BRANCHED COVERINGS

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1. INTRODUCTION

In this paper we consider branched coverings of compact manifolds. A map f of a compact n-manifold M onto an n-manifold N is a *branched covering* if $f^{-1}f(B_f) = B_f$ and $f \mid (M - B_f)$ is a finite-to-one covering map. Here B_f denotes the set of points of M at which f is not a local homeomorphism. If $f \mid f^{-1}f(B_f)$ is a homeomorphism, the branched covering is a Montgomery-Samelson fibering with zero codimension, and we call it an M-S *covering*. If $f \mid f^{-1}f(B_f)$ is a covering map, we call it a *singular covering*. If $f \mid (M - B_f)$ is a regular covering, we call f a *regular branched covering*. In Section 2, we prove some theorems about general branched coverings. In Section 3, we construct a special homology theory and use it to investigate the structure of the branch set for M-S coverings. In Section 4 we study branched coverings by spheres, and in Section 5 we study branched coverings onto spheres. Section 6 contains some examples and remarks involving smooth branched coverings. We call $f: M \to N$ *smooth* if both M and N are n-manifolds with a C^m structure and f is C^m. We call f *simplicial* if M and N can be triangulated so that f is simplicial with respect to the triangulations. For a survey of problems related to this paper, see [9].

2. BRANCHED COVERINGS

PROPOSITION 1. *Let* $f: X \to Y$ *be an open map from the compact, path-connected and locally path-connected space* X *to the connected and locally simply connected Hausdorff space* Y. *Suppose that* $q = \min \{\text{card } f^{-1}(y): y \in Y\}$ *is finite. Then* $f_\# \pi(X)$ *has at most* q *cosets in* $\pi(Y)$, *where* π *denotes the fundamental group and* $f_\#$ *denotes the homomorphism induced by* f.

Proof. Suppose that $f_\# \pi(X)$ has p cosets in $\pi(Y)$ and that $p > q$. Since f(X) is open and compact, hence closed, in Y, the mapping f is onto Y. Therefore Y is path-connected and locally path-connected. Let $g: Z \to Y$ be the covering map corresponding to $f_\# \pi(X)$, and let $h: X \to Z$ be the lift of f. The map h is open, because f is open and g is a local homeomorphism. It follows that h is onto Z. Since g is a p-to-1 map, we infer that, for each y in Y,

$$\text{card } f^{-1}(y) = \text{card } h^{-1}g^{-1}(y) \geq \text{card } g^{-1}(y) = p > q,$$

contrary to the choice of q.

COROLLARY 1.1. *If* $f: M \to N$ *is a singular covering,* $\dim B_f \leq n - 2$, *and* $f \mid B_f$ *is* p-to-1, *then* $f_\# \pi(M)$ *has at most* p *cosets in* $\pi(N)$.

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COROLLARY 1.2. *If* $f: M \to N$ *is an* M-S *covering and* $\dim B_f \leq n - 2$, *then* $f_\#$ *maps* $\pi(M)$ *onto* $\pi(N)$.

Remark 1. It follows from Corollary 2 that S^n $(n > 1)$ admits M-S coverings onto simply connected spaces only, and that S^n admits singular coverings onto spaces with a finite fundamental group only. The $(n - 1)$-fold suspension of a d-to-1 covering of S^1 by S^1 (an M-S covering of S^n by S^n) followed by a covering map onto a lens space is a singular covering of a space with cyclic fundamental group by S^n. It is easy to construct M-S coverings of one 3-dimensional lens space N by another such space M such that card $\pi(N)$ is any specified multiple of card $\pi(M)$. All these examples can be constructed so that $B_f = S^1$. One can also infer the impossibility of an M-S-covering of certain lens spaces by certain others. On the solid torus $S^1 \times D^2$, consider the map given by $g(z_1, z_2) = (z_1^3, z_2^3)$. By identifying the boundaries of two such spaces in the usual way, one obtains a singular covering of S^3 by S^3 that is 3-to-1 on B_f (a link), but such that $f_\# \pi(S^3)$ has precisely one coset in $\pi(S^3)$. We can construct another example by composing the 3-to-1 irregular covering from the surface M of genus 4 to the surface of genus 2 with the standard M-S covering from the surface of genus 2 to the torus N. The resulting map f is a singular branched covering, 3-to-1 on B_f; but $f_\#$ maps $\pi(M)$ onto $\pi(N)$.

THEOREM 1. *Let* $f: M \to N$ *be a singular covering, and suppose* $f \mid B_f$ *is a* q-to-1 *covering map. Then* $f_\# \pi(M)$ *has precisely* q *cosets in* $\pi(N)$ *if and only if* $f = g \circ h$, *where* h *is an* M-S *covering with* $B_h = B_f$, *and* g *is a* q-to-1 *covering map.*

Proof. First, suppose that $f_\# \pi(M)$ has precisely q cosets in $\pi(N)$. Let $g: Z \to N$ be the covering map corresponding to $f_\# \pi(M)$, and let $h: M \to Z$ be the lift of f. Then $f = g \circ h$; also, $B_f = B_h$, because g is a local homeomorphism; $h \mid B_h$ is one-to-one, because each point of N has q inverse images under g; and $f = g \circ h$ is q-to-1 on $B_f = B_h$. For the converse, assume that the factorization exists. Then $f_\# \pi(M) = g_\# h_\# \pi(M) \subset g_\# \pi(Z)$. Here Z is the domain of g. Therefore $f_\# \pi(M)$ has at least q cosets. It has at most q cosets, by Theorem 1, and hence it has precisely q cosets.

A similar argument proves the following result.

THEOREM 1'. *Let* $f: M \to N$ *be a branched covering, and let*

$$q = \min \{\text{card } f^{-1}(y): y \in N\}.$$

Then $f_\# \pi(M)$ *has precisely* q *cosets in* $\pi(N)$ *if and only if* $f = g \circ h$, *where* h *is a branched covering with* $1 = \min\{\text{card } h^{-1}(y) = y \in N\}$ *and* g *is a* q-to-1 *covering map.*

Notice that if such a factorization exists, then $h_\#$ maps $\pi(M)$ onto $\pi(Z)$, where Z denotes the range of h.

If $f \mid (M - B_f)$ is a regular covering map, then f is the orbit map of an action on M by a finite group. The restricted part of the action is B_f. This means that the study of regular branched coverings is a subset of the study of finite transformation groups with the unusual condition that the orbit space is a manifold.

THEOREM 2. *If* $f: M \to N$ *is a singular covering,* N *is simply connected,* fB_f *is a tamely (respectively, smoothly) embedded manifold, and* f *is simplicial (respectively, smooth), then* f *is a regular branched covering.*

Proof. Since N is simply connected, $\pi(N - fB_f)$ is generated by small loops around fB_f that are attached to the base-point by an arc. It follows from [5, Theorem

1.2] (respectively [4, Theorem 2.1]) that each such loop lifts to an arc, no matter what base-point is chosen in $M - B_f$. Therefore f is regular.

3. SPECIAL HOMOLOGY, AND THE STRUCTURE OF THE BRANCH SET FOR M-S COVERINGS

PROPOSITION 2. *Let* $f\colon X \to Y$ *be an open simplicial map of the complex* X *onto the complex* Y. *Suppose there exists a subcomplex* X_0 *of* X *such that* $f \mid (X - X_0)$ *is a* d-to-1 *covering map and* $f \mid f^{-1} f X_0$ *is a homeomorphism. Let* C(A) *denote the chain complex of* A *with integer coefficients. Then there exist homomorphisms* ρ *and* τ *of the graded group* C(X) *into itself with the following properties (here* σ *denotes* ρ *or* τ *and* σ' *denotes* τ *or* ρ, *respectively).*

(A) $\sigma \circ \sigma' = 0$;

(B) $\sigma[C(X_0)] = 0$;

(C) *if* $\sigma(c) = 0$, *there exist chains* $a \in C(X, X_0)$ *and* $b \in C(X_0)$ *such that* $c = \sigma'(a) + b$;

(D) *for each chain* $c \in C(X)$, $\partial\sigma(c) - \sigma\partial(c) \in C(X_0)$;

(E) $\sigma[C(X)] \cap C(X_0) = 0$;

(F) $\partial\sigma[C(X)] \subset C(X, X_0) \oplus dC(X_0)$; *and*

(G) *if* q *is an integer dividing* d, *and if* $\partial\sigma(c) \in qC(X)$, *then*

$$\partial c \in C(X, X_0) \oplus qC(X_0).$$

Proof. This follows immediately from [16, Proposition 7 and Definitions 1 and 2].

Definition 1. Let K denote one of the three complexes X, X_0, or (X, X_0). We define *the set of* σ*-chains of* K to be the kernel of $\sigma \mid K$, and we denote this set by $C^\sigma(K)$.

LEMMA 1. $C^\sigma(K)$ *is a sub-chain-group of* C(K).

Proof. $C^\sigma(K)$ is a subgroup of C(K) because it is the kernel of a homomorphism. Let $c \in C^\sigma(K)$. By Proposition 2(D) and since $\sigma(c) = 0$,

$$\sigma\partial(c) \in \partial\sigma(c) + C(X_0) = C(X_0).$$

Therefore $\sigma\partial(c) \in \operatorname{Im}\sigma \cap C(X_0) = 0$, by Proposition 2(E). This means that $\partial c \in C^\sigma(K)$.

Definition 2. Let G be an abelian group. The homology group of the chain group $C^\sigma(K) \otimes G$ is called *the* σ*-homology group of* K *with coefficients in* G, and it is denoted by $H^\sigma(K; G)$.

Remark 2. Let p be a prime dividing d. It follows from Proposition 2(F) that $H^\sigma(X; Z_p) = H^\sigma(X, X_0; Z_p) \oplus H(X; Z_p)$, where H denotes simplicial homology and Z_p denotes the integers modulo p.

Construction 1. Let $z \in H^\sigma_{m+1}(X, X_0; Z_p)$, and let w be a cycle in z. Pick a chain v in $C^\sigma(X, X_0)$ that maps onto w under the canonical homomorphism. Since $\sigma(v) = 0$, it follows from Proposition 2(C) that there exists a chain u with $v = \sigma'(u)$.

Since w is a cycle in $C^{\sigma}(X, X_0; Z_p)$, there exist chains $a \in C(X, X_0)$ and $b \in C(X_0)$ such that $\partial\sigma'(u) = pa + b$. Using Proposition 2(D), we see that

$$pa = \sigma'\partial(u) - [\partial\sigma'(u) - \sigma'\partial(u)] + b;$$

hence the equations

$$\sigma(pa) = p\sigma(a) = 0$$

follow from Proposition 2(A) and 2(B). Therefore $\sigma(a) = 0$, and by Proposition 2(C) there exists a chain $a' \in C(X, X_0)$ such that $a = \sigma'(a')$. We observe that

$$\sigma'(\partial u - pa') \in \operatorname{Im} \sigma' \cap C(X_0) = 0.$$

Furthermore, $\partial[\partial u - pa'] = p\partial a'$; hence the image of $\partial u - pa'$ in $C^{\sigma'}(X) \otimes Z_p$ is a cycle x. We define $\alpha(z)$ to be the class of x in $H_n^{\sigma'}(X; Z_p)$. It is a straightforward matter to verify that $\alpha(z)$ does not depend on the choices made after w is fixed. We can use Proposition 2(G) to prove that $\alpha(z)$ does not depend on the choice of w. Obviously, α is a homomorphism. Let $\beta: H_m^{\sigma'}(X; Z_p) \to H_m(X; Z_p)$ be the homomorphism induced by the inclusion $C_m^{\sigma}(X) \to C_m(X)$. Let

$$\gamma: H_m(X; Z_p) \to H_m^{\sigma}(X, X_0; Z_p)$$

be the homomorphism induced by the homomorphism $\sigma': C_m(X) \to C_m^{\sigma}(X, X_0)$.

PROPOSITION 3. *Let* $f: X \to Y$ *be an open simplicial map of the complex* X *onto the complex* Y. *Suppose there exists a subcomplex* X_0 *such that* $f \mid (X - X_0)$ *is a* d-to-1 *covering map and* $f \mid f^{-1}fX_0$ *is a homeomorphism. Let* p *be a prime dividing* d. *Then there exist graded* Z_p-*modules* $H^{\rho}(X)$, $H^{\rho}(X, X_0)$, $H^{\tau}(X)$, *and* $H^{\tau}(X, X_0)$ *with the following three properties.*

(a) *There exist exact sequences*

$$\cdots \to H_{m+1}^{\rho}(X, X_0) \to H_m^{\tau}(X) \to H_m(X) \to \cdots \tag{1}$$

and

$$\cdots \to H_{m+1}^{\tau}(X, X_0) \to H_m^{\rho}(X) \to H_m(X) \to \cdots, \tag{2}$$

(b) $H^{\tau}(X) = H^{\tau}(X, X_0) \oplus H(X_0)$ *and* $H^{\rho}(X) = H^{\rho}(X, X_0) \oplus H(X_0)$, *and*

(c) $H^{\tau}(X, X_0) = H(Y, fX_0)$, *where* H *denotes simplicial homology with coefficients in* Z_p.

Proof. The graded Z_p-modules are $H^{\sigma}(K; Z_p)$, for the allowable choices of σ and K. Part (a) follows from the fact that $\ker \alpha = \operatorname{Im} \gamma$, $\ker \beta = \operatorname{Im} \alpha$, and $\ker \gamma = \operatorname{Im} \beta$, which can be proved by the method of [6, Theorem 2.3]. Part (b) follows from Remark 2. Part (c) follows from the definitions of H and H^{τ}, Proposition 2(C), and [16, Construction 2 and Proposition 7].

PROPOSITION 4. (I) *Suppose that* $f: M \to N$ *is an* M-S *covering and* $\dim B_f \leq n - 2$. *Then there exists a cofinal family of coverings* A_λ *and* fA_λ *on* M *and* N, *respectively, such that the map* f_λ *induced by* f *on the nerves satisfies the hypotheses of Propositions* 2 *and* 3.

(II) *Let* α_λ, β_λ, *and* γ_λ *be the maps in the exact sequences obtained by applying Proposition* 3 *to* f_λ. *Then the projections between nerves can be chosen to commute with* α_λ, β_λ, *and* γ_λ.

Proof. (I) Let the cofinal family of coverings and the projections be those constructed in [11, Theorems 1 and 2]. (II) follows from [11, Lemmas 2 and 4] and the definitions of α_λ, β_λ, and γ_λ.

THEOREM 3. *Suppose that* $f: M \to N$ *is a* d-to-1 M-S *covering and* $\dim B_f \leq n - 2$. *Let* p *be any prime dividing* d. *Then there exist graded* Z_p-*modules* $H^\rho(M)$, $H^\rho(M, B_f)$, $H^\tau(M)$, *and* $H^\tau(M, B_f)$ *with the following three properties.*

(a) *There exist exact sequences*

$$(1) \qquad \cdots \to H^\rho_{m+1}(M, B_f) \to H^\tau_m(M) \to H_m(M) \to \cdots$$

and

$$(2) \qquad \cdots \to H^\tau_{m+1}(M, B_f) \to H^\rho_m(M) \to H_m(M) \to \cdots,$$

(b) $H^\tau(M) = H^\tau(M, B_f) \oplus H(B_f)$ *and* $H^\rho(M) = H^\rho(M, B_f) \oplus H(B_f)$, *and*

(c) $H^\tau(M, B_f) = H(N, fB_f)$, *where* H *denotes Čech homology with coefficients in* Z_p.

Proof. This theorem follows from Propositions 3 and 4 and known properties of inverse limits.

THEOREM 4. *Suppose that* $f: M \to N$ *is a* d-to-1 *singular covering such that* $f \mid B_f$ *is* q-to-1, *and that* $\dim B_f \leq n - 2$. *Suppose* $f_\# \pi(M)$ *has* q *cosets in* $\pi(N)$. *Then, for each positive integer* m *and each prime* p *dividing* d,

$$\sum_m^\infty \dim H_j(B_f; Z_p) \leq \sum_m^\infty \dim H_j(M; Z_p).$$

Proof. Factor f as $g \circ h$, where $B_h = B_f$ and h is an M-S covering, by Theorem 1. Now apply [11, Theorem 4] to h.

THEOREM 5. *Suppose that* $f: M \to N$ *is a* d-to-1 M-S *covering and* $\dim B_f \leq n - 2$. *Then, for each prime* p *dividing* d *and for each pair of integers* (m, k) *with* $m < k$,

$$\sum_m^{k-1} \dim H_j(B_f; Z_p) \leq \dim H_{k+1}(N; Z_p) + \sum_m^k \dim H_j(M; Z_p).$$

Proof. We use the fact that if $A \to B \to C$ is an exact sequence of vector spaces, then $\dim B \leq \dim A + \dim C$; also, we use the properties (a), (b), and (c) of Theorem 3, and the usual exact sequence for a pair. Write $b^\sigma_j(X)$ for $\dim H^\sigma_j(X; Z_p)$. Choose any integer k. Let $\sigma = \rho$ or τ and $\sigma' = \tau$ or ρ, respectively.

The inequalities

$$\begin{aligned} b_k(B_f) &\leq b^\tau_{k+1}(M, B_f) + b_k(M) - b^\rho_k(M, B_f) \\ &\leq b_{k+1}(N, B_f) + b_k(M) - [b^\tau_{k-1}(M) - b_{k-1}(M)] \\ &\leq b_{k+1}(N) + b_k(B_f) + b_k(M) + b_{k-1}(M) - b^\tau_{k-1}(M) \\ &\leq b_{k+1}(N) + b_k(B_f) + b_k(M) + b_{k-1}(M) - b^\tau_{k-1}(M, B_f) - b_{k-1}(B_f) \end{aligned}$$

imply that

$$b_{k-1}(B_f) \leq b_{k+1}(N) + b_k(M) + b_{k-1}(M) - b^{\tau}_{k-1}(M, B_f).$$

Suppose it has been shown that

$$\sum_{m+1}^{k-1} b_j(B_f) \leq b_{k+1}(N) + \sum_{m+1}^{k} b_j(M) - b^{\sigma}_{m+1}(M, B_f).$$

Then

$$\sum_{m+1}^{k-1} b_j(B_f) \leq b_{k+1}(N) + \sum_{m+1}^{k} b_j(M) - [b^{\sigma'}_m(M) - b_m(M)]$$

$$\leq b_{k+1}(N) + \sum_{m}^{k} b_j(M) - b^{\sigma'}_m(M)$$

$$\leq b_{k+1}(N) + \sum_{m}^{k} b_j(M) - b^{\sigma'}_m(M, B_f) - b_m(B_f);$$

that is,

$$\sum_{m}^{k-1} b_j(B_f) \leq b_{k+1}(N) + \sum_{m}^{k} b_j(M) - b^{\sigma'}_m(M, B_f),$$

for all $m < k$, where $\sigma' = \tau$ if and only if $k - m$ is odd. Since $b^{\sigma'}_m(M, B_f) \geq 0$ for each m, the theorem is proved.

THEOREM 6. *If* $f\colon M \to N$ *is a* d-to-1 M-S *covering and* $\dim B_f \leq n - 2$, *then, for each prime* p *dividing* d *and for each integer* k,

$$\dim H_k(N; Z_p) \leq \sum_{k}^{\infty} \dim H_j(M; Z_p) - \sum_{k+1}^{\infty} \dim H_j(B_f; Z_p).$$

Proof. Adopt the notation and the technique in the proof of Theorem 5. For each integer k,

$$b_k(N) \leq b_k(N, B_f) + b_k(B_f) = b^{\tau}_k(M)$$

$$\leq b^{\rho}_{k+1}(M, B_f) + b_k(M) = b^{\rho}_{k+1}(M) + b_k(M) - b_{k+1}(B_f).$$

Suppose that

$$b_k(N) \leq b^{\sigma}_m(M) + \sum_{k}^{m-1} b_j(M) - \sum_{k+1}^{m} b_j(B_j).$$

Then

$$b_k(N) \leq b^{\sigma'}_{m+1}(M, B_f) + b_m(M) + \sum_{k}^{m-1} b_j(M) - \sum_{k+1}^{m} b_j(B_f)$$

$$\leq b^{\sigma'}_{m+1}(M) + \sum_{k}^{m} b_j(M) - \sum_{k+1}^{m+1} b_j(B_f).$$

Since $b^{\sigma'}_{m+1}(M) = 0$ for all sufficiently large m, the theorem is proved.

Remark 3. The results of this section are valid for M-S coverings $f: X \to Y$ of n-dimensional compact metric spaces. If W is any compact n-dimensional metric space, M is the surface of genus 4, N is the surface of genus 2, $g: M \to N$ is an irregular covering, and $\circ$ denotes join, then $f = (\mathrm{id} \circ g): W \circ M \to W \circ N$ is an irregular M-S covering with $B_f = W$. Therefore the results of this section do not follow from the results of Smith Theory.

4. SINGULAR COVERINGS OF MANIFOLDS BY SPHERES

From now on, S^n denotes the n-sphere.

LEMMA 2. *Let* $g: S^n \to N$ *be a map of degree* d *onto the orientable manifold* N. *Then* $d \circ H_i(N; Z) = 0$ *for* $0 < i < n$.

Proof. Let u be the generator of $H_n(S^n; Z)$, and v the generator of $H_n(N; Z)$. Let x be an element of $H_i(N; Z)$ for $0 < i < n$. There exists an element y of $H^{n-i}(N; Z)$ with $v \cap y = x$. The relations

$$f_*[f^*(y) \cap u] = y \cap f_*(u) = y \cap dv = dz$$

imply that

$$d \circ H_i(N; Z) \subset f_*[H_i(S^n; Z)] = 0 \qquad (0 < i < n).$$

THEOREM 7. *Let* $f: S^n \to N$ *be a* d-to-1 M-S *covering. Suppose that* B_f *is a tamely* (*respectively, smoothly*) *embedded manifold and an integral homology* (n - 2)-*sphere, and that* f *is simplicial* (*respectively, smooth*). *Then* N *is a homotopy* n-*sphere, hence a topological* n-*sphere if* $n \neq 3, 4$.

Proof. We infer from Theorem 1 that N is simply connected, hence orientable. It follows from Lemma 2 that N has the Betti numbers of an n-sphere, and that its torsion numbers are divisors of d. By Theorem 2, f is the orbit map of a semifree action of a finite group G on S^n with fixed-point set B_f, and B_f is an integral (n - 2)-dimensional homology sphere; hence $G = Z_d$ [17, Corollary on p. 408]. Therefore, for each prime p dividing d, there is a Z_p-action on S^n with fixed-point set B_f. It follows ([1, Theorem 6.1 on p. 63]) that $H^i(N; Z_p) = 0$, for each such prime, and for $0 < i < n$. Therefore, all torsion numbers of N are zero, and it follows that N is a homotopy n-sphere.

LEMMA 3. *Let* $f: S^n \to N$ *be a singular covering. Suppose that* B_f *is a tamely* (*respectively, smoothly*) *embedded connected manifold and that* f *is simplicial* (*respectively, smooth*). *If* N *is simply connected, then* f *is an* M-S *covering.*

Proof. Theorem 2 implies that f is regular. Therefore f is the orbit map for the action of a finite group G on S^n. Since f is singular and B_f is connected, the stability group at a point is constant on B_f. Call it H. We write $f = g \circ h$; here h is the orbit map $S^n \to S^n/H = M$, and hence is an M-S covering, and g is the orbit map $M \to M/(G/H) = N$, and hence is a covering map. Since N is simply connected, g is a homeomorphism, and hence f is an M-S covering.

THEOREM 8. *If* $f: S^n \to N$ *is a singular covering and* fB_f *is tamely (respectively, smoothly) embedded and* f *is simplicial (respectively, smooth) and* $n \neq 3, 4$, *then the universal covering space of* N *is* S^n, *and* f *is the composition of an* M-S *covering followed by the covering map.*

Proof. It is clearly possible to lift f to a map g from S^n onto the universal covering space of N. By Lemma 3, g is an M-S covering, and by Theorem 7, its image is S^n.

5. M-S COVERINGS OF SPHERES BY MANIFOLDS

Consider a tame embedding $\phi: B \to S^n$ of an orientable (n - 2)-manifold in S^n. By Alexander duality, $H_1[S^n - \phi(B)] = Z$. It follows from the Hurewicz theorem that

$$0 \to [\pi\{M - \phi(B)\}, \pi\{M - \phi(B)\}] \to \pi[M - \phi(B)] \to Z \to 0$$

is exact, and since Z is free, this sequence is split. By Fox's theorem [7, Uniqueness Theorem], the covering of $S^n - \phi(B)$ corresponding to

$$dZ \oplus [\pi\{M - \phi(B)\}, \pi\{M - \phi(B)\}]$$

can be extended in a unique way to a branched covering $f: X \to S^n$ of S^n by the topological space X with $fB_f = \phi(B)$. We call this the d*-fold cyclic covering of* S^n *branched over* $\phi(B)$.

THEOREM 9. *Let* $f: X \to S^n$ *be a* d-to-1 *branched covering of* S^n, *with* $fB_f = \phi(B)$. *Then* f *is an* M-S *covering if and only if it is the* d*-fold cyclic covering of* S^n *branched over* $\phi(B)$.

Proof. Suppose that f is an M-S covering. Choose a point p in fB_f and a neighborhood U about p such that $(U, U \cap fB_f)$ is homeomorphic to (R^n, R^{n-2}) and such that a generator of the free part of $\pi[S^n - \phi(B)]$ is represented by a small loop around fB_f. Now $f^{-1}(U)$ is a connected neighborhood of $f^{-1}(p)$. The map $f \mid [f^{-1}(U) - B_f]$ is a d-to-1 covering map onto $U - fB_f$, and $U - fB_f$ is homotopically equivalent to S^1. The image of $\pi(X - B_f)$ in $\pi(S^n - fB_f)$ is therefore $dZ \oplus G$, where G is a subgroup of the commutator of $\pi[S^n - fB_f]$. Since $f \mid (X - B_f)$ is d-to-1, $f_{\#}\pi(X - B_f)$ must have d cosets in $\pi(S^n - fB_f)$. This means that G is all of the commutator, and therefore f is the d-fold cyclic covering of S^n branched over $\phi(B)$. Suppose that f is the d-fold cyclic covering of S^n branched over $\phi(B)$. Choose a point p in $\phi(B) = fB_f$ and a point q in $f^{-1}(p) \subset f^{-1}fB_f$. Choose a neighborhood U of p such that V, the component of $f^{-1}(U)$ containing q, contains no other elements of $f^{-1}(p)$. Choose a base-point in U and a small loop α around fB_f in U that represents a generator of the free part of $\pi(S^n - fB_f) = Z \oplus [\pi, \pi]$. The loop α lifts to a loop β in $V - f^{-1}fB_f$. Since f is a d-fold cyclic covering of S^n branched over fB_f, f maps β d-to-1 onto α, hence V d-to-1 onto U; therefore $V = f^{-1}(U)$ and q is all of $f^{-1}(p)$. Therefore, f is an M-S covering.

COROLLARY 9.1. *Let* $f: M \to S^n$ *be an* M-S *covering such that* fB_f *is a trivially knotted* S^p. *Then* $p = n - 2$, *and* f *is the* (n - 1)*-fold suspension of a* d-to-1 *covering map of* S^1 *on* S^1.

Proof. $S^n - fB_f$ is homotopically equivalent to S^{n-p+1}, and it admits a nontrivial covering. Therefore $p = n - 2$. By Theorem 9 and Fox's theorem, the map f is the only d-to-1 M-S covering branched over a trivially knotted S^{n-2}. Since the (n - 1)-fold suspension of a d-to-1 covering map g of S^1 onto S^1 is such a map, f is topologically equivalent to g. For a constructive proof of this theorem, see [16, Theorem 2].

Remark 4. Write $R^n = R^{n-2} \times C$, and define $f_d: R^{n-2} \times C \to R^{n-2} \times C$ by the equation $f_d(x, z) = (x, z^d)$. Since $\pi(R^n - R^{n-2}) = Z$, it follows from Fox's uniqueness theorem that f_d is the unique d-fold branched covering of R^n branched over R^{n-2}.

COROLLARY 9.2. *Let* $f: X \to S^n$ *be the* d*-fold cyclic branched covering of* S^n *branched over* $\phi(B)$. *Then the space* X *is an* n*-manifold.*

Proof. Since $f \mid (X - f^{-1} fB_f)$ is a covering map onto $S^n - \phi(B)$, points in $X - B_f$ have Euclidean neighborhoods. Consider a point q in B_f, and let $p = f(q)$. Choose a neighborhood U of p such that $(U, U \cap fB_f)$ is homeomorphic to (R^n, R^{n-2}). Let $V = f^{-1}(U)$. Since $f \mid V$ is the d-fold cyclic covering of R^n branched over R^{n-2}, it follows from Remark 4 that V is homeomorphic to R^n, hence V is a Euclidean neighborhood of q.

6. COVERINGS OF CERTAIN SMOOTH KNOTS

Let ϕ be a smooth embedding of a Brieskorn (n - 2)-sphere Σ^{n-2} in a Brieskorn n-sphere Σ^n (see [2] and [12]).

PROPOSITION 5. *There exist a unique differentiable manifold* M *and a smooth* M-S *covering* $f: M \to \Sigma^n$ *with* $fB_f = \phi(\Sigma^{n-2})$.

Proof. The normal bundle to the embedding ϕ is trivial; hence there exists a smooth embedding $\psi: \Sigma^{n-2} \times D^2 \to \Sigma^n$, where D^2 is the 2-disk. Let $f: M \to \Sigma^n$ be the topological d-fold cyclic branched covering of $\phi(\Sigma^{n-2})$. The manifold $M - f^{-1}[\psi(\Sigma^{n-2} \times D^2)]$ has a unique differentiable structure that makes

$$f \mid \{M - f^{-1}[\psi(\Sigma^{n-2} \times D^2)]\}$$

a smooth covering map. Since $f \mid f^{-1}[\psi(\Sigma^{n-2} \times D^2)]$ is the d-fold cyclic branched covering of the embedding

$$\phi: \Sigma^{n-2} \to \psi(\Sigma^{n-2} \times D^2),$$

it follows that $f^{-1}[\psi(\Sigma^{n-2} \times D^2)]$ is homeomorphic to $S^{n-2} \times D^2$, and f is topologically the map

$$(\mathrm{id} \times Z^d): S^{n-2} \times D^2 \to \Sigma^{n-2} \times D^2.$$

There is precisely one differentiable structure on $S^{n-2} \times D^2$ that makes this map smooth, namely that of $\Sigma^{n-2} \times D^2$. Appropriate identification on the boundaries of

$$\Sigma^{n-2} \times D^2 = f^{-1}[\psi(\Sigma^{n-2} \times D^2)] \quad \text{and} \quad M - f^{-1}[\psi(\Sigma^{n-2} \times D^2)]$$

produces a smooth M-S covering $f: M \to \Sigma^n$ with $fB_f = \phi(\Sigma^{n-2})$; f is the unique such map by construction.

Example 1. Let $\Sigma(2, -, 2, 3)$ be a (4m + 1)-Brieskorn sphere [2, Theorem 2]. The d-fold cyclic covering space of the nontrivial knot [2] $\Sigma(2, -, 2, 3)$ in S^{4m+3} is the Brieskorn variety $\Sigma(2, -, 2, 3, d)$, where the covering map f is that given by [14, Section 5]. This variety is a sphere if and only if $d \equiv \pm 1$ (6). Each Brieskorn sphere of dimension $4m+3$ appears as a branched covering space. In fact, if $d = 6j \pm 1$ and Σ_1 is the generator of bP_{4m+4}, then the covering space is $j\Sigma_1$. Furthermore, since $\pi_1(S^{4m+1} - \Sigma(2, -, 2, 3)) = Z$ [2, Lemma 6], the branched covering is regular, and therefore f is the orbit map of a semifree Z_d action. The action is clearly smooth, away from $f^{-1}[\Sigma(2, -, 2, 3)]$. Near $f^{-1}[\Sigma(2, -, 2, 3)]$, the action is given by

$$f = [\mathrm{id} \times \exp(2\pi i d\theta)]: \Sigma^{n-2} \times D^2 \to \Sigma^{n-2} \times D^2,$$

and hence it is smooth. The fixed-point set is clearly $f^{-1}(\Sigma^{n-2})$. These examples show that an M-S covering f of S^n by S^n can have an exotic S^{n-2} for B_f, and that the hypothesis of unknottedness of B_f in Corollary 9.1 is necessary. The orbit maps of the actions in [8] are also M-S coverings of S^n by S^n for which B_f is a knotted S^{n-2}.

Example 2. The Brieskorn variety $\Sigma(2, 2, 3, 5)$ is a homotopy 5-sphere by [2, Theorem 1], hence a 5-sphere. There exists a 2-to-1 M-S covering $f: \Sigma(2, 2, 3, 5) \to S^5$ with $B_f = \Sigma(2, 3, 5)$ [14, Section 5]. Now $\Sigma(2, 3, 5)$ is a Poincaré space. It follows that the suspension of f is an M-S covering of S^6 by S^6 that can be taken to be simplicial, while B_f is not a manifold. That this cannot happen in lower dimensions was proved in [10, Theorem 1] and [15, Corollary 2]. Repeated suspension produces examples of simplicial M-S coverings $f: S^n \to S^n$, where B_f is not a manifold for any $n \geq 6$.

Example 3. Consider the standard embedding $i_1: \Sigma(2, -, 2, 35) \subset S^{4m+3}$ and its smooth, cyclic, d-fold branched covering $h: \Sigma(2, -, 2, 35, d) \to S^{4m+3}$. Let $(-1)^{m+1}\Sigma_1$ be a generator of bP_{4m+4}. We denote by i_2 the embedding

$$\Sigma(2, -, 2, 35) \subset S^{4m+3} \# 16\Sigma_1$$

obtained by forming the connected sum away from $i_1(\Sigma)$. Its smooth, cyclic, d-fold branched covering is the connected sum $\Sigma(2, -, 2, 35, d) \# 16d\Sigma_1$ formed equivariantly with respect to the action of which h is the orbit map. Let i_3 denote the inclusion

$$\Sigma(2, -, 2, 5) \to \Sigma(2, -, 2, 5, 7) = 16\Sigma_1,$$

where the equality follows from [2, Theorem 3 and subsequent remarks]. Notice that $\Sigma(2, -, 2, 5)$ is diffeomorphic to $\Sigma(2, -, 2, 35)$, by [2, Theorem 2(ii)]. The domain of the 7-to-1 smooth cyclic branched covering of $i_3(\Sigma)$ is $\Sigma(2, -, 2, 5, 49) = 116\Sigma_1$. The domain of the 7-to-1 smooth cyclic branched covering of $i_2(\Sigma)$ is $\Sigma(2, -, 2, 35, 7) \# 112\Sigma_1$, and this is not a sphere [2, Theorem 1]. *A fortiori*, these spaces are not diffeomorphic, and therefore the embeddings are not equivalent. Let i_4 be the natural embedding $\Sigma(2, -, 2, 5) \subset S^{4m+3}$. We know that

$$\pi[S^{4m+3} - i_4\{\Sigma(2, -, 2, 5)\}] = Z,$$

so that its covering spaces are topologically determined by the degree of the covering map. The 7-to-1 smooth cyclic branched covering of $i_4[\Sigma(2, -, 2, 5)]$ is the map $f: \Sigma(2, -, 2, 5, 7) \to S^{4m+3}$, and $B_f = i_3[\Sigma(2, -, 2, 5)]$. Therefore the restriction of f produces a 7-to-1 covering map from $\Sigma(2, -, 2, 5, 7) - i_3[\Sigma(2, -, 2, 5)]$ onto $S^{4m+3} - i_4[\Sigma(2, -, 2, 5)]$. Also, the map g defined on

$$S^{4m+3} - i_1[\Sigma(2, -, 2, 35)]$$

by

$$g(z_1, -, z_{2m+2}) = \frac{1}{\|z\|}(z_1, \text{——}, z_{2m+1}, z_{2m+2}^7)$$

is a 7-to-1 covering map onto $S^{4m+3} - i_4[\Sigma(2, -, 2, 5)]$. Therefore

$$\Sigma(2, -, 2, 5, 7) - i_3[\Sigma(2, -, 2, 5)]$$

is homeomorphic to $S^{4m+3} - i_1[\Sigma(2, -, 2, 35)]$, and hence to

$$\{S^{4m+3} - i_1[\Sigma(2, -, 2, 35)]\} \# 16\Sigma_1 = \Sigma(2, -, 2, 5, 7) - i_2[\Sigma(2, -, 2, 35)].$$

It follows that $\pi\{\Sigma(2, -, 2, 5, 7) - i_2[\Sigma(2, -, 2, 35)]\} = Z$. In this situation, the techniques of [13, Theorem 2.1] show that the complements of the inequivalent knots i_2 and i_3 are diffeomorphic. It is known that a smooth knot of standard spheres has a complement diffeomorphic with the complement of at most one inequivalent smooth knot [3].

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