AN ISOMORPHIC CHARACTERIZATION OF L_p AND c_0-SPACES. II.

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In a previous paper of the same title [14], we have proved that a Banach space X is isomorphic either to c_0 or to an L_p-space $(1 \leq p < \infty)$ over a finite measure space if and only if it is a cyclic space $X = \mathrm{clm}\,\{Px_0 \mid P \in \mathscr{B}\}$ $(x_0 \in X)$ relative to a σ-complete Boolean algebra $\mathscr{B}$ of projections that has a *two-sided estimate.* The latter condition is to be interpreted in the sense that there exist a constant K and a function ψ (defined in the space of sequences of complex numbers) such that the inequalities

$$K^{-1}\psi(\{\|P_n x\|\}) \leq \|x\| \leq K\psi(\{\|P_n x\|\})$$

hold for each $x \in X$ and for each sequence of disjoint projections $P_n \in \mathscr{B}$ whose sum is the identity I.

Other characterizations in terms of Boolean algebras of projections with two-sided estimates have been obtained in [8] and [10] for the $\mathscr{L}_p$-spaces introduced by J. Lindenstrauss and A. Pełczyński in [7].

In the present paper we weaken this condition: we show that instead of the two-sided estimate for $\mathscr{B}$ we need merely the existence of a similar function ϕ, with values in $[0, \infty]$, such that a series $\sum_{n=1}^{\infty} P_n x_n$ is weakly convergent (not necessarily to a vector in X) if and only if $\phi(\{\|P_n x_n\|\}) < \infty$ for each sequence $\{x_n\}$ $(x_n \in X)$ and each sequence of disjoint projections $P_n \in \mathscr{B}$. We shall use this result to prove our main theorem, which is another isomorphic characterization of c_0 and L_p, this time involving the existence of positive projections on every sublattice of a σ-Dedekind complete (conditionally σ-complete) Banach lattice. This theorem represents an isomorphic version of a recent result of T. Ando [1].

1. CYCLIC SPACES ISOMORPHIC TO L_p AND c_0

For the notions and the terminology used in this paper, we refer the reader to [14] (see also [2], [3], and [7]).

We begin by showing that a cyclic space having enough subspaces isomorphic to ℓ_p $(1 \leq p < \infty)$ is in fact isomorphic itself to an L_p-space for the same p.

PROPOSITION 1. *A Banach space* X *is isomorphic to an* L_p*-space* $(1 \leq p < \infty)$ *over a finite measure space if and only if it is a cyclic space*

$$X = \mathfrak{M}(x_0) = \mathrm{clm}\,\{Ex_0 \mid E \in \mathcal{E}\}$$

relative to a σ*-complete Boolean algebra* $\mathcal{E}$ *of projections such that for each* $x \in X$ *and each infinite sequence of disjoint projections* $E_n \in \mathcal{E}$ $(E_n x \neq 0;\ n = 1, 2, \cdots)$ *the basis* $\{E_n x/\|E_n x\|\}$ *is equivalent to the natural basis of* ℓ_p.

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Proof. If τ is an isomorphism from $L_p(\Omega, \Sigma, \mu)$ $(1 \leq p < \infty,\ \mu(\Omega) < \infty)$ onto X, we can set

$$E(\sigma)x = \tau\chi_\sigma(\tau^{-1}x) \quad \left(x \in X,\ \sigma \in \Sigma\right).$$

Obviously, the projections $E(\sigma)$ $\left(\sigma \in \Sigma\right)$ form a σ-complete Boolean algebra $\mathcal{E}$ of projections such that the basis $\{E(\sigma_n)x/\|E(\sigma_n)x\|\}$ is equivalent to the natural basis of ℓ_p for each $x \in X$ and each infinite sequence of disjoint projections $E(\sigma_n) \in \mathcal{E}$ with $E(\sigma_n)x \neq 0$ $(n = 1, 2, \cdots)$. This proves the necessity. In order to prove the sufficiency, we remark first that $\mathcal{E}$ can be considered the range of a spectral measure $E(\sigma)$ defined on the Borel sets $\sigma \in \Sigma$ of a compact, topological Hausdorff space Ω (see W. G. Bade [3]). We write

$$\nu_x(\delta) = \sup\left\{\sum_k \|E(\delta_k)x\|^p\right\} \quad \left(\delta \in \Sigma;\ x \in X\right),$$

where the supremum is taken over all finite partitions $\{\delta_k\}$ of δ (if no finite upper bound exists, we set $\nu_x(\delta) = \infty$). For $\delta, \sigma \in \Sigma$ and $\delta \cap \sigma = \emptyset$, we have the inequalities

$$\nu_x^{1/p}(\delta \cup \sigma) = \sup_{\bigcup_i \eta_i = \delta\cup\sigma} \left\{\sum_i \|E(\eta_i)x\|^p\right\}^{1/p}$$

$$\leq \sup\left\{\sum_i (\|E(\eta_i \cap \delta)x\| + \|E(\eta_i \cap \sigma)x\|)^p\right\}^{1/p} \leq \nu_x^{1/p}(\delta) + \nu_x^{1/p}(\sigma).$$

Suppose there exists a sequence $\{x_n\}$ $(x_n \in X,\ \|x_n\| = 1)$ such that $\nu_{x_n}(\Omega) \geq n$ $(n = 1, 2, \cdots)$. Then there exist an index n_1 and a partition $\sigma_s^{(1)}$ $(s = 1, 2, \cdots, q_1)$ of Ω for which

$$\sum_{s=1}^{q_1} \|E(\sigma_s^{(1)})x_{n_1}\|^p \geq 4^p.$$

Then, since

$$\nu_{x_n}^{1/p}(\Omega) \leq \sum_{s=1}^{q_1} \nu_{x_n}^{1/p}(\sigma_s^{(1)}),$$

we can assume without loss of generality that $\lim_{n\to\infty} \nu_{x_n}(\sigma_{q_1}^{(1)}) = \infty$. Let $n_2 > n_1$ be an index such that

$$\nu_{x_{n_2}}\left(\sigma_{q_1}^{(1)}\right) \geq 1 + 4^{2p} + \left\|E\left(\sigma_{q_1}^{(1)}\right)x_{n_1}\right\|^p.$$

Then, in the second step, we can construct a partition $\sigma_s^{(2)}$ $(s = 1, 2, \cdots, q_2)$ of $\sigma_{q_1}^{(1)}$ such that

$$\sum_{s=1}^{q_2} \left\| E\left(\sigma_s^{(2)} \right) x_{n_2} \right\|^p \geq 4^{2p} + \left\| E\left(\sigma_{q_1}^{(1)} \right) x_{n_1} \right\|^p .$$

Repeating again the same arguments, we can assume that $\lim_{n\to\infty} \nu_{x_n}(\sigma_{q_2}^{(2)}) = \infty$, and we can choose an integer $n_3 > n_2$ for which

$$\nu_{x_{n_3}}\left(\sigma_{q_2}^{(2)} \right) \geq 1 + 4^{3p} + \left\| E\left(\sigma_{q_2}^{(2)} \right) x_{n_2} \right\|^p$$

and a partition $\sigma_s^{(3)}$ $(s = 1, 2, \cdots, q_3)$ of $\sigma_{q_2}^{(2)}$ such that

$$\sum_{s=1}^{q_3} \left\| E\left(\sigma_s^{(3)} \right) x_{n_3} \right\|^p \geq 4^{3p} + \left\| E\left(\sigma_{q_2}^{(2)} \right) x_{n_2} \right\|^p .$$

Continuing thus, we obtain a sequence of disjoint sets $\sigma_s^{(r)}$ $(1 \leq s \leq q_r - 1;$ $r = 1, 2, \cdots)$ with the property

$$\sum_{s=1}^{q_r - 1} \left\| E\left(\sigma_s^{(r)} \right) x_{n_r} \right\|^p \geq 4^{rp} + \left\| E\left(\sigma_{q_{r-1}}^{(r-1)} \right) x_{n_{r-1}} \right\|^p - \left\| E\left(\sigma_{q_r}^{(r)} \right) x_{n_r} \right\|^p .$$

Hence

$$\sum_{r=1}^{R} \sum_{s=1}^{q_r - 1} \left\| E\left(\sigma_s^{(r)} \right) x_{n_r} \right\|^p \geq 4^{Rp} - \left\| E\left(\sigma_{q_R}^{(R)} \right) x_{n_R} \right\|^p$$

$$\geq 4^{Rp} - M^p \quad (R = 1, 2, \cdots),$$

where M is a bound on the norms of the projections $E \in \mathcal{E}$. Set

$$x = \sum_{r=1}^{\infty} \sum_{s=1}^{q_r - 1} E\left(\sigma_s^{(r)} \right) x_{n_r} / 2^r$$

(the series converges, since $\left\| \sum_{s=1}^{q_r - 1} E\left(\sigma_s^{(r)} \right) x_{n_r} \right\| \leq M$ $(r = 1, 2, \cdots)$. Then

$$\sum_{r=1}^{R} \sum_{s=1}^{q_r - 1} \left\| E\left(\sigma_s^{(r)} \right) x \right\|^p \geq \frac{1}{2^{Rp}} \sum_{r=1}^{R} \sum_{s=1}^{q_r - 1} \left\| E\left(\sigma_s^{(r)} \right) x_{n_r} \right\|^p$$

$$\geq \frac{4^{Rp} - M^p}{2^{Rp}} \quad (R = 1, 2, \cdots),$$

and this contradicts the fact that the basis $\{E(\sigma_s^{(r)})x / \|E(\sigma_s^{(r)})x\|\}$ $(1 \leq s \leq q_r - 1;$ $r = 1, 2, \cdots)$ is equivalent to the natural basis of ℓ_p. In conclusion, we have just proved the existence of a constant A such that $\nu_x^{1/p}(\Omega) \leq A \|x\|$ $(x \in X)$, in other words, such that

$$\left(\sum_i \left\| E(\eta_i) x \right\|^p \right)^{1/p} \leq A \|x\|$$

for each $x \in X$ and each sequence of disjoint projections $E(\eta_i) \in \mathcal{E}$.

If $p = 1$, then, in view of a result of C. A. McCarthy and L. Tzafriri [12, Theorem 10], the space X is isomorphic to an L_1-space over a finite measure space, since the previous inequality shows that X is an $\mathscr{L}_1$-space in the sense of [7].

When $1 < p < \infty$, the cyclic space X is reflexive (see [15]), and therefore, for each $x^* \in X^*$, there exists an $x \in X$ such that $\|x\| = 1$ and $x^*x = \|x^*\|$. Consequently, for each partition $\{\delta_n\}$ of Ω we have the inequalities

$$\|x^*\| = x^*x = \sum_n E^*(\delta_n)x^* E(\delta_n)x \leq \sum_n \|E^*(\delta_n)x^*\| \; \|E(\delta_n)x\|$$
$$\leq \left(\sum_n \|E(\delta_n)x\|^p\right)^{1/p} \left(\sum_n \|E^*(\delta_n)x^*\|^q\right)^{1/q} \leq A\left(\sum_n \|E^*(\delta_n)x^*\|^q\right)^{1/q}$$

where $p^{-1} + q^{-1} = 1$. Thus, for each $x^* \in X^*$ and each partition $\{\delta_n\}$ of Ω,

$$\left(\sum_n \|E^*(\delta_n)x^*\|^q\right)^{1/q} \geq A^{-1} \|x^*\| .$$

By applying this inequality to a vector $\sum_n a_n \dfrac{E^*(\delta_n)x^*}{\|E^*(\delta_n)x^*\|}$, we obtain the inequality

$$\left(\sum_n |a_n|^q\right)^{1/q} \geq A^{-1} \left\|\sum_n a_n \frac{E^*(\delta_n)x^*}{\|E^*(\delta_n)x^*\|}\right\| ;$$

in other words, $(a_n) \in \ell_q$ implies the convergence in X^* of the series

$$\sum_{n=1}^{\infty} a_n \frac{E^*(\delta_n)x^*}{\|E^*(\delta_n)x^*\|} .$$

Assume now that a series $\sum_{n=1}^{\infty} b_n \dfrac{E^*(\delta_n)x^*}{\|E^*(\delta_n)x^*\|}$ converges in X^*, and choose $x_n = E(\delta_n)x_n \in X$ such that

$$E^*(\delta_n)x^* x_n = \|E^*(\delta_n)x^*\| \qquad (\|x_n\| = 1;\ n = 1, 2, \cdots) .$$

In view of our hypotheses, a series $\sum_{n=1}^{\infty} c_n x_n$ converges if and only if $(c_n) \in \ell_p$. Thus the relation

$$\sum_{n=1}^{\infty} b_n \frac{E^*(\delta_n)x^*}{\|E^*(\delta_n)x^*\|} \sum_{m=1}^{\infty} c_m x_m = \sum_{n=1}^{\infty} b_n c_n$$

shows that $\sum_{n=1}^{\infty} b_n c_n$ converges for every sequence $(c_n) \in \ell_p$. Hence $(b_n) \in \ell_q$, and the basis $\{E^*(\delta_n)x^*/\|E^*(\delta_n)x^*\|\}$ is equivalent to the natural basis of ℓ_q, for each $x^* \in X^*$ and each partition $\{\delta_n\}$ of Ω.

Because of reflexivity, X^* itself is a cyclic space (see W. G. Bade [3, Theorem 8.4], and by applying the conclusions of the first part of the proof we obtain the existence of a constant B such that

$$\left(\sum_j \|E^*(\eta_j)x^*\|^q\right)^{1/q} \leq B\|x^*\|$$

for each $x^* \in X^*$ and each sequence of disjoint projections $E^*(\eta_j) \in \mathcal{E}^*$ (the adjoint Boolean algebra of projections) whose sum is I^*.

It follows that X^* is a cyclic space relative to a Boolean algebra of projections having a two-sided estimate; consequently, X^* is isomorphic to L_q, and X is isomorphic to L_p over a finite measure space (see [14, Theorem 4]).

PROPOSITION 2. *A Banach space* X *is isomorphic to* c_0 *if and only if it is a cyclic space* $X = \mathfrak{M}(x_0) = \text{clm}\{Fx_0 | F \in \mathcal{F}\}$ *relative to a* σ*-complete Boolean algebra* $\mathcal{F}$ *of projections such that for each* $x \in X$ *and each infinite sequence of disjoint projections* $F_n \in \mathcal{F}$ $(F_n x \neq 0;\ n = 1, 2, \cdots)$ *the basis* $\{F_n x / \|F_n x\|\}$ *is equivalent to the natural basis of* c_0.

Proof. First notice that X^* contains no subspace isomorphic to c_0 (see C. Bessaga and A. Pełczyński [4, Theorem 4] and the proof of Lemma 4 of [15]). Hence, in view of [15, Lemma 4], $\mathcal{F}^* = \{F^* | F \in \mathcal{F}\}$ is a σ-complete Boolean algebra of projections on X^*, and X^* is a cyclic space relative to $\mathcal{F}^*$ (see W. G. Bade [3, Theorem 8.4]).

Suppose $\sum_{n=1}^{\infty} a_n E^*(\delta_n)x^* / \|E^*(\delta_n)x^*\|$ converges in X^*, and choose $x_n = E(\delta_n)x_n \in X$ ($\|x_n\| = 1$) so that

$$E^*(\delta_n)x^* x_n \geq \frac{\|E^*(\delta_n)x^*\|}{2} \qquad (n = 1, 2, \cdots)\,.$$

In view of [15, Lemma 1], the series $\sum_{n=1}^{\infty} |a_n| \dfrac{E^*(\delta_n)x^*}{\|E^*(\delta_n)x^*\|}$ is also convergent, and therefore

$$\infty > \sum_{n=1}^{\infty} |a_n| \frac{E^*(\delta_n)x^*}{\|E^*(\delta_n)x^*\|} \sum_{m=1}^{\infty} |d_m| x_m \geq \frac{1}{2} \sum_{n=1}^{\infty} |a_n|\, |d_n|$$

for each series $\sum_{m=1}^{\infty} |d_m| x_m$ belonging to X. Since $\sum_{m=1}^{\infty} |d_m| x_m$ converges if and only if $\{d_m\} \in c_0$, we conclude that $\{a_n\} \in \ell_1$.

We have now shown that X^* satisfies the conditions of Proposition 1 for $p = 1$ and is thus isomorphic to an L_1-space over a finite measure space. Consequently, X is a cyclic subspace of X^{**}, and the latter is an $\mathcal{L}_\infty$-space. We can now complete the proof by using a result of C. A. McCarthy and L. Tzafriri [12, Theorem 16] (instead of this theorem, one can also use A. Pełczyński [13, Theorem 4.1]).

Remark. The equivalence between two bases $\{u_n\}$ and $\{v_n\}$ generates an isomorphism between clm $\{u_n\}$ and clm $\{v_n\}$. It seems that the rich structure of a cyclic space enables us to skip conditions of uniform boundedness for the norms of the isomorphisms between clm $\{E(\sigma_n)x / \|E(\sigma_n)x\|\}$ and ℓ_p, respectively c_0, in Propositions 1 and 2.

Definition 3. A cyclic space $X = \mathfrak{M}(x_0)$ ($x_0 \in X$) relative to a σ-complete Boolean algebra of projections $\mathscr{B}$ is *weakly homogeneous* if there exists a function ϕ, with values in $[0, \infty]$ (defined in the space of sequences of complex numbers) such that the series $\sum_{n=1}^{\infty} P_n x_n$ is weakly convergent (not necessarily to a vector in X) if and only if the inequality $\phi(\{\|P_n x_n\|\}) < \infty$ holds for each $x_n \in X$ and for each sequence of disjoint projections $P_n \in \mathscr{B}$.

Remark. If $\mathscr{B}$ has a two-sided estimate with a constant K and function ψ (see [14, Definition 2] or the introduction to this paper), then X is weakly homogeneous with respect to the function

$$\phi(a_1, a_2, \cdots, a_n, \cdots) = \sup_N \psi(a_1, a_2, \cdots, a_N, 0, 0, \cdots) .$$

This follows from the fact that the series $\sum_{n=1}^{\infty} P_n x_n$ converges weakly if and only if $\sup_N \|\sum_{n=1}^{N} P_n x_n\| < \infty$ (the problem is discussed in [15]), that is, if and only if $\sup_N \psi(\|P_1 x_1\|, \cdots, \|P_N x_N\|, 0, 0, \cdots) < \infty$.

However, the function ϕ involved in the definition of weak homogeneity does not generally lead to a two-sided estimate. For instance, in $L_1(0, 1)$ we can set

$$\phi(\{a_n\}) = \begin{cases} 0 & \text{if } \sum_n |a_n| < \infty, \\ \infty & \text{if } \sum_n |a_n| = \infty. \end{cases}$$

THEOREM 4. *A Banach space* X *is isomorphic to* c_0 *or* L_p $(1 < p < \infty)$ *on some finite measure space if and only if it is a weakly homogeneous cyclic space* $X = \mathfrak{M}(x_0)$ $(x_0 \in X)$ *relative to a* σ*-complete Boolean algebra* $\mathscr{B}$ *of projections.*

Proof. The necessity follows from [14, Theorem 4] and the preceding remark. In order to prove the sufficiency, let us first assume that X contains no subspace isomorphic to c_0. Then, by [15, Lemmas 1 and 3], a series $\sum_{n=1}^{\infty} P_n x_n / \|P_n x_n\|$ converges strongly provided it is weakly convergent. Consider now a basis of the type $\{P_n x / \|P_n x\|\}$ ($x \in X$), and write

$$w_k = \sum_{n=p_k+1}^{p_{k+1}} \lambda_n \frac{P_n x}{\|P_n x\|} \qquad (k = 1, 2, \cdots) ,$$

where the λ_n are scalars such that $\|w_k\| = 1$ ($k = 1, 2, \cdots$) and $\{p_k\}$ is an increasing sequence of positive integers. Notice that a series $\sum_k a_k w_k$ converges (strongly) if and only if $\phi(\{a_k\}) < \infty$, that is, if and only if $\sum_n a_n P_n x / \|P_n x\|$ does so. In the terminology of [4], this means that the basis $\{P_n x / \|P_n x\|\}$ is perfectly homogeneous; hence, by M. Zippin [18], the basis is equivalent to the natural basis of ℓ_p for some p $(1 \leq p < \infty)$. Since it is obvious (because all the bases having this form are equivalent to the natural basis of ℓ_p for the same p $(1 \leq p < \infty)$), we can complete the proof in this case by using Proposition 1.

If X contains subspaces isomorphic to c_0, then there exist an $x \in X$ and a partition $\{\sigma_n\}$ ($n = 1, 2, \cdots$) such that the basis $\{P(\sigma_n)x / \|P(\sigma_n)x\|\}$ is equivalent to the natural basis of c_0 and the subspace $\operatorname{clm}\{P(\sigma_n)x \mid n = 1, 2, \cdots\}$ is complemented in X (see for example the remark at the end of [15]). Let Z be the closure of clm $\{P(\sigma_n)x/\|P(\sigma_n)x\|\}$ in X^{**} in the $\sigma(X^{**}, X^*)$-topology; one can easily see that Z is isomorphic to ℓ_∞, since it is in fact the set of all weakly convergent series of the type $\sum_{n=1}^{\infty} a_n P(\sigma_n)x/\|P(\sigma_n)x\|$.

In view of our hypotheses, a series having the form $\sum_n a_n P(\delta_n)y/\|P(\delta_n)y\|$ $(y \in X)$ converges in X^{**} in the $\sigma(X^{**}, X^*)$-topology if and only if

$$\sum_n a_n P(\sigma_n)x/\|P(\sigma_n)x\|$$

does so. The positivity of the mapping

$$\tau: \sum_n a_n \frac{P(\sigma_n)x}{\|P(\sigma_n)x\|} \to \sum_n a_n \frac{P(\delta_n)y}{\|P(\delta_n)y\|}$$

implies it is an isomorphism from the closure in X^{**} of clm $\{P(\delta_n)y \mid n = 1, 2, \cdots\}$ in the $\sigma(X^{**}, X^*)$-topology onto a space isomorphic to ℓ_∞. We can now complete the proof by using [12, Theorem 16 and Proposition 2].

2. APPLICATIONS TO BANACH LATTICES

The terminology used in this section will be that of W. A. J. Luxemburg and A. C. Zaanen [11]. Accordingly, a *normed Riesz space* is a lattice L_ρ endowed with a norm ρ satisfying the condition $\rho(u) \leq \rho(v)$ if $|u| < |v|$. The space L_ρ is called *σ-Dedekind-complete* if every order-bounded sequence has a least upper bound.

For the convenience of the reader, we reproduce here a result proved in [16].

LEMMA 5 [16, Lemma 15]. *Let L_ρ be a normed Riesz space that is σ-Dedekind-complete and satisfies the condition*

(*) $x_n \downarrow 0$ *implies* $\lim_{n \to \infty} \rho(x_n) = 0$ *for each decreasing sequence* $\{x_n\}$ $(x_n \in L_\rho;\ n = 1, 2, \cdots)$.

Then there exists a family $\mathcal{E}$ of projections of L_ρ such that $\mathcal{E}$ restricted to the invariant subspace $\mathfrak{M}(x) = \mathrm{clm}\{Ex \mid E \in \mathcal{E}\}$ $(x \in L_\rho)$ is a σ-complete Boolean algebra of projections and L_ρ can be decomposed into a direct sum (not necessarily countable) of cyclic spaces $\mathfrak{M}(x_\alpha) = \mathrm{clm}\{Ex_\alpha \mid E \in \mathcal{E}\}$ $(x_\alpha \in L_\rho)$, with the property $x_\alpha \wedge x_\beta = 0$ for $\alpha \neq \beta$.

Moreover, if W is a separable subspace of L_ρ, then x_0 can be chosen so that $W \subset \mathfrak{M}(x_0)$.

The following result is a version of Proposition 1 for nonseparable spaces.

PROPOSITION 6. *A σ-Dedekind-complete, normed Riesz space L_ρ is isomorphic to an L_p-space for some p $(1 \leq p < \infty)$ provided for each sequence of disjoint elements $x_n \in L_\rho$ $(x_n \neq 0;\ n = 1, 2, \cdots)$ the basis $\{x_n/\rho(x_n)\}$ is equivalent to the natural basis of ℓ_p.*

Proof. First notice that L_ρ must satisfy condition (*) of Lemma 5 (see [16, Theorem 20 and its proof]). Therefore, in view of Proposition 1, the cyclic subspaces $\mathfrak{M}(x_\alpha)$ defined by Lemma 5 for L_ρ are respectively isomorphic to L_p-spaces $L_p^{(\alpha)}$; and, moreover, there exist constants A_α and B_α such that

$$A_\alpha \rho(x) \leq \left(\sum_n \rho^p(E_n x)\right)^{1/p} \leq B_\alpha \rho(x) \qquad (x \in \mathfrak{M}(x_\alpha)),$$

for each sequence of disjoint projections $E_n \in \mathcal{E}$ $(n = 1, 2, \cdots)$ satisfying the condition $\left(\sum_n E_n\right) x = x$ ($\mathcal{E}$ is defined by Lemma 5). By the last part of Lemma 5, one can choose constants A and B, independent of α, such that

$$A\rho(x) \leq \left(\sum_n \rho^p(E_n x)\right)^{1/p} \leq B\rho(x) \qquad (x \in L_\rho)$$

for each sequence of disjoint projections $E_n \in \mathcal{E}$ for which $\left(\sum_n E_n\right) x = x$. Thus L_ρ is isomorphic to the direct sum in the ℓ_p-sense of the spaces $L_p(\alpha)$, that is, L_ρ itself is isomorphic to an L_p-space.

We can prove a version of Proposition 2 for nonseparable spaces in a similar manner, provided we assume L_ρ satisfies condition (*). Without this condition, the following result is not correct.

PROPOSITION 7. *A σ-Dedekind-complete, normed Riesz space L_ρ satisfying condition* (*) (*of Lemma* 5) *is isomorphic to* $c_0(\Gamma)$ (*defined in* [14]) *for some abstract set* Γ *provided for each sequence of disjoint elements* $x_n \in L_\rho$ $(x_n \neq 0;\ n = 1, 2, \cdots)$ *the basis* $\{x_n/\rho(x_n)\}$ *is equivalent to the natural basis of* c_0.

The spaces $c_0(\Gamma)$ and $L_p\left(\Omega, \sum, \mu\right)$ $(1 \leq p < \infty)$ are σ-Dedekind-complete, normed Riesz spaces with remarkable characteristic properties. For instance, it is well known that they are the only Banach lattices (up to an isometry) in which the condition

$$x_1 \wedge x_2 = y_1 \wedge y_2 = 0 \qquad (\rho(x_i) = \rho(y_i),\ i = 1, 2)$$

implies that $\rho(x_1 + x_2) = \rho(y_1 + y_2)$ (see F. Bohnenblust [5]). The isomorphic version of this result is stated in Theorem 4 (see also [14, Theorem 4]) for cyclic spaces, and it can easily be restated in normed Riesz spaces.

Using Bohnenblust's theorem, T. Ando [1] has recently shown that $c_0(\Gamma)$ and L_p $(1 \leq p < \infty)$ are the only Banach lattices (again up to an isometry) in which every sublattice is the range of a *positive contractive* projection. In the remainder of this paper, we shall present an isomorphic version of Ando's theorem by dropping the condition imposed on the norms of the projections.

THEOREM 8. *A σ-Dedekind-complete, normed Riesz space L_ρ is isomorphic either to* $c_0(\Gamma)$ *for some abstract set* Γ, *or to* L_p $(1 \leq p < \infty)$ *on some measure space, provided every closed sublattice of* L_ρ *is the range of a positive projection.*

Proof. Let $\{u_n\}$ and $\{v_n\}$ be two sequences of positive elements in L_ρ for which the conditions

(a) $\rho(u_n) = \rho(v_n) = 1$ $\qquad (n = 1, 2, \cdots)$,

(b) $u_n \wedge u_m = v_n \wedge v_m = 0$ for $n \neq m$,

(c) $u_n \wedge v_m = 0$ for all n and m

hold.

The first step will be to prove that $\{u_n\}$ and $\{v_n\}$ are equivalent unconditional bases. Evidently, it suffices to show that a series $\sum_{n=1}^{\infty} \alpha_n u_n$ is convergent if $\sum_{n=1}^{\infty} \alpha_n v_n$ converges. We shall do this by considering two different cases.

Case I. The subspace $\operatorname{clm} \{u_n\}$ is weakly sequentially complete. Suppose there exists a convergent series $\sum_{n=1}^{\infty} \beta_n v_n$ such that $\sum_{n=1}^{\infty} \beta_n u_n$ diverges. By the properties of the norm, we can assume without loss of generality that $\beta_n \geq 0$ $(n = 1, 2, \cdots)$. If the series $\sum_{n=1}^{\infty} \eta_n \beta_n u_n$ converges for each sequence $\{\eta_n\} \in c_0$, then by C. Bessaga and A. Pełczyński [4, Lemma 2 and Theorem 5] the series $\sum_{n=1}^{\infty} \beta_n u_n$ is weakly unconditionally convergent and $\operatorname{clm}_n \{u_n\}$ contains a subspace isomorphic to c_0; this contradicts our assumption of weak completeness. Consequently, we can assume the existence of a sequence $\{\eta_n\} \in c_0$ such that $\sum_{n=1}^{\infty} \eta_n \beta_n u_n$ is still divergent. Consider now the positive projection P whose range is the closed sublattice generated by $|\eta_n| u_n + v_n$ $(n = 1, 2, \cdots)$. Since P is positive and

$$\rho(|\eta_n| u_n + v_n) \geq \rho(v_n) = 1 \qquad (n = 1, 2, \cdots),$$

there exist numbers c_n and d_n $(0 \leq c_n \leq \|P\|$ and $0 \leq d_n \leq \|P\|$ for $n = 1, 2, \cdots)$ such that

$$Pu_n = c_n(|\eta_n| u_n + v_n) \quad \text{and} \quad Pv_n = d_n(|\eta_n| u_n + v_n) \qquad (n = 1, 2, \cdots).$$

Obviously, the condition $c_n |\eta_n| + d_n = 1$ $(n = 1, 2, \cdots)$ implies that $\lim_{n \to \infty} d_n = 1$. Hence, the convergence of the series $\sum_{n=1}^{\infty} \beta_n v_n$ implies the convergence of $\sum_{n=1}^{\infty} |\eta_n| \beta_n u_n$ (apply again P on $\sum_{n=1}^{\infty} \beta_n v_n$ and take into account that $\inf_n d_n > 0$). This contradiction proves our assertion in the first case.

Case II. The subspace $\operatorname{clm}_n \{u_n\}$ is not weakly sequentially complete. Since $\{u_n\}$ is an unconditional basic sequence, some sequence of positive disjoint elements w_j $(w_j \in \operatorname{clm}_n \{u_n\},\ j = 1, 2, \cdots)$ is equivalent to the natural basis of c_0 (see R. C. James [6, Lemma 1 and the proof thereafter]). We shall prove that $\{v_n\}$ is equivalent to the usual basis of c_0. Indeed, let $\sum_{n=1}^{\infty} \kappa_n v_n$ be a divergent series for which $\lim_{n \to \infty} \kappa_n = 0$, let P be the positive projection whose range is the closed sublattice generated by $|\kappa_n|^{1/2} v_n + w_n$ $(n = 1, 2, \cdots)$. Repeating arguments already used in this proof, we can show that the convergence of the series $\sum_{n=1}^{\infty} |\kappa_n|^{1/2} w_n$ (recall that $\{|\kappa_n|^{1/2}\} \in c_0$) implies the convergence of $\sum_{n=1}^{\infty} |\kappa_n| v_n$, which is contradictory. Now, replacing $\{v_n\}$ by $\{u_n\}$, we can prove that $\{u_n\}$ has the same property; hence $\{u_n\}$ and $\{v_n\}$ are both equivalent to the usual basis of c_0. This proves completely the Case II.

In order to finish the proof of the theorem, we consider a sequence $\{x_n\}$ of positive normalized disjoint elements. Set

$$y_k = \sum_{n=p_k+1}^{p_{k+1}} x_{2n} \quad (k = 1, 2, \cdots),$$

where $\{p_k\}$ is an increasing sequence of integers. By the previous part of the proof used for $\{x_{2k-1}\}$ and $\{y_k/\rho(y_k)\}$ (respectively, $\{x_{2k-1}\}$ and $\{x_{2k}\}$), we conclude that $\{x_{2k}\}$ and $\{y_k/\rho(y_k)\}$ are equivalent bases for any choice of the sequence $\{p_k\}$. Thus, by a result of M. Zippin [18] (in the formulation found in J. Lindenstrauss and M. Zippin [10, Lemma 2]), it follows that $\{x_{2n}\}$ is equivalent to the usual basis of c_0 or ℓ_p for some p $(1 \leq p < \infty)$. Since any two sequences of disjoint elements of L_ρ can always be imbedded in a sublattice generated by a third sequence of *positive* disjoint elements, we can apply either Proposition 6 or 7, thus proving the theorem completely. In the case when Proposition 7 is used, one should notice that condition (*) of Lemma 5 is satisfied (see T. Ando [1, Theorem 1] or [16, Theorem 20]).

Remark. In [17], we have used arguments similar to those in the proof of the preceding theorem to obtain an isomorphic characterization of L_1-spaces in terms of conditional expectations.

REFERENCES

1. T. Ando, *Banachverbände und positive Projektionen.* Math. Z. 109 (1969), 121-130.

2. W. G. Bade, *On Boolean algebras of projections and algebras of operators.* Trans. Amer. Math. Soc. 80 (1955), 345-360.

3. ———, *A multiplicity theory for Boolean algebras of projections in Banach spaces.* Trans. Amer. Math. Soc. 92 (1959), 508-530.

4. C. Bessaga and A. Pełczyński, *On bases and unconditional convergence of series in Banach spaces.* Studia Math. 17 (1958), 151-164.

5. H. F. Bohnenblust, *An axiomatic characterization of* L_p*-spaces.* Duke Math. J. 6 (1940), 627-640.

6. R. C. James, *Bases and reflexivity of Banach spaces.* Ann. of Math. (2) 52 (1950), 518-527.

7. J. Lindenstrauss and A. Pełczyński, *Absolutely summing operators in* $\mathscr{L}_p$*-spaces and their applications.* Studia Math. 29 (1968), 275-326.

8. J. Lindenstrauss and H. P. Rosenthal, *The* $\mathscr{L}_p$ *spaces.* Israel J. Math. 7 (1969), 325-349.

9. J. Lindenstrauss and M. Zippin, *Banach spaces with a unique unconditional basis.* J. Functional Analysis 3 (1969), 115-125.

10. ———, *Banach spaces with sufficiently many Boolean algebras of projections.* J. Math. Anal. Appl. 25 (1969), 309-320.

11. W. A. J. Luxemburg and A. C. Zaanen, *Notes on Banach function spaces.* I. Nederl. Akad. Wetensch. Proc. Ser. A 66 (1963), 135-147.

12. C. A. McCarthy and L. Tzafriri, *Projections in* $\mathscr{L}_1$ *and* $\mathscr{L}_\infty$*-spaces.* Pacific J. Math. 26 (1968), 529-546.

13. A. Pełczyński, *On Banach spaces containing* $L_1(\mu)$. Studia Math. 30 (1968), 231-246.

14. L. Tzafriri, *An isomorphic characterization of* L_p *and* c_0*-spaces.* Studia Math. 32 (1969), 295-304.

15. ———, *Reflexivity of cyclic Banach spaces.* Proc. Amer. Math. Soc. 22 (1969), 61-68.

16. ———, *Reflexivity in Banach lattices and their subspaces.* J. Functional Analysis (to appear).

17. ———, *Conditional expectations and an isomorphic characterization of* L_1*-spaces.* Proc. Amer. Math. Soc. (to appear).

18. M. Zippin, *On perfectly homogeneous bases in Banach spaces.* Israel J. Math. 4 (1966), 265-272.

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