A THEOREM ON HOMOTOPY-COMMUTATIVITY

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In [6], higher forms of homotopy-commutativity, C_n-forms, were defined for associative H-spaces. It was shown that an associative H-space admits a C_n-form if and only if its Hopf fibration $X \to E_1 \to SX$ extends to a fibration $X \to E_n \to (SX)_n$, where $(SX)_n$ denotes the n-fold James' reduced product space of the suspension of X. A C_2-form is simply a commuting homotopy for X. It is the purpose of this paper to show that the above result for $n = 2$ holds also for homotopy-commutative H-spaces that are not necessarily associative, but only homotopy-associative.

THEOREM. *Let* X *be a homotopy-associative* H-*space. Then* X *is homotopy-commutative if and only if the Hopf fibration extends to a fibration* $X \to E_2 \to (SX)_2$.

In fact, our proof will show that the "if" part of the theorem holds even *without* associativity requirements on X. We shall begin with the demonstration of this part of the theorem, then define the construction that establishes the reverse implication. We then conclude with a corollary and some illustrative applications.

Let X be an H-space, with multiplication $m: X^2 \to X$, and let $X \to E_1 \xrightarrow{p} SX$ denote the Hopf fibration for X. Since X is null-homotopic in E_1, there exists a retraction $r: \Omega SX \to X$ such that if $i: X \to \Omega SX$ denotes the usual inclusion, then ri is homotopic to the identity map of X. Furthermore, if $n: X^2 \to X$ is given by $n(x, y) = r(i(x) + i(y))$, then n is homotopic to m. (For details on these well-known facts, see [2, pp. 201-205] or [5].) Now assume that p extends to $X \to E_2 \xrightarrow{p'} (SX)_2$. Then r extends to $r': \Omega(SX)_2 \to X$. Let $j: \Omega SX \to \Omega(SX)_2$ denote the inclusion. The homotopies that are commonly used to show that the loop space of an H-space is homotopy-commutative can also serve to define a homotopy $Q': I \times (\Omega SX)^2 \to \Omega(SX)_2$ between $j(a) + j(b)$ and $j(b) + j(a)$. Let $\tilde{Q}: I \times X^2 \to X$ be the composition $r' \circ Q' \circ (1 \times i^2)$. Then $\tilde{Q}$ can be deformed to $Q: I \times X^2 \to X$, which is a commuting homotopy for m. Hence, X is homotopy-commutative.

Now let X be a homotopy-associative, homotopy-commutative H-space. As in [6, pp. 194-195], let K_n be the convex hull in R^n of the orbit of the point $(1, 2, \cdots, n)$ under permutation of the coordinates. [See [3] for a picture of K_n $(n \leq 4)$ and for verification of the following facts.] The boundary of K_n is the union of $(n - 2)$-cells that are in one-to-one correspondence with the (ℓ, m)-shuffles of the set $\{1, 2, \cdots, n\}$ $(1 \leq \ell, m \leq n - 1)$. If (A_ℓ, B_m) is such an (ℓ, m)-shuffle, then the cell of $Bd(K_n)$ corresponding to it is the image of $K_\ell \times K_m$ by a one-to-one linear map $V(A_\ell, B_m): K_\ell \times K_m \to Bd(K_n)$. There are maps $s_j: K_{n+1} \to K_n$ $(j = 1, \cdots, n + 1)$ that interact with each other and with the $V(A_\ell, B_m)$'s somewhat in the manner of degeneracy operators. We shall be concerned with K_n only for $n = 1$, 2, and 3.

We begin the construction of E_n $(n \leq 2)$ by setting $E_0 = X$ and choosing for $a_1: X \to X$ the identity map. Let $Q: I \times X^2 \to X$ and $M: I \times X^3 \to X$ be commuting and associating homotopies for X. Let

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$$Z_1 = (K_2 \times X \times (e)) \cup (\mathrm{Bd}(K_2) \times X^2) \subset K_2 \times X^2$$

and

$$Z_2 = (K_3 \times X \times (X \vee X)) \cup (\mathrm{Bd}(K_3) \times X^3) \subset K_3 \times X^3.$$

Construct a relative homeomorphism a_i: $(K_i \times X^i, Z_{i-1}) \to (E_{i-1}, E_{i-2})$ $(i = 2, 3)$ inductively as follows. Define $a_i \mid Z_i$ by the rules

(1) if $x_j = e$, then $a_i(\tau, x, x_1, \cdots, x_{i-1}) = a_{i-1}(s_j(\tau), x, \cdots, \hat{x}_j, \cdots)$;

(2) if (A_r, B_s) is an (r, s)-shuffle of $(1, \cdots, i)$ and $i \in A_r$, then

$$a_i(V(A_r, B_s)[\rho, \sigma], x, x_1, \cdots, x_{i-1}) = a_r(\rho, x, x_{A'(1)}, \cdots, x_{A'(r-1)}),$$

where $A'_{r-1} = A_r - (i)$;

(3) if (A_r, B_s) is an (r, s)-shuffle of $(1, \cdots, i)$, $i \in B_s$, and $A_r \neq (1, 2)$, then

$$a_i(V(A_r, B_s)[\rho, \sigma], x, x_1, \cdots, x_{i-1}) = a_s(\sigma, x \cdot x_{A(1)}, x_{B'(1)}, \cdots, x_{B'(s-1)}),$$

where $B'_{s-1} = B_s - (i)$; and

$$\text{(3')} \quad a_3(V((1, 2), (3))[(t+1, 2-t), *], x, x_1, x_2) = \begin{cases} M(3t, x, x_1, x_2) & (0 \leq 3t \leq 1), \\ x \cdot Q(3t-1, x_1, x_2) & (1 \leq 3t \leq 2), \\ M(3-3t, x, x_2, x_1) & (2 \leq 3t \leq 3). \end{cases}$$

Now let

$$W_1 = (K_2 \times (e)) \cup (\mathrm{Bd}(K_2) \times X) \subset K_2 \times X$$

and

$$W_2 = (K_3 \times (X \vee X)) \cup (\mathrm{Bd}(K_3) \times X^2) \subset K_3 \times X^2.$$

Let B_0 be a point, and let b_i: $(K_i \times X^{i-1}, W_{i-1}) \to (B_{i-1}, B_{i-2})$ $(i = 2, 3)$ be a relative homeomorphism, where $b_i \mid W_{i-1}$ is defined by formulas (1) to (3) with x omitted. Finally, let p_i: $E_i \to B_i$ $(i = 1, 2)$ be induced by the projection

$$K_{i+1} \times X \times X^i \to K_{i+1} \times (e) \times X^i = K_{i+1} \times X^i.$$

It follows from standard techniques (see [4, p. 286]) that p_i is a fibration, and we note that p_1: $E_1 \to B_1$ coincides with the Hopf fibration for X. It remains to verify that B_2 is the homotopy type of $(SX)_2$, and in this case the lack of strict associativity of X does not alter the proof given on p. 203 of [6].

The explicit form of the attaching maps that define E_2 makes it easy to calculate the homology of E_2. As an example, we offer the following corollary (it can easily be proved by calculation).

COROLLARY 1. *Let* X *be a homotopy-associative, homotopy-commutative* H-*space. Suppose that* $H_i(X) = 0$ $(0 < i < n)$ *and that* $H_n(X) = Z$. *Then* $H_i(E_2) = 0$ $(0 < i < 2n+1)$ *and* $H_{2n+1}(E_2) = Z_2$.

COROLLARY 2. *Under the hypotheses of Corollary* 1, $\pi_{2n+2}(S^2 X)$ *contains* Z_2 *as a direct summand.*

Proof. The base space $B_2 = (SX)_2$ coincides with $(SX)_\infty = \Omega S^2 X$ up through the $(3n+2)$-skeleton. Thus $\pi_{2n+2}(S^2 X) = \pi_{2n+1}(\Omega S^2 X) = \pi_{2n+1}((SX)_2)$. Since X is null-homotopic in E_2, $\pi_{2n+1}((SX)_2)$ is isomorphic to $\pi_{2n}(X) \oplus \pi_{2n+1}(E_2)$. But $\pi_{2n+1}(E_2) = Z_2$, by Corollary 1 and the Hurewicz theorem.

We offer two examples of spaces for which the conclusion of Corollary 2 does not hold. Let X_1 be S^3 with its usual multiplication, and let X_2 be S^7, made homotopy-associative by killing the 2- and 3-components of $\pi_k(S^7)$ $(k \geq 21)$ (see [1]). Then Corollary 2 furnishes an alternate proof that X_1 and X_2 are not homotopy-commutative, since $\pi_8(S^2 X_1) = \pi_8(S^5) = Z_{24}$ and $\pi_{16}(S^2 X_2) = \pi_{16}(S^9) = Z_{240}$, and since neither of these groups admits Z_2 as a direct summand.

As an example of a space that does satisfy Corollary 2, consider $X = S^1$. In the splitting given in the proof of Corollary 2, $\pi_2(S^1) = 0$, so that

$$\pi_4(S^3) = \pi_4(S^2 X) = Z_2.$$

This offers an alternate calculation of $\pi_4(S^3)$.

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