A CLASSIFICATION OF HYPERELLIPTIC RIEMANN SURFACES WITH AUTOMORPHISMS BY MEANS OF CHARACTERISTIC RIEMANN MATRICES

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It has been shown [5] that a hyperelliptic Riemann surface S of even genus g has an automorphism (conformal self-homeomorphism) σ of order 2 other than the interchange ι of sheets if and only if S has a Riemann matrix of the form

$$\frac{1}{2}\begin{pmatrix} \hat{M} & I \\ I & -\tilde{M}^{-1} \end{pmatrix} \quad \text{or, equivalently,} \quad \frac{1}{2}\begin{pmatrix} \tilde{M}+\hat{M} & \tilde{M}-\hat{M} \\ \tilde{M}-\hat{M} & \tilde{M}+\hat{M} \end{pmatrix},$$

where all the entries are submatrices of order g/2, and where I is the multiplicative identity matrix. Furthermore, $\tilde{M}$ and $\hat{M}$ are Riemann matrices for the quotient surfaces S/σ and $S/\iota\sigma$, respectively, which are elliptic or hyperelliptic; in the latter case, the natural projections map the hyperelliptic branch points (Weierstrass points) of S over the Riemann sphere P to the hyperelliptic branch points of the respective quotient surfaces over P. A similar result holds for odd genus. The object of this paper is to complete the classification of hyperelliptic Riemann surfaces with automorphisms by means of characteristic Riemann matrices.

Let S be a compact Riemann surface of genus $g > 0$. A set of (independent) one-cycles (a_i, b_i) $(i = 1, \cdots, g)$ satisfying the conditions

$$\delta(a_i, b_j) = \delta_{ij} \quad \text{and} \quad \delta(a_i, a_j) = 0 = \delta(b_i, b_j),$$

where δ is the bilinear, skew-symmetric intersection number, is called a set of *retrosections* for S, and the corresponding homology basis is said to be *canonical.* If $\omega_1, \cdots, \omega_g$ form a basis for the holomorphic differentials on S, then the $g \times 2g$ matrix

$$(A\ B) = \left(\left(\int_{a_j} \omega_i\right)\left(\int_{b_j} \omega_i\right)\right)$$

is called a *period matrix* for S. By a change of basis for the holomorphic differentials, the matrix A can be reduced to the multiplicative identity (the new basis is said to be *normalized* with respect to (a_i, b_i)), and then B becomes $A^{-1}B$, which is symmetric, has positive-definite imaginary part, and is called the *Riemann matrix for* S *with respect to* (a_i, b_i). Torelli's theorem says that if the Riemann matrix for a surface S with respect to (a_i, b_i) is the same as the Riemann matrix for a surface S' with respect to (a_i', b_i'), then some conformal homeomorphism from S onto S' takes either a_i to a_i' and b_i to b_i', or a_i to $-a_i'$ and b_i to $-b_i'$ (in the sense that homologous cycles are identified; see [4, pp. 27-28] and [3]). If S' (and therefore S) is hyperelliptic, then conformality of one map implies conformality of the other,

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since the two maps then differ by the interchange of sheets on S', which is conformal. We note that if σ is an automorphism of order n on S, then S is an n-sheeted, branched, analytic covering of the quotient surface S/σ under the natural projection π. In addition to Torelli's theorem, we use a result due to A. Hurwitz [1, p. 257], which says that if a hyperelliptic Riemann surface S of genus g has an automorphism σ, then S has an equation of the form either

$$w^2 = zf(z^n), \qquad \text{with } \sigma: (z, w) \to (\varepsilon z, \sqrt{\varepsilon}\,\eta w), \text{ or}$$

$$w^2 = f(z^n), \qquad \text{with } \sigma: (z, w) \to (\varepsilon z, \eta w),$$

where $\varepsilon^n = 1$ and $\eta = \pm 1$. In either case, w^2 is of degree $2g+1$ or $2g+2$ in z. We may assume that ε is a primitive n^{th} root of unity; for if ε is a primitive k^{th} root of unity, then k divides n, say $mk = n$, and we consider $g(z^k) = f(z^{mk})$. Note that $(\sqrt{\varepsilon}\,\eta)^n = \pm 1$, so that σ is of order n or 2n. When n is even, $\sqrt{\varepsilon}^n = -1$, since we assume that ε is primitive. We may also assume that $\sqrt{\varepsilon}^n = +1$ when n is odd, since the case $\sqrt{\varepsilon}^n = -1$ merely interchanges the roles of σ and $\iota\sigma$, where $\iota: (z, w) \to (z, -w)$ is the interchange of sheets on S. In order to eliminate both the identity mapping and ι from consideration, we assume throughout that $n > 1$. Finally, we adopt the convention of denoting the case $\eta = +1$ by σ, and then $\eta = -1$ corresponds to $\iota\sigma$.

Consider first the case where $w^2 = zf(z^n)$, σ maps (z, w) onto $(\varepsilon z, \sqrt{\varepsilon} w)$, and n is odd. Then σ is of order n, and $\iota\sigma$ is of order 2n. Two points (z_1, w_1) and (z_2, w_2) of S are in the same orbit of σ if and only if

$$(z_1^n, z_1^{(n-1)/2} w_1) = (z_2^n, z_2^{(n-1)/2} w_2),$$

so that the natural projection $\tilde{\pi}: S \to S/\sigma$ is given by

$$\tilde{\pi}: (z, w) \to (z^n, z^{(n-1)/2} w) \equiv (\tilde{z}, \tilde{w}),$$

from which it follows that S/σ has the equation $\tilde{w}^2 = \tilde{z}f(\tilde{z})$. Two points (z_1, w_1) and (z_2, w_2) of S are in the same orbit of $\iota\sigma$ if and only if

$$(z_1^n, z_1^{n-1} w_1^2) = (z_2^n, z_2^{n-1} w_2^2),$$

so that the projection $\hat{\pi}: S \to S/\iota\sigma$ is given by

$$\hat{\pi}: (z, w) \to (z^n, z^{n-1} w^2) \equiv (\hat{z}, \hat{w}),$$

and $S/\iota\sigma$ has the equation $\hat{w} = \hat{z} f(\hat{z})$, that is, $S/\iota\sigma$ has genus $\hat{g} = 0$.

The other cases are similar. We list all the possibilities in Table 1.

Case	S	Parity of n	order of σ	$(\tilde{z}, \tilde{w})$	S/σ	order of $\iota\sigma$	$(\hat{z}, \hat{w})$	$S/\iota\sigma$
1	$w^2 = z f(z^n)$	odd	n	$(z^n, z^{(n-1)/2} w)$	$\tilde{w}^2 = \tilde{z} f(\tilde{z})$	2n	$(z^n, z^{n-1} w^2)$	$\hat{w} = \hat{z} f(\hat{z})$
2	$w^2 = z f(z^n)$	even	2n	$(z^n, z^{n-1} w^2)$	$\tilde{w} = \tilde{z} f(\tilde{z})$	2n	$(z^n, z^{n-1} w^2)$	$\hat{w} = \hat{z} f(\hat{z})$
3	$w^2 = f(z^n)$	odd	n	(z^n, w)	$\tilde{w}^2 = f(\tilde{z})$	2n	(z^n, w^2)	$\hat{w} = f(\hat{z})$
4	$w^2 = f(z^n)$	even	n	(z^n, w)	$\tilde{w}^2 = f(\tilde{z})$	n	$(z^n, z^{n/2} w)$	$\hat{w}^2 = \hat{z} f(\hat{z})$

Table 1.

We note that in all cases the quotient surface is either rational, elliptic, or hyperelliptic. Furthermore, when the quotient surface is hyperelliptic, the hyperelliptic branch points (Weierstrass points) of S *over* P *map by the natural projection into the hyperelliptic branch points of the quotient surface over* P.

In Case 1, if w^2 is of degree $2g+1$, then $\tilde{w}^2$ is of degree $(2g+n)/n$, and this is odd since n is odd. Hence $\tilde{w}^2$ is of degree $2\tilde{g}+1$, where $\tilde{g}$ is the genus of S/σ, and $n = g/\tilde{g}$. The surface $S/\iota\sigma$ has genus $\hat{g} = 0$. We refer to this possibility as Case 1.1. If w^2 is of degree $2g+2$, then $\tilde{w}^2$ is of degree $2\tilde{g}+2$, $n = (2g+1)/(2\tilde{g}+1)$, and again $\hat{g} = 0$. We refer to this possibility as Case 1.2.

The other cases are similar. We list all the possibilities in Table 2.

Case	degree w^2	degree $\tilde{w}^2$	n	degree $\hat{w}^2$	n
1.1	$2g+1$	$2\tilde{g}+1$	$g/\tilde{g}$	$(\hat{g}=0)$	-
2.1	$2g+1$	$(\tilde{g}=0)$	-	$(\hat{g}=0)$	(2g/n even)
3.1	$2g+2$	$2\tilde{g}+2$	$(g+1)/(\tilde{g}+1)$	$(\hat{g}=0)$	-
4.1	$2g+2$	$2\tilde{g}+2$	$(g+1)/(\tilde{g}+1)$	$2\hat{g}+1$	$(g+1)/\hat{g}$
1.2	$2g+2$	$2\tilde{g}+2$	$(2g+1)/(2\tilde{g}+1)$	$(\hat{g}=0)$	-
2.2	$2g+1$	$(\tilde{g}=0)$	-	$(\hat{g}=0)$	(2g/n odd)
3.2	$2g+1$	$2\tilde{g}+1$	$(2g+1)/(2\tilde{g}+1)$	$(\hat{g}=0)$	-
4.2	$2g+2$	$2\tilde{g}+1$	$(2g+2)/(2\tilde{g}+1)$	$2\hat{g}+2$	$(2g+2)/(2\hat{g}+1)$

Table 2.

We note that Cases 1.2 and 3.2 are equivalent. Indeed, the conformal homeomorphism $(z, w) \to (1/z, w/z^{g+1}) \equiv (Z, W)$ maps a surface of Type 1.2 onto a surface of Type 3.2.

Before determining the characteristic matrices, we introduce the following notation. A "cyclic" matrix of the form

$$\begin{pmatrix} M_0 & M_1 & M_2 & \cdots & M_{n-2} & M_{n-1} \\ M_{n-1} & M_0 & M_1 & \cdots & M_{n-3} & M_{n-2} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ M_1 & M_2 & M_3 & \cdots & M_{n-1} & M_0 \end{pmatrix},$$

where the M_k are $p \times q$ submatrices, will be denoted by

$$\langle M_0, \cdots, M_{n-1}; p \times q \rangle . \tag{1}$$

A matrix obtained from (1) by replacing the submatrices below the main block-diagonal by their negatives will be denoted by

$$[M_0, \cdots, M_{n-1}; p \times q] . \tag{2}$$

A matrix obtained from (1) or (2) by the deletion of the r^{th} block-column ($r = 1, \cdots, n$) will be denoted by

$$\langle M_0, \cdots, M_{n-1}; p \times q \rangle_r \quad \text{or} \quad [M_0, \cdots, M_{n-1}; p \times q]_r,$$

respectively. A superscript r indicates that the r^{th} block-row has been deleted. If the order of the submatrices is understood from the context, then $p \times q$ will be omitted from the notation. Finally, if C_r denotes the r^{th} block-column of a matrix M, then M^* denotes the matrix whose r^{th} block-column is $C_1 + C_2 + \cdots + C_r$.

Case 1.1

$w^2 = z(z^n - r_1^n) \cdots (z^n - r_{2g/n}^n)$, n odd.

$\sigma: (z, w) \to (\varepsilon z, \sqrt{\varepsilon} w)$ (order n), $\quad \tilde{w}^2 = \tilde{z}(\tilde{z} - r_1^n) \cdots (\tilde{z} - r_{2\tilde{g}}^n)$, $\quad n = g/\tilde{g}$.

$\iota\sigma: (z, w) \to (\varepsilon z, -\sqrt{\varepsilon} w)$ (order 2n), $\hat{w} = \hat{z}(\hat{z} - r_1^n) \cdots (\hat{z} - r_{2g/n}^n)$, $\hat{g} = 0$.

Without loss in generality, assume $\sqrt{\varepsilon} = -(\cos(\pi/n) + i \sin(\pi/n))$. We represent S as a two-sheeted, branched covering of the Riemann sphere P in the usual manner. There are $2\tilde{g}$ circular orbits of branch points with n branch points $\varepsilon^k r_i$ ($k = 0, \cdots, n-1$) in the i^{th} orbit ($i = 1, \cdots, 2\tilde{g}$). Also, 0 and ∞ are branch points. Let each pair

$$(\varepsilon^k r_{2i-1}, \varepsilon^k r_{2i}) \qquad (i = 1, \cdots, \tilde{g};\ k = 0, \cdots, n-1)$$

as well as $(0, \infty)$ determine a branch cut. Let b_i be a loop about the cut (r_{2i-1}, r_{2i}), and let a_i be a loop that passes from one sheet to the other through $(0, \infty)$ and (r_{2i-1}, r_{2i}) ($i = 1, \cdots, \tilde{g}$) as in Figure 1.1.

An inspection of Figure 1.1 shows that $\delta(a_i, \sigma a_i) = 1$ or -1. Again by the figure, if $\delta(a_i, \sigma a_i) = 1$, then $\delta(a_i, \sigma^k a_i) = 1$, and if $\delta(a_i, \sigma a_i) = -1$, then $\delta(a_i, \sigma^k a_i) = (-1)^k$ ($k = 1, \cdots, n-1$). In either case, $\delta(a_i, \sigma^{n-1} a_i) = 1$, since n is odd. But then $\delta(\sigma a_i, a_i) = 1$, since σ preserves intersection number and is of order n. Hence, $\delta(a_i, \sigma a_i) = -1$. It can therefore be seen that

$$\delta(\sigma^k b_i, \sigma^m b_j) = 0,$$

$$\delta(\sigma^k a_i, \sigma^m b_j) = \delta_{ij}\, \delta_{km}, \text{ and}$$

$$\delta(\sigma^k a_i, \sigma^m a_j) = (1 - \delta_{km})(-1)^{m+k} \qquad (m \geq k)$$

($i, j = 1, \cdots, \tilde{g};\ k, m = 0, \cdots, n-1$). With the notation

$$a_{k,i} \equiv \sigma^k a_i + \sum_{r=1}^{\tilde{g}} \sum_{s=1}^{n-1} (-1)^s \sigma^{s+k} b_r,$$

the cycles $(a_{k,i}, \sigma^k b_i)$ ($i = 1, \cdots, \tilde{g};\ k = 0, \cdots, n-1$) form a set of retrosections for S, and $(\tilde{\pi} a_{0,i}, \tilde{\pi} b_i)$ ($i = 1, \cdots, \tilde{g}$) form a set of retrosections for S/σ. Note that $\sigma a_{k,i} = a_{(k+1) \bmod n, i}$. Hence, the retrosections $(a_{k,i}, \sigma^k b_i)$ are of the form

(1) $$(\sigma^k a_{0,i}, \sigma^k b_i) \qquad (i = 1, \cdots, \tilde{g};\ k = 0, \cdots, n-1).$$

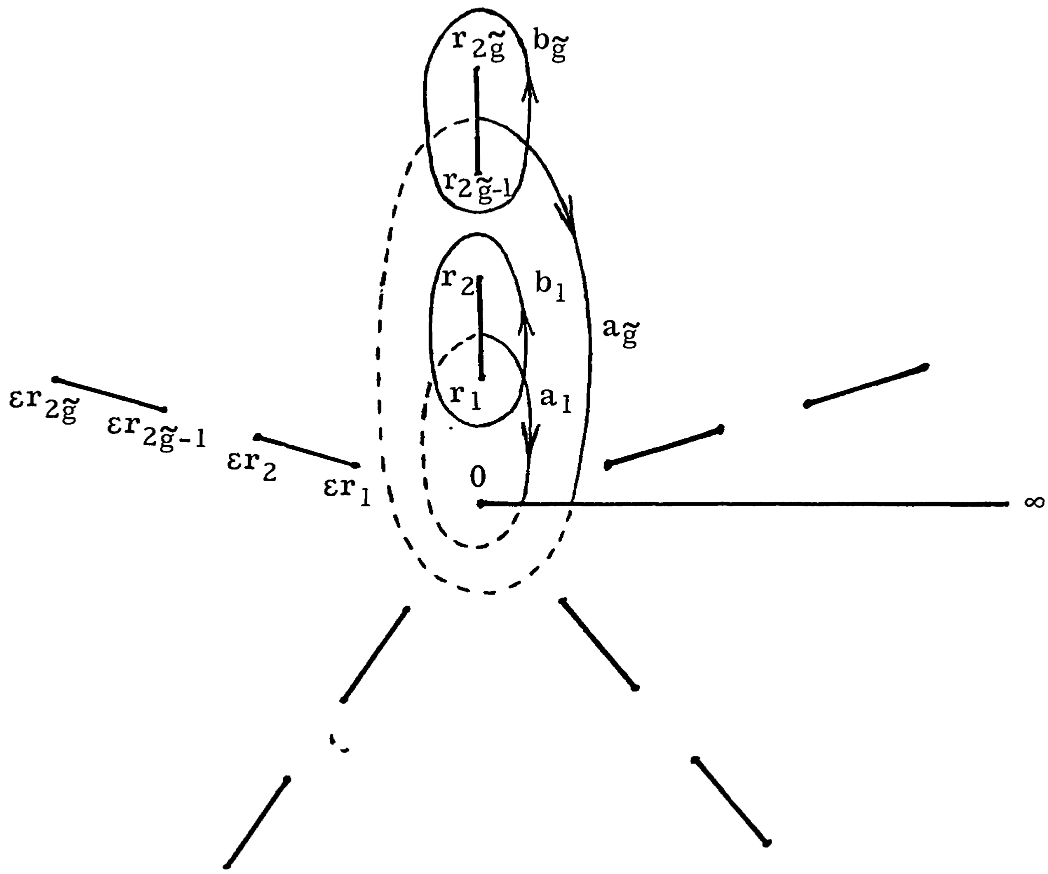

Figure 1.1. Orbits $1, 2, \cdots, 2\tilde{g} - 1, 2\tilde{g}$; $n = 5$.

Let ω_i $(i = 1, \cdots, \tilde{g})$ be holomorphic differentials on S satisfying the relation

$$\int_{\sigma^k a_{0,j}} \omega_i = \delta_{ij}\,\delta_{0k}$$

$(k = 0, \cdots, n-1;\ j = 1, \cdots, \tilde{g})$. Then the differentials $\sigma^k \omega_i$ $(i = 1, \cdots, \tilde{g};$ $k = 0, \cdots, n-1)$ form a basis for the holomorphic differentials on S, normalized with respect to (1) (if $\omega = df$ at P, then $\sigma\omega \equiv d(f\sigma^{-1})$ at σP, so that $\int_x \omega = \int_{\sigma x} \sigma\omega$). If M_k denotes the $\tilde{g} \times \tilde{g}$ matrix $(m_{i,j})_k = \left(\int_{\sigma^k b_j} \omega_i\right)$, then the corresponding Riemann matrix for S is

$$(2) \qquad \left\langle M_0, \cdots, M_{n-1} \right\rangle,$$

where $M_k = M_{n-k}^t$ since every Riemann matrix is symmetric. Furthermore, the differentials $\tilde{\omega}_i \equiv \sum_{k=0}^{n-1} \sigma^k \omega_i$ $(i = 1, \cdots, \tilde{g})$ are invariant with respect to σ and are therefore defined on the quotient surface S/σ. They are normalized there with respect to $(\tilde{\pi} a_{0,i}, \tilde{\pi} b_i)$ $(i = 1, \cdots, \tilde{g})$, and the corresponding Riemann matrix for S/σ is $\sum_{k=0}^{n-1} M_k$. Finally, if in the process above we replace σ by $\iota\sigma$, then the corresponding Riemann matrix for S is

$$[W_0, \cdots, W_{n-1}],$$

where $W_k = (-1)^k M_k$.

Conversely, suppose that a hyperelliptic Riemann surface S has a Riemann matrix of the form (2) with respect to some retrosections $(a_{k,i}, b_{k,i})$ $(i = 1, \cdots, g'; k = 0, \cdots, n-1)$ $(n > 1)$. Then S has the same Riemann matrix with respect to

$$(a_{(k+1) \bmod n,\, i},\ b_{(k+1) \bmod n,\, i})\ (i = 1, \cdots, g';\ k = 0, \cdots, n-1).$$

Hence, by Torelli's theorem (with $S = S'$), the retrosections $(a_{k,i}, b_{k,i})$ are of the form $(\sigma^k a_{0,i}, \sigma^k b_{0,i})$ $(i = 1, \cdots, g'; k = 0, \cdots, n-1)$, where σ is an automorphism on S. Furthermore, σ induces an automorphism of order n on the first homology group of S, and therefore σ is of order n [2, p. 737]. It is easily verified that the corresponding normalized differentials are of the form $\sigma^k \omega_i$ $(i = 1, \cdots, g'; k = 0, \cdots, n-1)$. As before, the differentials $\tilde{\omega}_i \equiv \sum_{k=0}^{n-1} \sigma^k \omega_i$ $(i = 1, \cdots, g')$ are defined and are linearly independent on the quotient surface S/σ, so that S/σ has genus $\tilde{g} \geq g'$. On the other hand, each holomorphic differential $\tilde{\omega}$ on S/σ can be lifted to a holomorphic differential ω on S that is invariant with respect to σ. Then

$$\omega = \sum_{i=1}^{g'} \sum_{k=0}^{n-1} c_{k,i} \sigma^k \omega_i ;$$

but $\omega = \sigma\omega$ implies that $c_{k,i} = c_{m,i}$ $(k, m = 0, \cdots, n-1; i = 1, \cdots, g')$. Hence, $\omega = \sum_{i=1}^{g'} c_{0,i} \tilde{\omega}_i$, so that $g' = \tilde{g}$. An inspection of Table 2 shows that if n is odd, then S is of Type 1.1, and if n is even, then $n = 2$ and S is of Type 4.2 (the existence of such a matrix for a surface of Type 4.2, when $n = 2$, will be established in the corresponding section). We summarize:

THEOREM 1.1. *Let* S *be a hyperelliptic Riemann surface, and let* $n > 1$ *be odd. Then* S *is of Type* 1.1 *if and only if* S *has a Riemann matrix of the form*

$$M = \langle M_0, \cdots, M_{n-1} ; \tilde{g} \times \tilde{g} \rangle,$$

where $M_k = M_{n-k}^t$. *Furthermore,* M *can be chosen so that* $\sum_{k=0}^{n-1} M_k$ *is a Riemann matrix for the quotient surface* S/σ.

Case 2.1

$w^2 = z(z^n - r_1^n) \cdots (z^n - r_{2g/n}^n)$, n even, 2g/n even.

$\sigma: (z, w) \to (\varepsilon z, \sqrt{\varepsilon} w)$ (order 2n), $\quad \tilde{w} = \tilde{z}(\tilde{z} - r_1^n) \cdots (\tilde{z} - r_{2g/n}^n)$, $\tilde{g} = 0$.

$\iota\sigma: (z, w) \to (\varepsilon z, -\sqrt{\varepsilon} w)$ (order 2n), $\hat{w} = \hat{z}(\hat{z} - r_1^n) \cdots (\hat{z} - r_{2g/n}^n)$, $\hat{g} = 0$.

Case 2.1 is similar to Case 1.1 in that there are an even number (2g/n) of branch orbits and $(0, \infty)$ is a branch cut. If we define (a_i, b_i) $(i = 1, \cdots, g/n)$ as in Case 1.1, then again $\delta(a_i, \sigma a_i) = 1$ or -1, but the argument used previously to show that in fact $\delta(a_i, \sigma a_i) = -1$ now breaks down, since n is even and σ is of order 2n. However, since $\delta(a_i, \sigma a_i) = -\delta(a_i, \iota\sigma a_i)$, we may assume (by a relabeling, if necessary) that $\delta(a_i, \sigma a_i) = -1$. Then, if $a_{0,i}$, ω_i, and M_k $(i = 1, \cdots, g/n; k = 0, \cdots, n-1)$ are defined as in Case 1.1, the corresponding Riemann matrix for S is

(1) $$[M_0, \cdots, M_{n-1}],$$

where $M_k = -M^t_{n-k}$ $(k = 1, \cdots, n/2)$ by symmetry. The difference between (1) of the present case and (2) of Case 1.1 is due to the fact that now $\sigma^n b_i = -b_i$, whereas $\sigma^n b_i = b_i$ in Case 1.1. If we replace σ by $\iota\sigma$, then the corresponding Riemann matrix for S is

$$[W_0, \cdots, W_{n-1}],$$

where $W_k = (-1)^k M_k$.

Conversely, if a hyperelliptic Riemann surface S of genus g has a Riemann matrix of the form (1) with respect to some retrosections $(a_{k,i}, b_{k,i})$ $(i = 1, \cdots, g/n;\ k = 0, \cdots, n-1)$ $(n > 1)$, then S has the same Riemann matrix with respect to

$$(a_{k+1,i}, b_{k+1,i}) \quad (i = 1, \cdots, g/n;\ k = 0, \cdots, n-2) \qquad \text{and} \qquad (-a_0, -b_0).$$

Proceeding as in Case 1.1, we see that S has an automorphism σ of order 2n. If n is even, then $\iota\sigma$ is also of order 2n and S is of Type 2.1. If n is odd, then $\iota\sigma$ is of order n and S is of Type 1.1 (with σ and $\iota\sigma$ interchanged). We summarize:

THEOREM 2.1. *Let* S *be a hyperelliptic Riemann surface of genus* g, *and let* $n > 1$ *be even. Then* S *is of Type* 2.1 *if and only if* S *has a Riemann matrix of the form*

$$[M_0, \cdots, M_{n-1};\ g/n \times g/n],$$

where $M_k = -M^t_{n-k}$.

Case 3.1

$$w^2 = (z^n - r_1^n) \cdots (z^n - r^n_{(2g+2)/n}), \ n \text{ odd}.$$

$$\sigma\colon (z, w) \to (\varepsilon z, w) \text{ (order } n),\ \tilde{w}^2 = (\tilde{z} - r_1^n) \cdots (\tilde{z} - r^n_{2\tilde{g}+2}),\ n = (g+1)/(\tilde{g}+1).$$

$$\iota\sigma\colon (z, w) \to (\varepsilon z, -w) \text{ (order } 2n),\ \hat{w} = (\hat{z} - r_1^n) \cdots (\hat{z} - r^n_{(2g+2)/n}),\ \hat{g} = 0.$$

If in Figure 3.1 we choose the cycles x and y so that $\delta(x, y) = 1$, then $\delta(\sigma x, y) = 1$ or -1. However, if $\delta(\sigma x, y) = 1$, then $\sum_{m=0}^{n-1} (-1)^m \sigma^m x \sim 0$, which implies that $x \sim -x$, since n is odd and σ is of order n. Hence $\delta(\sigma x, y) = -1$. We see that the pairs

$$(1) \qquad \begin{aligned} &(\sigma^k a_i, \sigma^k b_i) && (i = 1, \cdots, \tilde{g};\ k = 0, \cdots, n-1), \\ &(x + \sigma x + \cdots + \sigma^m x, \sigma^m y) && (m = 0, \cdots, n-2) \end{aligned}$$

form a set of retrosections for S, and the pairs $(\tilde{\pi} a_i, \tilde{\pi} b_i)$ $(i = 1, \cdots, \tilde{g})$ form a set of retrosections for S/σ. Furthermore,

$$(2) \qquad x + \sigma x + \cdots + \sigma^{n-1} x \sim 0 \sim y + \sigma y + \cdots + \sigma^{n-1} y$$

on S, so that $\tilde{\pi} x \sim 0 \sim \tilde{\pi} y$ on S/σ. Let ω_i $(i = 1, \cdots, \tilde{g})$ and Ω be holomorphic differentials on S satisfying the conditions

$$\int_{\sigma^k a_j} \omega_i = \delta_{ij}\delta_{0k}, \qquad \int_{x+\sigma x+\cdots+\sigma^m x} \omega_i = 0,$$

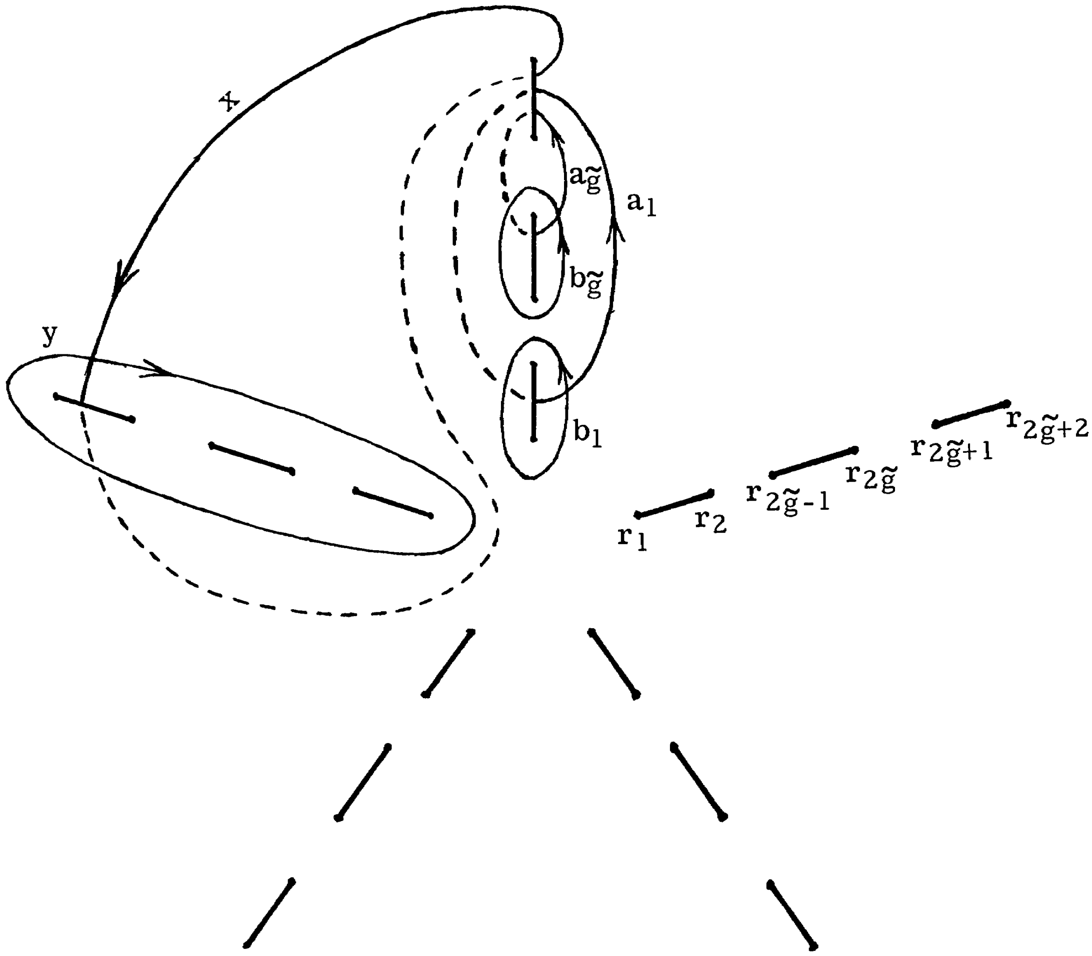

Figure 3.1. Orbits 1, 2, ⋯, $2\tilde{g} + 1$, $2\tilde{g} + 2$; n = 5.

$$\int_{\sigma^k a_j} \Omega = 0, \qquad \int_{x+\sigma x+\cdots+\sigma^m x} \Omega = \delta_{m0}$$

($j = 1, \cdots, \tilde{g}$; $k = 0, \cdots, n - 1$; $m = 0, \cdots, n - 2$). Then $\sigma^k \omega_i$ ($i = 1, \cdots, \tilde{g}$; $k = 0, \cdots, n - 1$) and $\sigma^m \Omega$ ($m = 0, \cdots, n - 2$) form a basis for the holomorphic differentials on S, normalized with respect to (1). If we define M_k as in Case 1.1, and denote by X_m the $\tilde{g} \times 1$ matrix $(x_i)_m = \left(\int_{\sigma^m y} \omega_i \right)$ and by Y_m the element $\int_{\sigma^m y} \Omega$, then the corresponding Riemann matrix for S is

(3)
$$\begin{array}{cc} & \begin{array}{cc} \sigma^k b_i & \sigma^m y \end{array} \\ \begin{array}{c} \sigma^k \omega_i \\ \sigma^m \Omega \end{array} & \begin{pmatrix} M & X \\ X^t & Y \end{pmatrix} \end{array},$$

where

$$M \ (= M^t) = \langle M_0, \cdots, M_{n-1} \rangle, \qquad X = \langle X_0, \cdots, X_{n-1} \rangle_n,$$
$$Y \ (= Y^t) = \langle Y_0, \cdots, Y_{n-1} \rangle_n^n,$$

and, by (2), $\sum_{m=0}^{n-1} X_m = 0 = \sum_{m=0}^{n-1} Y_m$. As before, $\sum_{k=0}^{n-1} M_k$ is a Riemann matrix for S/σ. If in the process above we replace σ by $\iota\sigma$, then the corresponding Riemann matrix for S is

$$\begin{pmatrix} W & U \\ U^t & V \end{pmatrix},$$

where, with the notation $W_k = (-1)^k M_k$, $U_m = (-1)^m X_m$, and $V_m = (-1)^m Y_m$, the entries W, U, and V are $W = [W_0, \cdots, W_{n-1}]$, $U = [U_0, \cdots, U_{n-1}]_n$, and $V = [V_0, \cdots, V_{n-1}]_n^n$.

Conversely, suppose that with respect to some retrosections $(a_{k,i}, b_{k,i})$ $(i = 1, \cdots, g'; k = 0, \cdots, n - 1)$ and (x_m, y_m) $(m = 0, \cdots, n - 2)$, a hyperelliptic Riemann surface S has a Riemann matrix of the form (3). Then S has the same Riemann matrix with respect to

$$(a_{(k+1)\bmod n,\, i}, b_{(k+1)\bmod n,\, i}) \quad (i = 1, \cdots, g';\ k = 0, \cdots, n-1),$$

$$(x_m - x_0, y_m) \quad (m = 1, \cdots, n-2), \qquad \text{and} \qquad -(x_0, y_0 + y_1 + \cdots + y_{n-2}) .$$

Hence, as before, S has an automorphism σ of order n. It can be seen that if we denote the cycle x_0 by x, then the retrosections (x_m, y_m) $(m = 0, \cdots, n - 2)$ are of the form (1), where (2) holds, so that $\tilde{\pi} x_m \sim 0$ on S/σ. Furthermore, the subspace of holomorphic differentials generated by the corresponding normalized differentials Ω_m $(= \sigma^m \Omega_0)$ $(m = 0, \cdots, n - 2)$ is invariant with respect to σ. However, if $\Omega = \sum_{m=0}^{n-2} d_m \Omega_m$ is invariant with respect to σ, then Ω is defined on S/σ, and then

$$0 = \int_{\tilde{\pi} x_r} \Omega = \int_{x_r} \Omega = d_r \quad (r = 0, \cdots, n-2).$$

It follows, as in Case 1.1, that the genus of S/σ is equal to g'. If n is odd, then $\iota\sigma$ is of order 2n, so that (by Table 2) S is of Type 3.1. If n is even, then $\iota\sigma$ is also of order n, and S is of Type 4.1 (the existence of such a matrix for a surface of Type 4.1 is established in the next section). We summarize:

THEOREM 3.1. *Let* S *be a hyperelliptic Riemann surface, and let* n *be odd* $(n > 1)$. *Then* S *is of Type* 3.1 *if and only if* S *has a Riemann matrix of the form*

$$\begin{pmatrix} M & X \\ X^t & Y \end{pmatrix},$$

where

$$M\ (= M^t) = \langle M_0, \cdots, M_{n-1};\ \tilde{g} \times \tilde{g} \rangle, \qquad X = \langle X_0, \cdots, X_{n-1};\ \tilde{g} \times 1 \rangle_n,$$

$$Y\ (= Y^t) = \langle Y_0, \cdots, Y_{n-1};\ 1 \times 1 \rangle_n^n,$$

and $\sum_{m=0}^{n-1} X_m = 0 = \sum_{m=0}^{n-1} Y_m$. *Furthermore, the matrix can be chosen so that* $\sum_{k=0}^{n-1} M_k$ *is a Riemann matrix for* S/σ.

Case 4.1

$w^2 = (z^n - r_1^n) \cdots (z^n - r_{(2g+2)/n}^n)$, n even.

σ: $(z, w) \to (\varepsilon z, w)$ (order n), $\tilde{w}^2 = (\tilde{z} - r_1^n) \cdots (\tilde{z} - r_{2\tilde{g}+2}^n)$, $n = (g+1)/(\tilde{g}+1)$.

$\iota\sigma$: $(z, w) \to (\varepsilon z, -w)$ (order n), $\hat{w}^2 = (\hat{z} - r_1^n) \cdots (\hat{z} - r_{2\hat{g}+2}^n)$, $n = (g+1)/\hat{g}$.

The case $n = 2$ (g odd) of [5] is contained in this case. Case 4.1 is similar to Case 3.1 in that there are an even number $(2\tilde{g} + 2)$ of branch orbits and neither 0 nor ∞ is a branch point. If we adjust Figure 3.1 so that each orbit contains an even number n of branch points, we see that $\delta(\sigma x, y)$ has one of the values 1 and -1. If $\delta(\sigma x, y) = 1$, then

$$0 \sim \sum_{m=0}^{n-1} (-1)^m \sigma^m x = \sum_{m=0}^{n-1} (\iota\sigma)^m x,$$

which implies that $\hat{\pi} x \sim 0$. But an inspection of the adjusted Figure 3.1 shows that $(\hat{\pi} x, \hat{\pi} y) = (2\alpha, \beta)$, where $\delta(\alpha, \beta) = 1$ on $S/\iota\sigma$. Hence, $\delta(\sigma x, y) = -1$. Proceeding exactly as in Case 3.1, we again see that with respect to the retrosections

$$(\sigma^k a_i, \sigma^k b_i) \qquad (i = 1, \cdots, \tilde{g};\ k = 0, \cdots, n-1),$$

$$(x + \sigma x + \cdots + \sigma^m x, \sigma^m y) \qquad (m = 0, \cdots, n-2)$$

S has a Riemann matrix of the form

$$\text{(1)} \qquad \begin{array}{c} \\ \sigma^k \omega_i \\ \sigma^m \Omega \end{array} \begin{array}{c} \sigma^k b_i \quad \sigma^m y \\ \begin{pmatrix} M & X \\ X^t & Y \end{pmatrix} \end{array},$$

where

$$M\ (= M^t) = \langle M_0, \cdots, M_{n-1} \rangle, \qquad X = \langle X_0, \cdots, X_{n-1} \rangle_n,$$

$$Y\ (= Y^t) = \langle Y_0, \cdots, Y_{n-1} \rangle_n^n,$$

and $\sum_{m=0}^{n-1} X_m = 0 = \sum_{m=0}^{n-1} Y_m$. Also, $\sum_{k=0}^{n-1} M_k$ is a Riemann matrix for S/σ with respect to $(\tilde{\pi} a_i, \tilde{\pi} b_i)$ $(i = 1, \cdots, \tilde{g})$.

If in the process above we replace σ by $\iota\sigma$, then the corresponding Riemann matrix for S is

$$\text{(2)} \qquad \begin{pmatrix} W & U \\ U^t & V \end{pmatrix},$$

where, if W_k, U_m, and V_m are defined as in Case 3.1, then

$$W = \langle W_0, \cdots, W_{n-1} \rangle, \qquad U = \langle U_0, \cdots, U_{n-1} \rangle_n, \qquad V = \langle V_0, \cdots, V_{n-1} \rangle_n^n.$$

We again note that $(\hat{\pi} x, \hat{\pi} y) = (2\alpha, \beta)$, where $\delta(\alpha, \beta) = 1$ on $S/\iota\sigma$. In fact, the pairs

$$(\hat{\pi} a_i, \hat{\pi} b_i) \quad (i = 1, \cdots, \hat{g} - 1) \quad \text{and} \quad (\alpha, \beta) \tag{3}$$

form a set of retrosections for $S/\iota\sigma$. The differentials

$$\hat{\omega}_i \equiv \sum_{k=0}^{n-1} (\iota\sigma)^k \omega_i \quad (i = 1, \cdots, \hat{g} - 1) \quad \text{and} \quad \hat{\Omega} \equiv \sum_{m=0}^{n-1} (\iota\sigma)^m \Omega$$

are invariant with respect to $\iota\sigma$. They form a basis for the holomorphic differentials on $S/\iota\sigma$, normalized with respect to (3). The corresponding Riemann matrix for $S/\iota\sigma$ is

$$\begin{matrix} & \hat{\pi} b_i & \beta \\ \hat{\omega}_i & \hat{W} & \hat{U} \\ \hat{\Omega} & \hat{U}^t & \hat{V} \end{matrix}\,,$$

where $\hat{W} = \sum_{k=0}^{n-1} W_k$, $\hat{U} = \sum_{m=0}^{n-1} U_m$, and $\hat{V} = \sum_{m=0}^{n-1} V_m$. We note that $\hat{V} \neq 0$, since $\hat{V}$ has positive-definite imaginary part, and $\hat{U} \neq 0$ by a result of H. H. Martens [3, p. 109]. Hence, (2) does not have the same properties as (1).

Conversely, if a hyperelliptic Riemann surface S has a Riemann matrix of the form (1), then the technique of Case 3.1 shows that S has an automorphism σ of order n. If n is even, then S is of Type 4.1, and if n is odd, then S is of Type 3.1. We have established the following result.

THEOREM 4.1. *Let* S *be a hyperelliptic Riemann surface, and let* $n > 1$ *be even. Then* S *is of Type* 4.1 *if and only if* S *has a Riemann matrix of the form*

$$\begin{pmatrix} M & X \\ X^t & Y \end{pmatrix},$$

where

$$M\ (= M^t) = \langle M_0, \cdots, M_{n-1};\ \tilde{g} \times \tilde{g} \rangle, \qquad X = \langle X_0, \cdots, X_{n-1};\ \tilde{g} \times 1 \rangle_n,$$

$$Y\ (= Y^t) = \langle Y_0, \cdots, Y_{n-1};\ 1 \times 1 \rangle_n^n,$$

and $\sum_{m=0}^{n-1} X_m = 0 = \sum_{m=0}^{n-1} Y_m$. *Furthermore, the matrix can be chosen so that* $\sum_{k=0}^{n-1} M_k$ *is a Riemann matrix for* S/σ, *and*

$$\begin{pmatrix} \hat{M} & \hat{X} \\ \hat{X}^t & \hat{Y} \end{pmatrix},$$

where $\hat{M} = \sum_{k=0}^{n-1} (-1)^k M_k$, $\hat{X} = \sum_{m=0}^{n-1} (-1)^m X_m$, *and* $\hat{Y} = \sum_{m=0}^{n-1} (-1)^m Y_m$, *is a Riemann matrix for* $S/\iota\sigma$.

Case 1.2 (3.2)

$$w^2 = z(z^n - r_1^n) \cdots (z^n - r^n_{(2g+1)/n}) .$$

σ: $(z, w) \to (\varepsilon z, \sqrt{\varepsilon} w)$ (order n), $\quad \tilde{w}^2 = \tilde{z}(\tilde{z} - r_1^n) \cdots (\tilde{z} - r^n_{2\tilde{g}+1}),$

$$n = (2g + 1)/(2\tilde{g} + 1) .$$

$\iota\sigma$: $(z, w) \to (\varepsilon z, -\sqrt{\varepsilon} w)$ (order 2n), $\quad \hat{w} = \hat{z}(\hat{z} - r_1^n) \cdots (\hat{z} - r^n_{(2g+1)/n}),\ \hat{g} = 0 .$

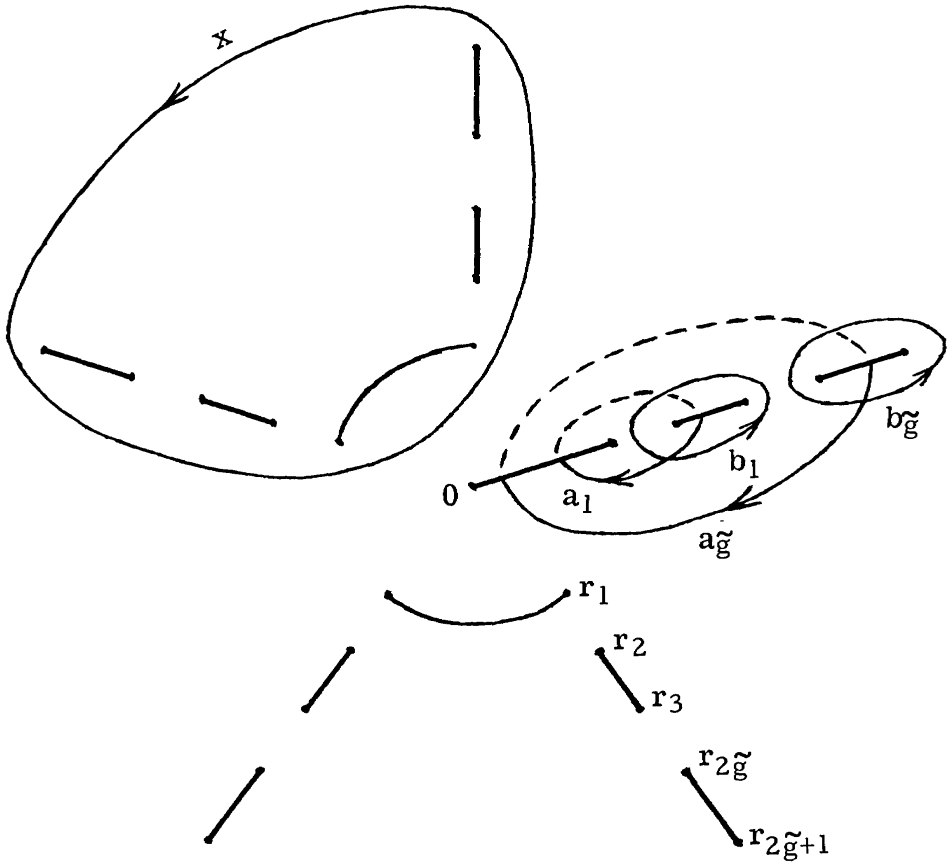

Figure 1.2. Orbits $1, 2, \cdots, 2\tilde{g}, 2\tilde{g} + 1$; $n = 5$.

Let the branch cuts and homology cycles for S over P be chosen as in Figure 1.2. The argument used in Case 1.1 to determine $\delta(a_i, \sigma a_i)$ can be applied here to show that $\delta(x, \sigma x) = -1$. We see then that the pairs

$$(1) \qquad \begin{aligned} &(\sigma^k a_i, \sigma^k b_i) && (i = 1, \cdots, \tilde{g};\ k = 0, \cdots, n - 1), \\ &(x + \sigma^2 x + \cdots + \sigma^{2m} x, -\sigma^{2m+1} x) && (m = 0, \cdots, (n - 3)/2) \end{aligned}$$

form a set of retrosections for S, and the pairs $(\tilde{\pi} a_i, \tilde{\pi} b_i)$ $(i = 1, \cdots, \tilde{g})$ form a set of retrosections for S/σ. Furthermore,

$$(2) \qquad x + \sigma x + \cdots + \sigma^{n-1} x \sim 0$$

on S, so that $\tilde{\pi} x \sim 0$ on S/σ. Now let ω_i $(i = 1, \cdots, \tilde{g})$ and Ω_m $(m = 0, \cdots, (n - 3)/2)$ be holomorphic differentials on S satisfying the conditions

$$\int_{\sigma^k a_j} \omega_i = \delta_{ij}\,\delta_{0k}, \qquad \int_{\sigma^k a_j} \Omega_m = 0, \qquad \int_{x+\sigma^2 x+\cdots+\sigma^{2r} x} \Omega_m = \delta_{rm}$$

($j = 1, \cdots, \tilde{g}$; $k = 0, \cdots, n-1$; $r = 0, \cdots, (n-3)/2$). Then $\sigma^k \omega_i$ ($i = 1, \cdots, \tilde{g}$; $k = 0, \cdots, n-1$) and the Ω_m form a basis (not normalized) for the holomorphic differentials on S. If M_k is as in Case 1.1, and if X_m denotes the $\tilde{g} \times 1$ matrix $(x_i)_m = \left(\int_{\sigma^m x} \omega_i\right)$, then the corresponding period matrix for S is

$$(3) \qquad (A \,\|\, B) = \begin{array}{c} \\ \sigma^k \omega_i \\ \Omega_m \end{array} \left(\begin{array}{c|c||c|c} \sigma^k a_i & x+\sigma^2 x+\cdots+\sigma^{2m} x & \sigma^k b_i & -\sigma^{2m+1} x \\ \hline I_{n\tilde{g} \times n\tilde{g}} & X' & M & X \\ \hline 0 & I_{(n-1)/2 \times (n-1)/2} & * & Y \end{array} \right),$$

where

$$M = \langle M_0, \cdots, M_{n-1} \rangle, \qquad X = -\langle X_0, \cdots, X_{n-1} \rangle_{1,3,\cdots,n},$$

$$X' = (\langle X_0, \cdots, X_{n-1} \rangle_{2,4,\cdots,n-1,n})^*,$$

and, by (2), $\sum_{m=0}^{n-1} X_m = 0$. By applying σ to the retrosections

$$(x + \sigma^2 x + \cdots + \sigma^{2m} x,\ -\sigma^{2m+1} x) \qquad (m = 0, \cdots, (n-3)/2)$$

and using (2), we see that Y is invariant under the change in retrosections $(x_m, y_m) \to (x'_m, y'_m)$, where

$$(4) \qquad \begin{aligned} (x'_m, y'_m) &= -(y_0 + y_1 + \cdots + y_m,\ x_{m+1} - x_m) && (m = 0, \cdots, (n-5)/2), \\ (x'_m, y'_m) &= -(y_0 + y_1 + \cdots + y_m,\ y_0 + y_1 + \cdots + y_m - x_m) && (m = (n-3)/2). \end{aligned}$$

The corresponding Riemann matrix for S is

$$(5) \qquad A^{-1} B = \left(\begin{array}{c|c} M - X'(X - X'Y)^t & X - X'Y \\ \hline (X - X'Y)^t & Y \end{array} \right),$$

and, as before, the $\tilde{g} \times \tilde{g}$ matrix $\sum_{k=0}^{n-1} M_k$ is a Riemann matrix for S/σ. We note that in Case 3.1 (4.1) the matrix corresponding to X' is the zero matrix; in other words, the period matrix corresponding to (3) is normalized. However, in this case $X = 0$ if $X' = 0$, and then (5) reduces to a direct sum, which is impossible by the result of Martens (Case 4.1). Finally, if we replace σ by $\iota\sigma$, and if we again denote $(-1)^k M_k$ by W_k and $(-1)^m X_m$ by U_m, then the corresponding Riemann matrix for S is

$$\left(\begin{array}{c|c} W - U'(U - U'Y)^t & U - U'Y \\ \hline (U - U'Y)^t & Y \end{array} \right),$$

where $W = [W_0, \cdots, W_{n-1}]$, $U = [U_0, \cdots, U_{n-1}]_{1,3,\cdots,n}$, and $U' = ([U_0, \cdots, U_{n-1}]_{2,4,\cdots,n-1,n})^*$.

Conversely, suppose that with respect to some retrosections $(a_{k,i}, b_{k,i})$ $(i = 1, \cdots, g'; k = 0, \cdots, n - 1)$ and (x_m, y_m) $(m = 0, \cdots, (n - 3)/2)$, a hyperelliptic Riemann surface S has a Riemann matrix of the form (5). Then S has the same Riemann matrix with respect to $(a_{(k+1)\mathrm{mod}\ n, i}, b_{(k+1)\mathrm{mod}\ n, i})$ $(i = 1, \cdots, g';$ $k = 0, \cdots, n - 1)$ and (x'_m, y'_m) of (4). To see this, we assume first that the Riemann matrix in question comes from a period matrix of the form (3); this is possible, since for each set of retrosections, any nonsingular matrix A determines a basis of holomorphic differentials, and B is then uniquely determined. Now let

$$\omega_{k,i} \ (i = 1, \cdots, g'; k = 0, \cdots, n - 1) \quad \text{and} \quad \Omega_m \ (m = 0, \cdots, (n - 3)/2)$$

be the differentials whose integration over the original retrosections $(a_{k,i}, b_{k,i})$ and (x_m, y_m) gives rise to (3). Then the differentials $\omega_{(k+1)\mathrm{mod}\ n, i}$, integrated over the new retrosections $(a_{(k+1)\mathrm{mod}\ n, i}, b_{(k+1)\mathrm{mod}\ n, i})$ and (x'_m, y'_m), keep $(I\ X'\ M\ X)$ of (3) fixed, by the properties of M, X, and X'. Furthermore, since the (x'_m, y'_m) are linear combinations of the original (x_m, y_m), the corresponding normalized differentials Ω'_m that preserve Y must be linear combinations of the Ω_m. Hence, the Ω'_m, integrated over the new retrosections, keep 0 as well as I and Y in $(0\ I * Y)$ of (3) fixed. But, since any Riemann matrix is symmetric, (*) must be equal to $(X - X'Y)^t$; that is, the remaining entries in the period matrix (3) determine (*), so that all of (3), and therefore (5), is held fixed. Hence, as before, S has an automorphism σ of order n (odd). We can adapt the technique of Case 3.1 to show that g' is equal to the genus of S/σ, so that S is of Type 1.2 (3.2). We have established the following theorem.

THEOREM 1.2 (3.2). *A hyperelliptic Riemann surface* S *is of Type* 1.2 (3.2) *if and only if* S *has a Riemann matrix of the form*

$$\left(\begin{array}{c|c} M - X'(X - X'Y)^t & X - X'Y \\ \hline (X - X'Y)^t & Y \end{array}\right),$$

where

$$M = \langle M_0, \cdots, M_{n-1}; \tilde{g} \times \tilde{g} \rangle, \qquad X = -\langle X_0, \cdots, X_{n-1}; \tilde{g} \times 1 \rangle_{1,3,\cdots,n},$$

$$X' = (\langle X_0, \cdots, X_{n-1}; \tilde{g} \times 1 \rangle_{2,4,\cdots,n-1,n})^*,$$

$\sum_{m=0}^{n-1} X_m = 0$, *and* Y *is invariant under the change in retrosections* $(x_m, y_m) \to (x'_m, y'_m)$, *where*

$$(x'_m, y'_m) = -(y_0 + y_1 + \cdots + y_m, x_{m+1} - x_m) \qquad (m = 0, \cdots, (n - 5)/2),$$

$$(x'_m, y'_m) = -(y_0 + y_1 + \cdots + y_m, y_0 + y_1 + \cdots + y_m - x_m) \quad (m = (n - 3)/2).$$

Furthermore, the matrix can be chosen so that $\sum_{k=0}^{n-1} M_k$ *is a Riemann matrix for* S/σ.

Case 2.2

$w^2 = z(z^n - r_1^n) \cdots (z^n - r_{2g/n}^n)$, n even, 2g/n odd.

σ: $(z, w) \to (\varepsilon z, \sqrt{\varepsilon} w)$ (order 2n), $\tilde{w} = \tilde{z}(\tilde{z} - r_1^n) \cdots (\tilde{z} - r_{2g/n}^n)$, $\tilde{g} = 0$.

$\iota\sigma$: $(z, w) \to (\varepsilon z, -\sqrt{\varepsilon} w)$ (order 2n), $\hat{w} = \hat{z}(\hat{z} - r_1^n) \cdots (\hat{z} - r_{2g/n}^n)$, $\hat{g} = 0$.

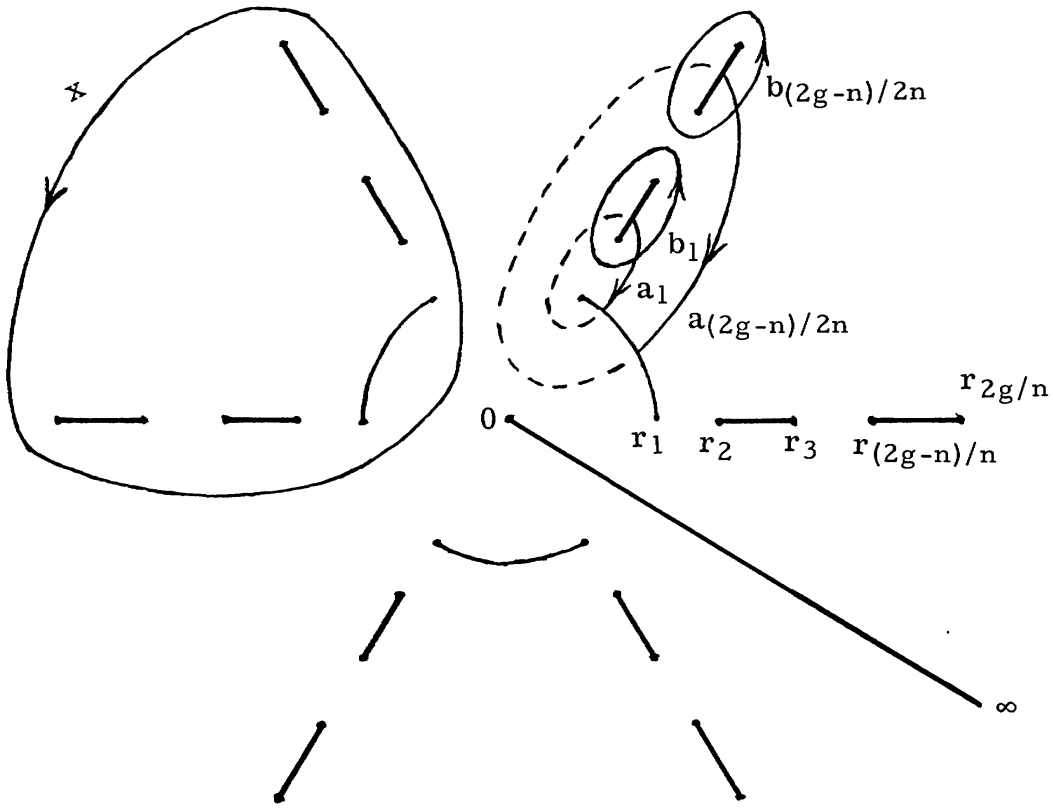

Figure 2.2. Orbits 1, 2, ⋯, (2g - n)/n, 2g/n; 2g/n odd, n = 6.

An inspection of Figure 2.2 shows that $\delta(x, \sigma x) = 1$ or -1. Since $\delta(x, \sigma x) = -\delta(x, \iota\sigma x)$, we may assume (by a relabeling, if necessary) that $\delta(x, \sigma x) = -1$. Then, as in Case 1.2, the pairs

$$(\sigma^k a_i, \sigma^k b_i) \qquad (i = 1, \cdots, (2g - n)/2;\ k = 0, \cdots, n - 1),$$

$$(x + \sigma^2 x + \cdots + \sigma^{2m} x, -\sigma^{2m+1} x) \qquad (m = 0, \cdots, (n - 2)/2)$$

form a set of retrosections for S. However, (2) of Case 1.2 does not hold here. Proceeding as in Case 1.2, we find that the corresponding Riemann matrix for S is

$$(1) \qquad \left(\begin{array}{c|c} M - X'(X - X'Y)^t & X - X'Y \\ \hline (X - X'Y)^t & Y \end{array}\right);$$

here $M = [M_0, \cdots, M_{n-1}]$, $X = -[X_0, \cdots, X_{n-1}]_{1,3,\ldots,n-1}$, $X' = ([X_0, \cdots, X_{n-1}]_{2,4,\ldots,n})^*$, and Y is invariant under the change in retrosections $(x_m, y_m) \to (x'_m, y'_m)$, where

$$(x'_m, y'_m) = (y_0 + y_1 + \cdots + y_m, x_{m+1} - x_m) \qquad (m = 0, \cdots, (n-4)/2),$$

$$(x'_{(n-2)/2}, y'_{(n-2)/2}) = -(y_0 + y_1 + \cdots + y_{(n-2)/2}, -x_0).$$

If we replace σ by $\iota\sigma$, then the corresponding Riemann matrix for S is

$$\left(\begin{array}{c|c} W - U'(U - U'Y)^t & U - U'Y \\ \hline (U - U'Y)^t & Y \end{array}\right),$$

where $W = [W_0, \cdots, W_{n-1}]$, $U = [U_0, \cdots, U_{n-1}]_{1,3,\cdots,n-1}$, and $U' = ([U_0, \cdots, U_{n-1}]_{2,4,\cdots,n})^*$.

Conversely, if a hyperelliptic Riemann surface S has a Riemann matrix of the form (1), then the technique of Case 1.2 can be adapted to show that S is of Type 2.2. We can state our result as follows.

THEOREM 2.2. *A hyperelliptic Riemann surface* S *is of Type* 2.2 *if and only if* S *has a Riemann matrix of the form*

$$\left(\begin{array}{c|c} M - X'(X - X'Y)^t & X - X'Y \\ \hline (X - X'Y)^t & Y \end{array}\right),$$

where $M = [M_0, \cdots, M_{n-1}; (2g-n)/2n \times (2g-n)/2n]$,

$X = -[X_0, \cdots, X_{n-1}; (2g-n)/2n \times 1]_{1,3,\cdots,n-1}, \qquad X' = ([X_0, \cdots, X_{n-1}]_{2,4,\cdots,n})^*,$

and Y *is invariant under the change in retrosections* $(x_m, y_m) \to (x'_m, y'_m)$, *where*

$$(x'_m, y'_m) = -(y_0 + y_1 + \cdots + y_m, x_{m+1} - x_m) \qquad (m = 0, \cdots, (n-4)/2),$$

$$(x'_{(n-2)/2}, y'_{(n-2)/2}) = -(y_0 + y_1 + \cdots + y_{(n-2)/2}, -x_0).$$

Case 4.2

$w^2 = (z^n - r_1^n) \cdots (z^n - r_{(2g+2)/n}^n)$, n even.

$\sigma: (z, w) \to (\varepsilon z, w)$ (order n), $\quad \tilde{w}^2 = (\tilde{z} - r_1^n) \cdots (\tilde{z} - r_{2\tilde{g}+1}^n)$,

$$n = (2g+2)/(2\tilde{g}+1).$$

$\iota\sigma: (z, w) \to (\varepsilon z, -w)$ (order n), $\quad \hat{w}^2 = \hat{z}(\hat{z} - r_1^n) \cdots (\hat{z} - r_{2\hat{g}+1}^n)$,

$$n = (2g+2)/(2\hat{g}+1).$$

The case $n = 2$ (g even) of [5] is contained in this case. Case 4.2 is similar to Case 2.2 in that there are an odd number of branch orbits ($2\tilde{g} + 1$ in this case, $2g/n$ in Case 2.2) with an even number of branch points (n) in each orbit. Figure 2.2, with the cut $(0, \infty)$ deleted, can be used for this case. Assume first that $n > 2$. Then, proceeding as in Case 2.2, we see that the pairs

$$(1)\quad \begin{array}{ll} (\sigma^k a_i,\ \sigma^k b_i) & (i = 1, \cdots, \tilde{g};\ k = 0, \cdots, n-1), \\ (x + \sigma^2 x + \cdots + \sigma^{2m} x,\ -\sigma^{2m+1} x) & (m = 0, \cdots, (n-4)/2) \end{array}$$

form a set of retrosections for S, the pairs $(\tilde{\pi} a_i, \tilde{\pi} b_i)$ $(i = 1, \cdots, \tilde{g})$ form a set of retrosections for S/σ, and the pairs $(\hat{\pi} a_i, \hat{\pi} b_i)$ $(i = 1, \cdots, \hat{g} = \tilde{g})$ form a set of retrosections for $S/\iota\sigma$. Furthermore,

$$x + \sigma^2 x + \cdots + \sigma^{n-2} x \sim 0$$

on S, so that $\tilde{\pi} x \sim 0$ on S/σ and $\hat{\pi} x \sim 0$ on $S/\iota\sigma$. Now, proceeding as in Case 1.2, we see that the corresponding Riemann matrix for S is

$$(2)\quad \left(\begin{array}{c|c} M - X'(X - X'Y)^t & X - X'Y \\ \hline (X - X'Y)^t & Y \end{array}\right);$$

here $M = \langle M_0, \cdots, M_{n-1} \rangle$,

$$X = -\langle X_0, \cdots, X_{n-1} \rangle_{1,3,\cdots,n-1,n}, \qquad X' = (\langle X_0, \cdots, X_{n-1} \rangle_{2,4,\cdots,n-2,n-1,n})^*,$$

$$\sum_{m=0}^{(n-2)/2} X_{2m} = 0 = \sum_{m=0}^{(n-2)/2} X_{2m+1},$$

and Y is invariant under the change in retrosections $(x_m, y_m) \to (x'_m, y'_m)$, where

$$(x'_m, y'_m) = -(y_0 + y_1 + \cdots + y_m,\ x_{m+1} - x_m) \qquad (m = 0, \cdots, (n-6)/2),$$

$$(x'_{(n-4)/2}, y'_{(n-4)/2}) = -(y_0 + y_1 + \cdots + y_{(n-4)/2},\ -x_{(n-4)/2}).$$

As before, the matrix $\sum_{k=0}^{n-1} M_k$ is a Riemann matrix for S/σ with respect to $(\tilde{\pi} a_i, \tilde{\pi} b_i)$ $(i = 1, \cdots, \tilde{g})$. If σ is replaced by $\iota\sigma$, then the corresponding Riemann matrix for S is

$$(3)\quad \left(\begin{array}{c|c} W - U'(U - U'Y)^t & U - U'Y \\ \hline (U - U'Y)^t & Y \end{array}\right),$$

where

$$W = \langle W_0, \cdots, W_{n-1} \rangle, \qquad U = \langle U_0, \cdots, U_{n-1} \rangle_{1,3,\cdots,n-1,n},$$

$$U' = (\langle U_0, \cdots, U_{n-1} \rangle_{2,4,\cdots,n-2,n-1,n})^*,$$

$$\sum_{m=0}^{(n-2)/2} U_{2m} = 0 = \sum_{m=0}^{(n-2)/2} U_{2m+1}.$$

Also, $\sum_{k=0}^{n-1} W_k$ is a Riemann matrix for $S/\iota\sigma$ with respect to $(\hat{\pi} a_i, \hat{\pi} b_i)$ $(i = 1, \cdots, \hat{g})$. If $n = 2$, then the x-retrosections of (1) do not appear. The matrix (2) becomes simply $M = \langle M_0, M_1 \rangle$, and (3) becomes $W = \langle M_0, -M_1 \rangle$, where

$M_0 + M_1$ is a Riemann matrix for S/σ, and $M_0 - M_1$ is a Riemann matrix for $S/\iota\sigma$. This is essentially the result for the case n = 2 (g even) of [5].

Conversely, if a hyperelliptic Riemann surface S has a Riemann matrix of the form (2), then we can adapt the technique of Case 1.2 to show that S is of Type 4.2. Hence, our final classification theorem is as follows.

THEOREM 4.2. *A hyperelliptic Riemann surface* S *is of Type* 4.2 *if and only if* S *has a Riemann matrix of the form*

$$\langle M_0, M_1; \tilde{g}\times\tilde{g}\rangle \quad (n = 2), \qquad \text{or} \qquad \left(\begin{array}{c|c} M - X'(X - X'Y)^t & X - X'Y \\ \hline (X - X'Y)^t & Y \end{array}\right) \quad (n > 2),$$

where

$$M = \langle M_0, \cdots, M_{n-1}; \tilde{g}\times\tilde{g}\rangle, \qquad X = -\langle X_0, \cdots, X_{n-1}; \tilde{g}\times 1\rangle_{1,3,\cdots,n-1,n},$$

$$X' = (\langle X_0, \cdots, X_{n-1}\rangle_{2,4,\cdots,n-2,n-1,n})^*,$$

$$\sum_{m=0}^{(n-2)/2} X_{2m} = 0 = \sum_{m=0}^{(n-2)/2} X_{2m+1},$$

and Y *is invariant under the change in retrosections* $(x_m, y_m) \to (x'_m, y'_m)$, *where*

$$(x'_m, y'_m) = -(y_0 + y_1 + \cdots + y_m, x_{m+1} - x_m) \qquad (m = 0, \cdots, (n-6)/2),$$

$$(x'_{(n-4)/2}, y'_{(n-4)/2}) = -(y_0 + y_1 + \cdots + y_{(n-4)/2}, -x_{(n-4)/2}).$$

Furthermore, the matrix can be chosen so that $\sum_{k=0}^{n-1} M_k$ *is a Riemann matrix for* S/σ *and* $\sum_{k=0}^{n-1} (-1)^k M_k$ *is a Riemann matrix for* $S/\iota\sigma$.

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