |\tan\sigma|$ and $\beta = \alpha_\Omega(n)$.) Since $h(\Omega) \geq h(\Omega_n) = h(\Omega_n')$, we get the inequality

$$h(\Omega) \geq \pi/[\alpha_\Omega(n) \cos^2\sigma].$$

Letting $n \to \infty$, we obtain the inequality $h(\Omega) \geq \pi/A \cos^2\sigma$. This, together with inequality (3), implies that $h(\Omega) = \pi/A \cos^2\sigma$.

We now show that if $A > 0$, then $f \notin H_p$ $(p = \pi/A \cos^2\sigma)$. For suppose that F is analytic on Δ, with $F(\Delta) \subseteq f(\Delta)$. If $f \in H_q$, then $|f|^q$ has a harmonic majorant u. Consequently,

$$|F(z)|^q = |f[f^{-1}(F(z))]|^q \leq u[f^{-1}(F(z))],$$

and thus $F \in H_q$. In particular, if F is the function defined in equation (2), then $F(\Delta) \subseteq f(\Delta)$ and $F \notin H_p$ $(p = \pi/A \cos^2\sigma)$. Therefore $f \notin H_p$ $(p = \pi/A \cos^2\sigma)$.

4. SOME APPLICATIONS

Let f be spiral-like of order σ on Δ. Then, since $f \in H_p$ $(0 < p < \pi/A \cos^2\sigma)$, we obtain the following results from known theorems about H_p.

THEOREM 2. *If* $M(r, f) = \max_{|z|=r} |f(z)|$, *then*

$$\lim_{r \to 1} [(1 - r)^{1/p} M(r, f)] = 0 \qquad (0 < p < \pi/A \cos^2 \sigma) .$$

(See Hardy and Littlewood [2].)

Thus if either $\sigma \neq 0$ or $A < 2\pi$, then $M(r, f) = o[(1 - r)^{-2}]$. If $\sigma = 0$ and $A = 2\pi$, then $f(z) = z(1 - z e^{i\theta})^{-2}$ and thus $M(r, f) = O[(1 - r)^{-2}]$. We suspect that if $A \neq 0$, then

$$M(r, f) = O[(1 - r)^{-1/p}] \qquad (p = \pi/A \cos^2 \sigma) .$$

THEOREM 3. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ $(z \in \Delta)$.

(i) *If* $\pi/A \cos^2 \sigma > 1$, *then* $a_n \to 0$.

(ii) *If* $\pi/A \cos^2 \sigma \leq 1$, *then* $|a_n| = o(n^{1/p-1})$ $(0 < p < \pi/A \cos^2 \sigma)$.

(See Privalov [4, pp. 110-114].)

From statement (ii) of Theorem 3 we see that if either $\sigma \neq 0$ or $A < 2\pi$, then $|a_n|/n \to 0$. Therefore, the only spiral-like functions whose Taylor coefficients do not satisfy the condition $|a_n|/n \to 0$ are the Koebe functions $z(1 - ze^{i\theta})^{-2}$, and $|a_n| = n$ in this case. Again we suspect that if $\pi/A \cos^2 \sigma \leq 1$, then

$$|a_n| = O[n^{1/p-1}] \qquad (p = \pi/A \cos^2 \sigma) .$$

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Wayne State University
Detroit, Michigan 48202