INTERTWINING ANALYTIC TOEPLITZ OPERATORS

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Let A and B be bounded linear operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively. We say that a bounded linear operator X from $\mathscr{H}$ into $\mathscr{K}$ *intertwines* A and B if XA = BX. B. Sz.-Nagy and C. Foiaş [8] have shown that any intertwining operator between two contractions extends to an intertwining operator between their coisometric extensions (see also [3]). However, an intertwining operator between two subnormal operators need not extend to an intertwining operator between their normal extensions (example: $A = T_z$ and $B = 0$; see also [1]). R. G. Douglas and C. Pearcy [2] gave a necessary and sufficient condition that there be no nonzero intertwining operator between two normal operators; because of the theorem of Fugledge and Putnam, this condition is symmetric in the two operators. However, there exist operators for which this property is not symmetric. That is, there exist operators A and B such that there are nonzero operators intertwining A and B but no nonzero operators intertwining B and A (example: $A = T_z$ and $B = 0$). These two phenomena make the study of intertwining operators between analytic Toeplitz operators of interest. In this note, we obtain an asymmetric, sufficient condition for the nonexistence of nonzero intertwining operators between two analytic Toeplitz operators. By means of this result, we then obtain an example of an operator whose commutant is abelian but that does not have a cyclic vector.

For convenience, we consider H^2 to be the Hilbert space of analytic functions in the unit disk for which the functions $f_r(\theta) = f(re^{i\theta})$ are bounded in the L^2-norm, and H^∞ to be the linear manifold of bounded functions in H^2. For $\phi \in H^\infty$, T_ϕ (or $T_{\phi(z)}$) is the *analytic Toeplitz operator* on H^2 defined by the relation $(T_\phi f)(z) = \phi(z) f(z)$. We shall denote the spectrum of T_ϕ by $\sigma(T_\phi)$ and the set $\{\phi(z): |z| < 1\}$ by range (ϕ). Then $\sigma(T_\phi)$ = closure (range (ϕ)) [4, Problems 26 and 197]. If $\phi \in H^2$, then the function $\bar{\phi}$ defined by $\bar{\phi}(z) = \overline{\phi(\bar{z})}$ is also in H^2. For $|\lambda| < 1$, define $h_\lambda \in H^2$ by the relation $h_\lambda(z) = (1 - \lambda z)^{-1}$. Then $T_\phi^* h_\lambda = \bar{\phi}(\lambda) h_\lambda$ for $\phi \in H^\infty$ [7].

LEMMA. *If* Λ *is an uncountable subset of the disk* $|\lambda| < 1$, *then* $\{h_\lambda: \lambda \in \Lambda\}$ *spans* H^2.

Proof. Suppose $f \in H^2$ is orthogonal to $\{h_\lambda: \lambda \in \Lambda\}$. Then $f(\bar{\lambda}) = (f, h_\lambda) = 0$ for $\lambda \in \Lambda$. If Λ is uncountable, then $f \equiv 0$, since f is analytic. Hence $\{h_\lambda: \lambda \in \Lambda\}$ spans H^2.

THEOREM. *Let* $\phi, \psi \in H^\infty$. *If* range $(\psi) \not\subseteq \sigma(T_\phi)$, *then the only bounded linear operator* X *satisfying the condition* $XT_\phi = T_\psi X$ *is* $X = 0$.

Proof. Let $N \equiv$ range $(\psi) \cap \mathbb{C} \setminus \sigma(T_\phi)$. Then N is either a nonempty open set or a singleton, depending on whether range (ψ) is an open set or a singleton (that is, whether ψ is nonconstant or constant). In either case, $\psi^{-1}(N)$ is a nonempty *open* subset in $\{z: |z| < 1\}$, and hence uncountable. By our lemma, $\{h_\lambda: \bar{\lambda} \in \psi^{-1}(N)\}$

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spans H^2. Suppose X satisfies $XT_\phi = T_\psi X$. Then $T_\phi^* X^* = X^* T_\psi^*$; evaluating at h_λ, we find that

$$T_\phi^*(X^* h_\lambda) = \overline{\psi}(\lambda)(X^* h_\lambda),$$

since h_λ is an eigenvector for T_ψ^*. Consequently, for each $|\lambda| < 1$, either $\overline{\psi}(\lambda)$ is an eigenvalue for T_ϕ^* or $X^* h_\lambda = 0$. Since $\bar\lambda \in \psi^{-1}(N)$ implies that $\psi(\bar\lambda) \notin \sigma(T_\phi)$, or equivalently, that $\overline{\psi}(\lambda) \notin \sigma(T_\phi^*)$, the number $\overline{\psi}(\lambda)$ cannot be an eigenvalue for T_ϕ^*. Thus $X^* h_\lambda = 0$ for $\bar\lambda \in \psi^{-1}(N)$. Since $\{h_\lambda : \bar\lambda \in \psi^{-1}(N)\}$ spans H^2, $X^* = 0$, or equivalently, $X = 0$.

A theorem of M. Rosenblum [5] states that if A and B are operators such that $\sigma(A) \cap \sigma(B) = \emptyset$ and $XA = BX$, then $X = 0$. In the case of analytic Toeplitz operators, our theorem is stronger, because it allows for a possible overlapping of the spectrums.

COROLLARY 1. *Let* $\phi, \psi \in H^\infty$. *If*

(i) interior (closure (range (ϕ))) = range (ϕ), *and*

(ii) range ($\overline{\psi}$) $\not\subseteq$ point spectrum (T_ϕ^*),

then the only bounded linear operator X *satisfying the condition* $XT_\phi = T_\psi X$ *is* $X = 0$.

Proof. First observe that (i) implies $\mathbb{C} \setminus$ range ($\bar\phi$) = closure ($\mathbb{C} \setminus \sigma(T_\phi^*)$). Consequently

$$\text{closure } (\mathbb{C} \setminus \sigma(T_\phi^*)) \supseteq \mathbb{C} \setminus \text{p. s. } (T_\phi^*) \supseteq \mathbb{C} \setminus \sigma(T_\phi^*).$$

It then follows from (ii) that $N \equiv$ range ($\overline{\psi}$) $\cap$ $C \setminus$ p. s. (T_ϕ^*) either contains a nonempty open set or else is a singleton, depending on whether ψ is nonconstant or constant. The proof now proceeds exactly as above, with the use of the uncountable set $\overline{\psi}^{-1}(N)$.

COROLLARY 2. *If* $\phi \in H^\infty$, $\phi \not\equiv 0$, *and* $|\alpha| > 1$, *then* $XT_\phi = T_{\alpha\phi} X$ *implies* $X = 0$.

Proof. Since $|\alpha| > 1$, range ($\alpha\phi$) $\not\subseteq \sigma(T_\phi)$.

COROLLARY 3. *If* $\phi \in H^\infty$ *and* $\alpha \neq 0$, *then* $XT_\phi = T_{\alpha+\phi} X$ *implies* $X = 0$.

Proof. Since $\alpha \neq 0$, range (ϕ) $\not\subseteq \sigma(T_{\alpha+\phi})$.

We comment on Corollary 3. For an operator A on a Hilbert space $\mathscr{H}$, define Δ_A, the *derivation* of A from $\mathscr{L}(\mathscr{H})$ into $\mathscr{L}(\mathscr{H})$, by the equation $\Delta_A(B) = BA - AB$. In this setting, Corollary 3 states that 0 is the only eigenvalue of the derivation of an analytic Toeplitz operator. A consequence of a theorem of D. Kleinecke is that if $XT_\phi = T_{\alpha+\phi} X$, then X is quasinilpotent. Corollary 3 is stronger, because it states that X must be 0 [4, Problem 184].

We remark that our condition range (ψ) $\not\subseteq \sigma(T_\phi)$ is not necessary; example: $\psi = 0$ and ϕ is an outer function with $0 \in \sigma(T_\phi)$. We also remark that the condition range (ψ) $\not\subseteq$ range (ϕ) is not sufficient; example: $\psi(z) = z$ and ϕ is a singular inner function. However, we conjecture that the condition range ($\overline{\psi}$) $\not\subseteq$ p. s. (T_ϕ^*) is both necessary and sufficient for the nonexistence of a nonzero intertwining operator. The necessity seems very difficult; however, we can verify it in a few cases.

PROPOSITION. *Let* $\phi, \psi \in H^\infty$, *and let* ϕ *be univalent. If*

$$\text{range}\,(\psi) \subseteq \text{range}\,(\phi),$$

then there exists a nonzero bounded linear operator X *satisfying the condition* $XT_\phi = T_\psi X$.

Proof. Since ϕ is one-to-one and range $(\psi) \subseteq$ range (ϕ), the function $F(z) \equiv \phi^{-1}(\psi(z))$ is analytic in $\{|z| < 1\}$ and $\{F(z): |z| < 1\} \subset \{\zeta: |\zeta| < 1\}$. If we define X on H^2 by $(Xf)(z) = f(\phi^{-1}(\psi(z)))$, then Theorem 1 in [6, p. 348] implies that X is a nonzero bounded linear operator on H^2. A computation shows that $XT_\phi = T_\psi X$.

At the International Symposium on Operator Theory at Indiana University, B. Sz.-Nagy asked what the relations are between the two conditions

(i) T and T^* have cyclic vectors,

(ii) $\{T\}' \equiv \{S \in \mathscr{L}(\mathscr{H}): ST = TS\}$ is abelian.

We shall use our Theorem to show that, in general, (ii) does not imply (i). That is, we shall present an example of an operator whose commutant is abelian but that does not have a cyclic vector.

For $\alpha \neq 0$, let $T = T_z \oplus T_{\alpha+z}$ on $H^2 \oplus H^2$. Then $\{T\}'$ is abelian. For if $S \in \{T\}'$, then

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}$$

with

$$S_1 \in \{T_z\}', \qquad S_4 \in \{T_{\alpha+z}\}' = \{T_z\}', \qquad T_z S_2 = S_2 T_{\alpha+z}, \qquad T_{\alpha+z} S_3 = S_3 T_z.$$

Applying Corollary 3, we note that $S_2 = S_3 = 0$ and hence $\{T\}' = \{T_z\}' \oplus \{T_z\}'$. Since $\{T_z\}'$ is abelian [7], $\{T\}'$ is abelian.

Next we show that for $0 < |\alpha| < 2$, T does not have a cyclic vector, although for $|\alpha| > 2$, T does have a cyclic vector. We first consider the case $0 < |\alpha| < 2$. Suppose $f \oplus g$ is a cyclic vector for $T = T_z \oplus T_{\alpha+z}$. Then there exist polynomials $\{p_n\}$ such that

$$\| p_n(T_z \oplus T_{\alpha+z})(f \oplus g) - (f \oplus 0)\|^2 < 1/n,$$

or equivalently, that

$$\|p_n(T_z)f - f\|^2 + \|p_n(T_{\alpha+z})g\|^2 < 1/n.$$

Now, for each $\rho \in \{z: |z| < 1 \text{ and } |z - \alpha| < 1\}$,

$$(*) \qquad |p_n(\rho) - 1|^2 |f(\rho)|^2 K_\rho + |p_n(\rho)|^2 |g(\rho - \alpha)|^2 K_{\rho-\alpha} < 1/n,$$

where for $|z| < 1$, $K_z = (1 - |z|^2)$ is a nonzero constant depending only on z [4, Problem 31]. Consider the set

$$Z = \{z: |z| < 1, |z - \alpha| < 1, f(z) \neq 0, g(z - \alpha) \neq 0\}.$$

Since f and g are cyclic for T_z and $T_{\alpha+z}$, respectively, they are analytic functions that are not identically zero. Hence Z is a nonempty set for $0 < |z| < 2$. Letting $\rho \in Z$ and considering (*), we conclude that $p_n(\rho) \to 1$ and $p_n(\rho) \to 0$, which contradicts the assumption that T has a cyclic vector. Hence, for $|\alpha| < 2$, T has no cyclic vector. In case $|\alpha| > 2$, the operator T has a cyclic vector because of the relation

$$\mathscr{A}_{T_z \oplus T_{\alpha+z}} = \mathscr{A}_{T_z} \oplus \mathscr{A}_{T_{\alpha+z}},$$

where $\mathscr{A}_A$ is the closure in the weak operator topology of the algebra of polynomials in A. The assertion about the algebra generated by $T_z \oplus T_{\alpha+z}$ holds because the spectrums of T_z and $T_{\alpha+z}$ are disjoint disks.

A. Shields has pointed out that in case $0 < |\alpha| < 1$, the noncyclicity of T follows from the fact that range (T) is a closed subspace of codimension at least 2.

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