AN UNSPLITTABLE TRIANGULATION

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In this note I show that there exists a combinatorial n-manifold (that is, a simplicial complex in which the star of every vertex is a combinatorial n-ball) that cannot be expressed as the union of two combinatorial sub-n-manifolds (sub-complexes) that meet only in sub-complexes of their boundaries. The example demonstrates the futility of direct attempts to prove results about n-manifolds by induction on the number of n-simplexes in a combinatorial triangulation, and it emphasises the necessity of taking subdivisions of complexes if reasonable results are desired. In fact, my example is a triangulation of the n-ball ($n > 3$); note that no such example exists for a manifold without boundary, because an n-simplex and the closure of its complement form a possible splitting. The example depends on the following lemma.

LEMMA. *There exists a simplicial complex* K *having more than one* 3-*simplex, which triangulates the* 3-*ball, such that* K *does not contain a disc* D *properly imbedded as a sub-complex* (*in other words, such that* $D \cap \partial K = \partial D$).

Proof. Suppose the converse, namely that every such complex K does contain such a disc D, which thus divides K into two 3-balls meeting in D. Let P_n denote the proposition that each complex L that triangulates the 3-ball and that has at most n 3-simplexes collapses simplicially to $\partial L - A$ (in symbols: $L \searrow^{s} (\partial L - A)$) for each 2-simplex A contained in ∂L. P_1 is trivially true. Suppose P_n is true. Let K triangulate the 3-ball and have $n + 1$ 3-simplexes. By the first supposition, $K \supset D$, where D is a disc sub-complex dividing K into two sub-complexes K_1 and K_2 that triangulate 3-balls and that contain at most n 3-simplexes. Let A be a 2-simplex in ∂K; without loss of generality, assume $A \subset \partial K_1$. The proposition P_n implies that $K_1 \searrow^{s} \partial K_1 - A$; hence

$$K \searrow^{s} K_2 \cup (\partial K - A),$$

and if B is a 2-simplex in D, then $K_2 \searrow^{s} (\partial K_2 - B)$, by P_n. One can easily show that $D - B \searrow^{s} \partial D$. Composing these collapses, one finds that $K \searrow^{s} (\partial K - A)$. This establishes P_{n+1}, so that P_n seems to be true for all n. However, with the notation of P_n, $\partial L - A$ is a disc, and hence it is simplicially collapsible; therefore L is simplicially collapsible. Thus every complex triangulating a 3-ball collapses simplicially, and this is false by the example of R. H. Bing [1], [2]. Hence the original supposition cannot be true.

PROPOSITION. *For each* $n \geq 4$, *there exists a combinatorial* n-*manifold* M *having more than one* n-*simplex such that* M *cannot be expressed as a union*

$$M = M_1 \cup M_2,$$

where M_1 *and* M_2 *are combinatorial* n-*manifolds that are sub-complexes of* M *and where* $M_1 \cap M_2 = \partial M_1 \cap \partial M_2$.

Proof. Let M^4 denote $v * K$, the cone on a complex K with the properties stated in the lemma. If M can be expressed as $M_1 \cup M_2$ in the way indicated above, then

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M_1 and M_2 both contain 4-simplexes, and thus $v \in M_1 \cap M_2$. Each M_i is a cone with vertex v, and, to be a combinatorial manifold, each M_i must be a cone on a 3-ball sub-complex K_i of K. The relation

$$M_1 \cap M_2 = \partial M_1 \cap \partial M_2$$

implies that $K_1 \cap K_2 = \partial K_1 \cap \partial K_2$. Therefore $K_1 \cap K_2$ must be a disc properly imbedded in K. This contradicts the choice of K and proves the result. By taking repeated cones on M^4 and by using the same proof, I can obtain examples for each $n \geq 4$.

REFERENCES

1. R. H. Bing, *Some aspects of the topology of 3-manifolds related to the Poincaré conjecture*. Lectures on modern mathematics, Vol. 2, pp. 93-128. Wiley, New York, 1964.
2. R. E. Goodrick, *Non-simplicially collapsible triangulations of* I^n. Proc. Cambridge Philos. Soc. 64 (1968), 31-36.

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