INEQUALITIES FOR CONDENSERS, HYPERBOLIC CAPACITY, AND EXTREMAL LENGTHS

F. W. Gehring

1. INTRODUCTION

In Section 3 of this paper, we give a pair of elementary estimates for the p-capacity of a condenser in Euclidean n-space, taken with respect to an arbitrary metric g. Various choices for g yield a number of useful bounds for the conformal or n-capacity (Sections 4 and 5). We use two of these bounds to derive a distortion theorem for plane quasiconformal mappings (Section 6) and to obtain sharp bounds for the hyperbolic capacity of a plane set (Section 7). In Sections 8 and 9, we employ two other bounds to study the relation between the moduli of the two families of Jordan curves that link the interior and exterior, respectively, of a torus in 3-space.

2. NOTATION

We consider sets in Euclidean n-space R^n $(n \geq 2)$ and in its one-point compactification $\overline{R}^n$ obtained by adding the point ∞ to R^n. Points in R^n are treated as vectors, and for each $x \in R^n$ we let $|x|$ denote the norm of x. For each set $E \subset \overline{R}^n$, we let ∂E, $\overline{E}$, and C(E) denote the boundary, closure, and complement of E in $\overline{R}^n$, while for $E \subset R^n$ and $k \in (0, \infty)$, we let $m_k(E)$ denote the k-dimensional Hausdorff measure of E. In particular, m_n will denote Lebesgue measure in R^n.

A *condenser* R is a domain in R^n whose complement consists of two distinguished disjoint closed sets C_0 and C_1. R is a *ring* if, in addition, C_0 and C_1 are connected. For convenience of notation, we shall always assume that $\infty \in C_1$.

Suppose that g is a function that is positive and continuous in a condenser R. Then, for $p \in (1, \infty)$, we define the *p-capacity of* R *with respect to* g as

$$(1) \qquad \operatorname{cap}_p(R, g) = \inf_u \int_R |\operatorname{grad} u|^p g^{n-p} \, dm_n,$$

where the infimum is taken over all functions u that are continuous in $\overline{R}^n$ and ACT (absolutely continuous in the sense of Tonelli) in R^n, with $u = 0$ in C_0 and $u = 1$ in C_1. We call any such function u an *admissible function* for R. The usual p-capacity of R [27] is then simply the p-capacity of R with respect to the function $g = 1$, that is,

$$(2) \qquad \operatorname{cap}_p(R) = \operatorname{cap}_p(R, 1),$$

while for the conformal or n-capacity of R [17] we have the relation

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(3) $$\operatorname{cap}_n(R) = \operatorname{cap}_n(R, g)$$

for all functions g. The *conformal modulus* of R is given by the formula

(4) $$\operatorname{mod}_n(R) = \left(\frac{\omega_n}{\operatorname{cap}_n(R)}\right)^{1/(n-1)},$$

where $\omega_n = m_{n-1}(\partial B^n)$ and B^n is the unit ball $\{x: |x| < 1\}$ in R^n.

An admissible function u for a condenser R is said to be a *simple admissible function* if $0 \leq u \leq 1$ in R and if the set where $0 \leq u < 1$ is contained in the finite union of closed n-simplices, in each of which u is linear. Arguing as in [6] or [18], we see without difficulty that for each g and p,

(5) $$\operatorname{cap}_p(R, g) = \inf_u \int_R |\operatorname{grad} u|^p g^{n-p} dm_n,$$

where the infimum is taken over the subfamily of all simple admissible functions u for R.

Suppose next that g is positive and continuous in an open set $G \subset R^n$ and that ϕ is positive and continuous in the interval $(0, c)$, where

$$c = \int_G g^n dm_n, \qquad c \in (0, \infty].$$

We say that g is *ϕ-isoperimetric* in G if

(6) $$\int_{\partial F} g^{n-1} dm_{n-1} \geq \phi\left(\int_F g^n dm_n\right)$$

for each closed polyhedron (finite union of closed simplices) $F \subset G$. For example, the function $g = 1$ is ϕ-isoperimetric in R^n with

(7) $$\phi(t) = \omega_n^{1/n}(nt)^{(n-1)/n} \qquad (t \in (0, \infty)).$$

Another example: if $n = 2$ and g is the hyperbolic metric

$$g(x) = (1 - |x|^2)^{-1}$$

in B^2, then g is ϕ-isoperimetric in B^2 with

(8) $$\phi(t) = 2(t(\pi + t))^{1/2} \qquad (t \in (0, \infty)).$$

Also, if $n = 2$ and g is the spherical metric

$$g(x) = (1 + |x|^2)^{-1}$$

in R^2, then g is ϕ-isoperimetric in R^2, where

(9) $$\phi(t) = 2(t(\pi - t))^{1/2} \qquad (t \in (0, \pi)).$$

Finally, suppose that Γ is a family of continua in $\overline{R}^n$. We say that h is an *admissible density* for Γ if h is nonnegative and Borel measurable in R^n, with

$$\int_{\gamma - \{\infty\}} h\, dm_1 \geq 1$$

for each $\gamma \in \Gamma$, and we define the *conformal* or n-*modulus* of Γ as

$$M_n(\Gamma) = \inf_h \int_{R^n} h^n\, dm_n, \tag{10}$$

where the infimum is taken over all admissible densities h for Γ [25].

One can show that if R is a condenser in R^n, then

$$M_n(\Gamma) = \mathrm{cap}_n(R),$$

where Γ is the family of all continua in R joining C_0 to C_1 (see [7] and [26]).

3. BOUNDS FOR THE p-CAPACITY

Suppose that R is a condenser in R^n with $C(R) = C_0 \cup C_1$, that g is positive and continuous in R, and that $p \in (1, \infty)$. We derive here upper and lower bounds for $\mathrm{cap}_p(R, g)$.

For the lower bound, suppose that g can be extended to C_0 so that, for some function ϕ, g is ϕ-isoperimetric in $R \cup C_0$, and set

$$A_0 = \int_{C_0} g^n\, dm_n, \qquad A_1 = \int_{R \cup C_0} g^n\, dm_n.$$

We then have the following extension of a well-known inequality due to T. Carleman [4].

THEOREM 1.

$$\mathrm{cap}_p(R, g) \geq \left(\int_{A_0}^{A_1} \phi(t)^{p/(1-p)}\, dt \right)^{1-p}. \tag{11}$$

Proof. From (5), we see that it is sufficient to prove that

$$\int_R |\mathrm{grad}\, u|^p g^{n-p}\, dm_n \geq \left(\int_{A_0}^{A_1} \phi(t)^{p/(1-p)}\, dt \right)^{1-p} \tag{12}$$

for each simple admissible function u for R.

Choose such a function u; for $t \in [0, 1)$, let

$$F(t) = \{x: u(x) \leq t\}, \qquad F(1) = \{x: u(x) < 1\},$$

and for $t \in [0, 1]$, let

$$A(t) = \int_{F(t)} g^n dm_n .$$

Then A is increasing in $[0, 1]$, and $A_0 \leq A(0) < A(1) \leq A_1$. Choose $t \in (0, 1)$ so that $m_n(u^{-1}(t)) = 0$. Then, since $A < \infty$ in $[0, 1)$, Lebesgue's dominated-convergence theorem implies that

$$A(t) = \lim_{s \to t} A(s),$$

and because u is a simple admissible function, we conclude that A is continuous at all but a finite set of points in $(0, 1)$.

Suppose that $[a, b]$ is a subinterval of $(0, 1)$ in which A is continuous. We shall show that

$$(13)\quad b - a \leq \left(\int_{F(b)-F(a)} |\operatorname{grad} u|^p g^{n-p} dm_n \right)^{1/p} \left(\int_{A(a)}^{A(b)} \phi(t)^{p/(1-p)} dt \right)^{(p-1)/p} .$$

For this, let $a = t_0 < t_1 < \cdots < t_m = b$ be any subdivision of $[a, b]$, let m_k denote the minimum of $\phi \circ A$ in $[t_{k-1}, t_k]$, and set

$$A_k = A(t_k), \qquad E_k = F(t_k) - F(t_{k-1}) .$$

For $t \in (0, 1)$, $F(t)$ is a closed polyhedron in $R \cup C_0$, and $\partial F(t) \subset u^{-1}(t)$. Hence (6) implies that

$$\phi \circ A(t) \leq \int_{u^{-1}(t)} g^{n-1} dm_{n-1}$$

for $t \in [a, b]$, and with help of the co-area formula (see Theorem 3.1 in [5]) and Hölder's inequality, we obtain the estimates

$$(14)\quad \begin{aligned} m_k(t_k - t_{k-1}) &\leq \int_{t_{k-1}}^{t_k} \left(\int_{u^{-1}(t)} g^{n-1} dm_{n-1} \right) dt = \int_{E_k} |\operatorname{grad} u|\, g^{n-1} dm_n \\ &\leq \left(\int_{E_k} |\operatorname{grad} u|^p g^{n-p} dm_n \right)^{1/p} (A_k - A_{k-1})^{(p-1)/p} . \end{aligned}$$

Next, let M_k denote the maximum of $\phi^{p/(1-p)}$ in $[A_{k-1}, A_k]$. Then (14), together with a second application of Hölder's inequality, yields the inequality

$$b - a \leq \left(\int_{F(b)-F(a)} |\operatorname{grad} u|^p g^{n-p} dm_n \right)^{1/p} \left(\sum_1^m M_k(A_k - A_{k-1}) \right)^{(p-1)/p} ;$$

together with the continuity of ϕ and A, this implies (13).

Now, given $d \in (0, 1)$, we can choose disjoint subintervals $[a_k, b_k]$ of $(0, 1)$ such that A is continuous in each subinterval and

$$d \leq \sum_{1}^{m} (b_k - a_k). \tag{15}$$

From (13), (15), and Hölder's inequality we obtain the estimate

$$\int_R |\text{grad } u|^p g^{n-p} dm_n \geq d^p \left(\sum_{1}^{m} \int_{A(a_k)}^{A(b_k)} \phi(t)^{p/(1-p)} dt \right)^{1-p}$$

$$\geq d^p \left(\int_{A_0}^{A_1} \phi(t)^{p/(1-p)} dt \right)^{1-p};$$

if we let $d \to 1$, this yields (12), and the proof of Theorem 1 is complete.

For the upper bound, suppose for convenience that

$$\limsup_{x \to C_0} g(x) < \infty.$$

Then each $x \in R$ can be joined to C_0 by a polygonal arc $\gamma \subset R$ on which g is bounded, and we set

$$v(x) = \inf_{\gamma} \int_{\gamma} g \, dm_1,$$

where the infimum is taken over all such γ. It is easy to verify that v is Lipschitzian, with $|\text{grad } v| \leq g$ a.e. in R, and that $v(x) \to 0$ as $x \to C_0$. Let

$$T = \liminf_{x \to C_1} v(x) \geq 0,$$

and for $t \in (0, \infty)$, set

$$L(t) = \int_{v^{-1}(t)} g^{n-1} dm_{n-1}.$$

We then have the following extension of a result due to G. Pólya and G. Szegö [20].

THEOREM 2.

$$\text{cap}_p(R, g) \leq \left(\int_0^T L(t)^{1/(1-p)} dt \right)^{1-p}. \tag{16}$$

Proof. We may assume that $T > 0$ and that $\text{cap}_p(R, g) > 0$, for otherwise, (16) follows trivially. Next, for $t \in (0, T)$, let

$$R(t) = \{x \in R: v(x) < t\}, \qquad C_1(t) = C(R(t) \cup C_0),$$

and let

$$f(t) = \inf_u \int_{R(t)} |\text{grad } u|^p g^{n-p} dm_n,$$

where the infimum is taken over all admissible functions u that are continuous in $\overline{R}^n$ and ACT in R^n, with $u = 0$ in C_0 and $u = 1$ in $C_1(t)$. Then f is nonincreasing, and

(17) $$0 < \operatorname{cap}_p(R, g) \le f(t) < \infty$$

in $(0, T)$. Hence, in order to establish (16), we need only show that for each subinterval $[a, b]$ of $(0, T)$,

(18) $$f(b)^{1/(1-p)} \ge \int_a^b L(t)^{1/(1-p)} dt .$$

For (17) and (18) imply that

$$\operatorname{cap}_p(R, g) \le \left(\int_a^b L(t)^{1/(1-p)} dt \right)^{1-p},$$

and letting $a \to 0$ and $b \to T$, we obtain (16).

Choose $t_1, t_2 \in (0, T)$ with $t_1 < t_2$, let u_1 be an admissible function for $R(t_1)$, and set $u_2 = cu_1 + (1 - c)w$, where $c \in [0, 1]$ and

$$w(x) = \begin{cases} 0 & \text{if } x \in R(t_1) \cup C_0, \\ \dfrac{v(x) - t_1}{t_2 - t_1} & \text{if } x \in R(t_2) - R(t_1), \\ 1 & \text{if } x \in C_1(t_2). \end{cases}$$

Then u_2 is admissible for $R(t_2)$; also,

$$f(t_2) \le c^p \int_{R(t_1)} |\operatorname{grad} u_1|^p g^{n-p} dm_n + (1 - c)^p I,$$

where

$$I = (t_2 - t_1)^{-p} \int_{R(t_2)-R(t_1)} |\operatorname{grad} v|^p g^{n-p} dm_n ,$$

and taking the infimum over all such u_1, we find that

(19) $$f(t_2) \le c^p f(t_1) + (1 - c)^p I .$$

Since $I \in (0, \infty)$, we can choose c so that

$$c(f(t_1)^{1/(1-p)} + I^{1/(1-p)}) = f(t_1)^{1/(1-p)},$$

and with the help of (19) we obtain the inequality

(20) $$f(t_2)^{1/(1-p)} - f(t_1)^{1/(1-p)} \ge I^{1/(1-p)} .$$

Also, since $|\operatorname{grad} v| \le g$ a.e. in R, the co-area formula implies that

(21) $$I \leq (t_2 - t_1)^{-p} \int_{t_1}^{t_2} L(t)\,dt.$$

Now the function $f^{1/(1-p)}$ is nondecreasing in $(0, T)$. Moreover, since L is integrable over each closed subinterval of $(0, T)$, (20) and (21) imply that

$$\frac{d}{dt}(f(t)^{1/(1-p)}) \geq L(t)^{1/(1-p)} \quad \text{a.e. in } (0, T),$$

and thus that

$$f(b)^{1/(1-p)} > f(b)^{1/(1-p)} - f(a)^{1/(1-p)} \geq \int_a^b L(t)^{1/(1-p)}\,dt$$

for each subinterval $[a, b]$ of $(0, T)$. This completes the proof of (18), and hence of Theorem 2.

4. APPLICATIONS OF THEOREM 1

We give here several applications of Theorem 1 by making special choices for n, g, and p.

For the first application, suppose that R is a condenser in R^n with

$$m_n(C_0) = \tau_n a^n, \qquad m_n(R \cup C_0) = \tau_n b^n,$$

where $\tau_n = m_n(B^n)$.

COROLLARY 1.

(22) $$\text{cap}_p(R) \geq \omega_n\left(\int_a^b r^{q-1}\,dr\right)^{1-p} \quad \left(q = \frac{p-n}{p-1}\right).$$

Proof. Let $g = 1$ in R^n. Then g is ϕ-isoperimetric in $R \cup C_0$, where ϕ is the function given in (7), and (22) follows directly from (11).

If R^* is a ring in R^n, bounded by concentric spheres of radii a and b $(a < b)$, then

$$\text{cap}_p(R^*) = \omega_n\left(\int_a^b r^{q-1}\,dr\right)^{1-p} \quad \left(q = \frac{p-n}{p-1}\right).$$

Hence Corollary 1 simply says that

(23) $$\text{cap}_p(R) \geq \text{cap}_p(R^*),$$

where R^* is a spherical ring with $m_n(C_0^*) = m_n(C_0)$ and $m_n(R^* \cup C_0^*) = m_n(R \cup C_0)$. (See Lemma 2 in [8] or (9) in [10].)

For the second application, suppose that R is a condenser in the unit disk B^2 and that A_0 and A_1 denote the hyperbolic areas of C_0 and $R \cup C_0$.

COROLLARY 2.

$$(24)\qquad \operatorname{mod}_2(R) \le \begin{cases} \frac{1}{2}\log\left(\frac{A_0+\pi}{A_0}\frac{A_1}{A_1+\pi}\right) & \text{if } A_1 < \infty, \\ \frac{1}{2}\log\left(\frac{A_0+\pi}{A_0}\right) & \text{if } A_1 = \infty. \end{cases}$$

Proof. Let $g(x) = (1 - |x|^2)^{-1}$ in B^2. Then g is ϕ-isoperimetric in $R \cup C_0$, where ϕ is as given in (8), and we obtain (24) by substituting this function in (11).

For the third application, suppose that R is a condenser in R^2 and that A_0 and A_1 denote the spherical areas of C_0 and $R \cup C_0$.

COROLLARY 3.

$$(25)\qquad \operatorname{mod}_2(R) \le \frac{1}{2}\log\left(\frac{\pi - A_0}{A_0}\frac{A_1}{\pi - A_1}\right).$$

Proof. Set $g(x) = (1 + |x|^2)^{-1}$, and substitute the function given in (9) into (11).

Corollary 3 obviously holds for condensers $R \subset \overline{R}^2$, if we extend the definition of $\operatorname{cap}_2(R)$ by means of a Möbius transformation.

Inequalities (24) and (25) hold with equality when R is bounded by concentric circles in the hyperbolic metric of B^2 and in the spherical metric of R^2, respectively. Hence the conclusions of Corollaries 2 and 3 can be expressed by means of a single inequality like (23), where R^* is an equivalent annulus in the appropriate metric (see [14] and [16]).

Finally, for the fourth application, suppose that H is the upper half-plane $\{x: x_2 > 0\}$ in R^2, that R is a condenser in R^2 with $C_0 \subset H$ and $C_1 = C(H)$, that g is the hyperbolic metric in H,

$$g(x) = \frac{1}{2x_2},$$

and that A denotes the hyperbolic area of C_0.

COROLLARY 4.

$$(26)\qquad \operatorname{cap}_{3/2}(R, g) \ge 2\pi\left(\left(\frac{A+\pi}{A}\right)^{1/4} - \left(\frac{A}{A+\pi}\right)^{1/4}\right)^{-1}.$$

Proof. The function g is ϕ-isoperimetric in H, with ϕ as given in (8), and we obtain (26) from (11).

5. APPLICATIONS OF THEOREM 2

We consider next some applications of Theorem 2.

First, for $t \in (0, 1)$, let $R_G(t)$ denote the ring $R \subset R^n$ with $C_1 = C(B^n)$ and C_0 the closed segment joining the point $(t, 0, \cdots, 0)$ to the origin. Then the function

$$r_n(t) = \operatorname{mod}_n(R_G(t)) + \log t$$

is positive and nonincreasing in (0, 1) with

$$\log \lambda_n = \lim_{t\to 0} r_n(t) < \infty$$

(see [3], [6], and [18]). Unfortunately the values of the constants λ_n are unknown except when $n = 2$, in which case $\lambda_2 = 4$. We have, however, the following bounds for λ_3.

COROLLARY 5.

$$8 \le \lambda_3 \le 8e^{\left(\frac{\pi}{4}-\frac{1}{2}\log 2\right)} = 12.4 \cdots . \tag{27}$$

Proof. The upper bound in (27) was derived in [6]. For the lower bound, choose $t \in (0, 1)$ and let $g = 1$ in B^3. Then, in the notation of Theorem 2, $T = 1 - t$,

$$L(s) = 4\pi s^2 + 2\pi st$$

for $s \in (0, T)$, and from (16) we obtain the inequality

$$\mathrm{mod}_3(R_G(t)) \ge \int_0^{1-t} \left(s^2 + \frac{st}{2}\right)^{-1/2} ds .$$

An elementary calculation shows that $\lambda_3 \ge 8$.

Theorem 2 can be used in the same way to get lower bounds for λ_n for $n > 3$. Upper bounds have already been found in [3].

For the second application, suppose that R is a condenser in R^2 with $C_0 \subset B^2$ and $C_1 = C(B^2)$, and that P denotes the infimum of the hyperbolic lengths of the Jordan curves in B^2 that separate C_0 and C_1. We call P the *hyperbolic perimeter* of C_0 in B^2.

COROLLARY 6.

$$\mathrm{mod}_2(R) \ge \log \frac{(P^2+\pi^2)^{1/2}+\pi}{P} . \tag{28}$$

Proof. It is sufficient to show that for each $P' \in (P, \infty)$, (28) holds with P replaced by P'.

Now, if $P' \in (P, \infty)$, there exists a Jordan curve $\gamma \subset B^2$ that separates C_0 and C_1, and whose hyperbolic length does not exceed P'. Let D_0 denote the component of $C(\gamma)$ that contains C_0, and C_0' the closed hyperbolically convex hull of D_0. That is, let C_0' denote the intersection of all closed sets $F \subset B^2$ that are hyperbolically convex and contain D_0. Then $\gamma' = \partial C_0' \subset B^2$ is a Jordan curve that separates C_0 and C_1 and whose hyperbolic length does not exceed P'. Next, by joining a suitably chosen set of consecutive points on γ' by hyperbolic segments, we can find a hyperbolically convex polygon $\gamma'' \subset B^2$ that separates C_0 and C_1 and whose hyperbolic length again does not exceed P'. Let D_0'' denote the component of $C(\gamma'')$ that contains C_0, and R'' the ring bounded by $C_0'' = \overline{D_0''}$ and $C_1'' = C_1$. Then

$$\mathrm{mod}_2(R) \ge \mathrm{mod}_2(R'') , \tag{29}$$

and since the right-hand side of (28) is decreasing in P, we conclude from (29) and the remark above that it is sufficient to prove (28) in the special case where ∂C_0 is a hyperbolically convex polygon with hyperbolic length P.

For this case, set $g(x) = (1 - |x|^2)^{-1}$ in B^2. Then $T = \infty$, and it is not difficult to verify, either directly or with the help of the Gauss-Bonnet formula (see [2]), that

$$L(t) = P \cosh 2t + (\pi + 2A) \sinh 2t \qquad \text{for } t \in (0, \infty), \tag{30}$$

where A is the hyperbolic area of C_0. The isoperimetric inequality applied to C_0 implies that

$$P^2 \geq 4A(A + \pi), \tag{31}$$

and from (30) and (31) it follows that

$$\frac{dL(t)}{dt} \leq 2(L(t)^2 + \pi^2)^{1/2} \tag{32}$$

in $(0, \infty)$. If we now combine (16) and (32), we obtain the desired inequalities

$$\operatorname{mod}_2(R) \geq 2\pi \int_0^\infty \frac{dt}{L(t)} \geq \pi \int_P^\infty t^{-1}(t^2 + \pi^2)^{-1/2} dt = \log \frac{(P^2 + \pi^2)^{1/2} + \pi}{P}.$$

Inequality (28) holds with equality when C_0 is a closed disk in B^2.

Finally, for the third application, let H denote the upper half-plane in R^2, R a condenser in R^2 with $C_0 \subset H$ and $C_1 = C(H)$, g the hyperbolic metric in H, and P the hyperbolic perimeter of C_0 in H.

COROLLARY 7.

$$\operatorname{cap}_{3/2}(R, g) \leq \pi \left(\frac{(P^2 + \pi^2)^{1/2} - P}{2P} \right)^{-1/2}. \tag{33}$$

Proof. As in Corollary 6, it is sufficient to consider the case where ∂C_0 is a hyperbolically convex polygon with hyperbolic length P. Then $T = \infty$, $L(t)$ is given by (30), and (33) follows from (16) and (32) as above.

It is easy to verify that the right-hand sides of inequalities (26) and (33) are equal whenever (31) holds with equality. From this it follows that (26) and (33) both hold with equality when C_0 is a closed disk in H.

6. DISTORTION THEOREM

We apply here two of the previously derived bounds to obtain a distortion theorem for quasiconformal mappings.

For $K \in [1, \infty)$ and $t \in (0, \infty)$, let

$$\phi_K(t) = \frac{2t^K((t^2 + 1)^{1/2} + 1)^K}{((t^2 + 1)^{1/2} + 1)^{2K} - t^{2K}}, \qquad \psi_K(t) = \frac{t^{1/K}}{(t + 1)^{1/K} - t^{1/K}}. \tag{34}$$

Then ϕ_K and ψ_K are increasing in $(0, \infty)$,

$$\phi_K(t) \sim 2^{1-K} t^K \qquad \text{and} \qquad \psi_K(t) \sim t^{1/K}$$

as $t \to 0$, and

$$\phi_K(t) \sim \frac{t}{K} \quad \text{and} \quad \psi_K(t) \sim \frac{t}{K}$$

as $t \to \infty$.

Next, suppose that D is a simply connected domain of hyperbolic type in $\overline{R}^2$ and that f is a conformal mapping of D onto B^2. Then the function

$$g = \frac{|f'|}{1 - |f|^2}$$

is independent of the choice of f, and it defines the hyperbolic metric in D. Moreover, $E \subset D$ is a hyperbolic disk (circle) in D if and only if f(E) is a disk (circle) in B^2.

We have the following sharp bounds for the change in hyperbolic perimeter and area, under a quasiconformal mapping.

THEOREM 3. *Suppose that* f *is a* K-*quasiconformal mapping of* D *and that* E *is a closed hyperbolic disk in* D. *Then*

(35) $$\frac{P'}{\pi} \geq \phi_K\left(\frac{P}{\pi}\right), \quad \frac{A'}{\pi} \leq \psi_K\left(\frac{A}{\pi}\right),$$

where P *and* P' *denote the hyperbolic perimeters and* A *and* A' *the hyperbolic areas of* E *and* f(E) *in* D *and* f(D), *respectively.*

Proof. If we compose f with two appropriate auxiliary conformal mappings, we may assume without loss of generality that $D = f(D) = B^2$.

Let E be a closed disk in B^2, and let R be the ring in R^2 with $C_0 = E$ and $C_1 = C(B^2)$. Then

(36) $$\log \frac{(P^2 + \pi^2)^{1/2} + \pi}{P} = \mathrm{mod}_2(R) = \frac{1}{2} \log \frac{A + \pi}{A}$$

since E is a disk,

(37) $$\log \frac{(P'^2 + \pi^2)^{1/2} + \pi}{P'} \leq \mathrm{mod}_2(f(R)) \leq \frac{1}{2} \log \frac{A' + \pi}{A'}$$

by inequalities (24) and (28), and

(38) $$\frac{1}{K} \mathrm{mod}_2(R) \leq \mathrm{mod}_2(f(R)) \leq K \,\mathrm{mod}_2(R)$$

because f is K-quasiconformal. The inequalities in (35) then follow from(36), (37), and (38).

When E is a disk about the origin, the mapping $|x|^{K-1}x$ and its inverse show that the bounds in (35) are sharp. A preliminary Möbius transformation reduces the general situation to this special case.

Theorem 3 also yields a sharp bound for the change in the hyperbolic length of a hyperbolic circle $C \subset D$ under a K-quasiconformal mapping f of D. For if L and L' denote the hyperbolic lengths of C and f(C) in D and f(D), respectively, then $L = P$ and $L' \geq P'$, where P and P' are the hyperbolic perimeters of the sets bounded by C and f(C) in D and f(D), and with the help of (35) we obtain the inequality

$$\frac{L'}{\pi} \geq \phi_K\left(\frac{L}{\pi}\right).$$

We can combine the argument used in Theorem 3 with inequality (25) to relate the change in spherical area of two spherically concentric disks under a K-quasiconformal mapping. The case where $K = 1$ was considered in [12] (see also [14]).

7. HYPERBOLIC CAPACITY

Suppose that D is a simply connected domain of hyperbolic type in $\overline{R}^2$, and that f is a conformal mapping of D onto B^2. For each compact set $E \subset D$, the *hyperbolic capacity* of E in D is defined as

$$\text{caph}(E) = \lim_{n\to\infty}\left(\max_{z_1,\ldots,z_n \in E} \prod_{i\neq j}\left|\frac{f(z_i) - f(z_j)}{1 - \overline{f(z_i)}f(z_j)}\right|\right)^{1/(n(n-1))} \tag{39}$$

(see [23] or [24]). This quantity does not depend on the choice of the mapping f, and hence it is a conformal invariant. Moreover, if F is compact and $\partial E \subset F \subset E$, then (39) and the maximum principle imply that

$$\text{caph}(E) = \text{caph}(F). \tag{40}$$

Also, if $D = D(r)$ denotes the open disk of radius r about the origin, then for each compact set $E \subset R^2$,

$$\lim_{r\to\infty} r\,\text{caph}(E) = \text{cap}(E), \tag{41}$$

where cap(E) denotes the usual Euclidean capacity for E,

$$\text{cap}(E) = \lim_{n\to\infty}\left(\max_{z_1,\ldots,z_n \in E} \prod_{i\neq j}|z_i - z_j|\right)^{1/(n(n-1))}$$

(see [22]).

We give here sharp bounds for caph(E) in terms of the hyperbolic perimeter and hyperbolic area of E in D. (For earlier results in this direction, see [22].)

THEOREM 4. *For each compact set* $E \subset D$,

$$\left(\frac{A}{A+\pi}\right)^{1/2} \leq \text{caph}(E) \leq \frac{(P^2+\pi^2)^{1/2} - \pi}{P}, \tag{42}$$

where P *and* A *denote the hyperbolic perimeter and hyperbolic area of* E *in* D. *There is equality in both parts of* (42) *when* E *is a hyperbolic disk.*

Proof. By the conformal invariance of all quantities involved, it is sufficient to consider the special case where $D = B^2$.

Let R denote the component of $B^2 - E$ for which $\partial B^2 \subset \partial R$, and let $C_0 = B^2 - R$, $C_1 = C(B^2)$. Then R is a condenser in R^2 with $\partial C_0 \subset E \subset C_0$, and

$$\text{caph}(E) = \text{caph}(C_0), \tag{43}$$

by (40). It is also easy to see that P is equal to the hyperbolic perimeter of C_0 and that A does not exceed the hyperbolic area of C_0. Therefore the argument for Theorem 6 in [1] implies that

$$\mathrm{mod}_2(R) = \inf_{\mu} \int_{C_0} \int_{C_0} \log \left| \frac{1 - \bar{z}w}{z - w} \right| d\mu(z)\, d\mu(w),$$

where the infimum is taken over all nonnegative measures μ with support in C_0 and $\mu(C_0) = 1$, and from [23] or [24, pp. 94-96] we conclude that

(44) $$\mathrm{caph}(C_0) = e^{-\mathrm{mod}_2(R)}$$

(see also [13] and [15]). Finally, (42) follows if we combine (43) and (44) with inequalities (24) and (28).

When E is a closed disk, both (24) and (28) hold with equality. Hence the same is true of both parts of (42).

Suppose that E is a compact set in R^2, that P_0 is the infimum of the lengths of the Jordan curves in R^2 that separate E from ∞, and that A_0 is the area of E. We can then derive the usual estimates, [19] and [21], for the Euclidean capacity of E in terms of P_0 and A_0 from Theorem 4 as follows. For $r \in (0, \infty)$, let D denote the open disk of radius r about the origin, and for large r let P, A, and caph(E) denote the hyperbolic perimeter, hyperbolic area, and hyperbolic capacity of E in D. Then it is easy to verify that

(45) $$P_0 = \lim_{r \to \infty} rP, \qquad A_0 = \lim_{r \to \infty} r^2 A,$$

and combining (40), (42), and (45), we obtain the desired inequalities

$$\left(\frac{A_0}{\pi}\right)^{1/2} \leq \mathrm{cap}(E) \leq \frac{P_0}{2\pi}.$$

8. MODULI OF LINKING CURVE FAMILIES

A domain $D \subset R^3$ is said to be a *torus of revolution* if it is generated by revolving a plane Jordan domain E about a line L that lies in the plane of E and at positive distance from E. We say that D is a *circular torus* if, in addition, E is a disk. By means of the Schoenflies theorem, it is easy to show that for each pair of tori of revolution D_1 and D_2, there exists a homeomorphism f of $\overline{R}^3$ onto $\overline{R}^3$ with $f(D_1) = D_2$.

A domain $D \subset \overline{R}^3$ is said to be a *torus* if some homeomorphism f of $\overline{R}^3$ onto $\overline{R}^3$ maps D onto a torus of revolution. Since the exterior of each circular torus can be mapped conformally (by means of a Möbius transformation) onto a second circular torus, the exterior of each torus D is again a torus.

Given a torus $D \subset \overline{R}^3$, we denote by $\Gamma_i = \Gamma_i(D)$ and $\Gamma_e = \Gamma_e(D)$ the families of Jordan curves in D and $C(\overline{D})$ that are not homotopic to 0 in D and $C(\overline{D})$, respectively. We call these the *interior* and *exterior linking families* for D. Obviously

$$\Gamma_i(D) = \Gamma_e(C(\overline{D})), \qquad \Gamma_e(D) = \Gamma_i(C(\overline{D})).$$

The modulus of a family Γ of curves behaves like the electrical conductance of a system of homogeneous wires: it is large when the curves in Γ are plentiful or short, and small when the curves in Γ are few or long. Since each curve in Γ_i links each curve in Γ_e, it is natural to expect that the moduli of the families Γ_i and Γ_e vary inversely. That is, the modulus of one family is large only if the modulus of the other family is small. We have, in fact, the following result.

THEOREM 5. *There exists an extended real-valued function* ψ *that is nonnegative, continuous, and decreasing in* $[0, \infty]$, *with* $\psi(0) = \infty$, $\psi(\infty) = 0$, $\psi \circ \psi(t) = t$, *such that*

$$M_3(\Gamma_i) \leq \psi(M_3(\Gamma_e)), \qquad M_3(\Gamma_e) \leq \psi(M_3(\Gamma_i)) \tag{46}$$

for each torus $D \subset \overline{R}^3$.

Proof. Let

$$\phi(t) = \begin{cases} \min\left(\frac{\pi}{2}\left(\log \frac{1}{t}\right)^{-2}, \frac{\pi}{6}(1+t)^3\right) & \text{if } t \in [0, 1), \\ \frac{\pi}{6}(1+t)^3 & \text{if } t \in [1, \infty], \end{cases}$$

and set

$$\psi(t) = \phi\left(\frac{1}{\phi^{-1}(t)}\right). \tag{47}$$

Then ψ has all of the analytic properties listed above, and it remains to establish inequality (46).

Choose a torus $D \subset \overline{R}^3$, and let r_i and r_e denote the infima of the radii of the open balls in R^3 that contain at least one curve in Γ_i and Γ_e, respectively. Then $r_i, r_e \in (0, \infty)$, and it suffices to show that

$$M_3(\Gamma_i) \leq \phi(r_e/r_i), \qquad M_3(\Gamma_e) \leq \phi(r_i/r_e), \tag{48}$$

since (46) follows from (47) and (48).

For this, suppose first that $r_e < r_i$, and choose a and b so that $r_e < a < b < r_i$. Then there exist a point $y \in R^3$ and a curve $\alpha \in \Gamma_e$ that is contained in $\{x: |x - y| < a\}$. Let

$$h(x) = \begin{cases} \left(2 \log \frac{b}{a}\right)^{-1} \frac{1}{|x-y|} & \text{if } a < |x - y| < b, \\ 0 & \text{otherwise,} \end{cases}$$

and choose $\gamma \in \Gamma_i$. Since γ links α and is not contained in $\{x: |x - y| < b\}$, γ contains two arcs joining $\{x: |x - y| = a\}$ to $\{x: |x - y| = b\}$ and

$$\int_{\gamma - \{\infty\}} h \, dm_1 \geq 1.$$

Thus h is an admissible density for Γ_i,

$$M_3(\Gamma_i) \le \int_{R^3} h^3\, dm_3 = \frac{\pi}{2}\left(\log \frac{b}{a}\right)^{-2},$$

and letting $a \to r_e$ and $b \to r_i$, we find that

$$M_3(\Gamma_i) \le \frac{\pi}{2}\left(\log \frac{r_i}{r_e}\right)^{-2}. \tag{49}$$

Suppose next that r_i and r_e are arbitrary, let $a \in (r_e, \infty)$, and choose $y \in R^3$ and $\alpha \in \Gamma_e$ so that $\alpha \subset \{x: |x - y| < a\}$. As above, it is not difficult to show that the function

$$h(x) = \begin{cases} \frac{1}{2r_i} & \text{if } |x - y| < r_i + a, \\ 0 & \text{otherwise} \end{cases}$$

is an admissible density for Γ_i. Hence

$$M_3(\Gamma_i) \le \int_{R^3} h^3\, dm_3 = \frac{\pi}{6}\left(1 + \frac{a}{r_i}\right)^3,$$

and by letting $a \to r_e$, we obtain the inequality

$$M_3(\Gamma_i) \le \frac{\pi}{6}\left(1 + \frac{r_e}{r_i}\right)^3. \tag{50}$$

Finally (49) and (50) imply the first half of (48), while the second half follows by symmetry. This completes the proof of Theorem 5.

By means of Corollary 4, we can establish the following more precise version of Theorem 5 for tori of revolution. For this, let ψ_0 denote the extended real-valued function defined on $[0, \infty]$ by the equation

$$\psi_0(t) = \frac{1}{4\pi}\left(\left(\frac{\pi t + 1}{\pi t}\right)^{1/4} - \left(\frac{\pi t}{\pi t + 1}\right)^{1/4}\right)^2. \tag{51}$$

THEOREM 6. *The function* ψ_0 *is nonnegative, continuous, and decreasing in* $[0, \infty]$ *with* $\psi_0(0) = \infty$, $\psi_0(\infty) = 0$, $\psi_0 \circ \psi_0(t) = t$, *and*

$$M_3(\Gamma_i) \le \psi_0(M_3(\Gamma_e)), \quad M_3(\Gamma_e) \le \psi_0(M_3(\Gamma_i)) \tag{52}$$

for each torus $D \subset \overline{R}^3$ *for which either* D *or* $C(\overline{D})$ *is conformally equivalent to a torus of revolution. Moreover both parts of* (52) *hold with equality whenever* D (*and hence* $C(\overline{D})$) *is conformally equivalent to a circular torus.*

Proof. It is easy to verify that ψ_0 has all of the listed analytic properties. Next, for (52), let H denote the upper half-plane in R^2 whose boundary line is L. The fact that ψ_0 is decreasing with $\psi_0 \circ \psi_0(t) = t$ means that each half of (52) implies the other. Hence, by the conformal invariance of $M_3(\Gamma)$, it is sufficient to establish the second half of (52) for the special case where D is the torus of revolution generated by revolving a plane Jordan domain E about L, where $\overline{E} \subset H$.

For this, suppose that R is the ring in R^2 with $C_0 = \overline{E}$ and $C_1 = C(H)$, that g is the hyperbolic metric in H, and that P and A are the hyperbolic perimeter and hyperbolic area of $\overline{E}$ in H. Following the argument in [9], we can easily see that

$$M_3(\Gamma_i) \leq \frac{A}{\pi^2}, \tag{53}$$

with equality whenever ∂E has zero hyperbolic area. We also require the following result, whose proof we outline in Section 9.

THEOREM 7.

$$M_3(\Gamma_e) = \pi(\mathrm{cap}_{3/2}(R, g))^{-2}. \tag{54}$$

From (26), (33) and (54), we obtain for $M_3(\Gamma_e)$ the sharp estimates

$$\frac{1}{\pi}\frac{(P^2+\pi^2)^{1/2} - P}{2P} \leq M_3(\Gamma_e) \leq \frac{1}{4\pi}\left(\left(\frac{A+\pi}{A}\right)^{1/4} - \left(\frac{A}{A+\pi}\right)^{1/4}\right)^2, \tag{55}$$

and the second half of (52) then follows from (53) and (55).

When D is a circular torus, then E is a disk and (55) holds with equality throughout. Thus we get equality in the second half (and hence in the first half) of (52); this completes the proof of Theorem 6.

COROLLARY 8. *If* $D \subset \overline{R}^3$ *is a torus for which either* D *or* $C(\overline{D})$ *is conformally equivalent to a torus of revolution, then*

$$M_3(\Gamma_i)M_3(\Gamma_e)^2 < \frac{1}{16\pi^3}, \qquad M_3(\Gamma_i)^2 M_3(\Gamma_e) < \frac{1}{16\pi^3}. \tag{56}$$

Proof. It is easy to verify that $t\psi_0(t)^2$ is decreasing and $t^2\psi_0(t)$ is increasing in $(0, \infty)$, and that

$$\lim_{t\to 0} t\psi_0(t)^2 = \frac{1}{16\pi^3}, \qquad \lim_{t\to\infty} t^2\psi_0(t) = \frac{1}{16\pi^3}; \tag{57}$$

hence (52) implies (56).

For tori D that are conformally equivalent to circular tori, (57) implies that

$$\lim_{M_3(\Gamma_i)\to 0} M_3(\Gamma_i)M_3(\Gamma_e)^2 = \frac{1}{16\pi^3}, \qquad \lim_{M_3(\Gamma_i)\to\infty} M_3(\Gamma_i)^2 M_3(\Gamma_e) = \frac{1}{16\pi^3},$$

and hence the inequalities in (56) are asymptotically sharp.

Suppose that D is a torus of revolution in R^3. Theorem 6 shows that, for each value of the modulus of one linking family, the modulus of the other linking family is maximized when D is a circular torus. Thus, roughly speaking, the curves in the linking families of a torus of revolution are shortest and most plentiful when the torus is circular. It seems reasonable to expect that the circular tori have this extremal property in the family of all tori, and we are led to the following conjecture.

CONJECTURE. *Inequalities* (52) *and* (56) *hold for all tori* $D \subset \overline{R}^3$.

9. PROOF OF THEOREM 7

We conclude this paper with a sketch of the proof of equation (54).

We begin by showing that

$$M_3(\Gamma_e) \geq \pi(\mathrm{cap}_{3/2}(R, g))^{-2}. \tag{58}$$

For this it is sufficient to show that

$$\int_{R^3} h^3\, dm_3 \geq \pi\left(\int_R |\mathrm{grad}\, u|^{3/2} g^{1/2}\, dm_2\right)^{-2} \tag{59}$$

for each admissible density h for Γ_e and each simple admissible function u for R. Moreover, since Γ_e is symmetric in the line L, it suffices to consider densities h with this property, in which case (59) reduces to

$$\int_H h^3 \frac{1}{g}\, dm_2 \geq \left(\int_R |\mathrm{grad}\, u|^{3/2} g^{1/2}\, dm_2\right)^{-2}. \tag{60}$$

Choose h and u. For each $t \in (0, 1)$, $u^{-1}(t)$ contains a polygonal Jordan curve $\gamma \subset R$ that separates C_0 and C_1, and hence is not homotopic to 0 in $C(\overline{D})$. Thus $\gamma \in \Gamma_e$ and

$$1 \leq \int_\gamma h\, dm_1 \leq \int_{u^{-1}(t)} h\, dm_1 .$$

The co-area formula then implies that

$$1 \leq \int_0^1 \left(\int_{u^{-1}(t)} h\, dm_1\right) dt = \int_R h\, |\mathrm{grad}\, u|\, dm_2,$$

and with the help of Hölder's inequality we obtain the desired inequality

$$1 \leq \left(\int_H h^3 \frac{1}{g}\, dm_2\right)\left(\int_R |\mathrm{grad}\, u|^{3/2} g^{1/2}\, dm_2\right)^2 .$$

It remains to show that

$$M_3(\Gamma_e) \leq \pi(\mathrm{cap}_{3/2}(R, g))^{-2}. \tag{61}$$

The argument for this inequality closely resembles the proof of Lemma 1 in [10], and hence we omit some details. By the uniform convexity of $L_p(\mu)$, with $p = 3/2$ and $d\mu = g^{1/2}\, dm_2$, there exists an extremal vector-valued function $v = (v_1, v_2)$ defined in R such that

$$\int_R |v|^{3/2} g^{1/2}\, dm_2 = \mathrm{cap}_{3/2}(R, g), \tag{62}$$

and such that

(63) $$\int_R |v|^{-1/2}(v \cdot \operatorname{grad} w)\, g^{1/2}\, dm_2 = 0$$

for all functions w that are continuous in $\overline{R}^2$ and ACT in R^2 with $w = 0$ in $C_0 \cup C_1$ and

$$\int_R |\operatorname{grad} w|^{3/2} g^{1/2}\, dm_2 < \infty .$$

Set $|v| = 0$ in $C_0 \cup C_1$, choose $b \in (0, \infty)$, and let Γ denote the subfamily of all rectifiable $\gamma \in \Gamma_e$ whose distance from $\overline{D} \cup L$ is at least b. Next, for each $a \in (0, b)$, let

(64) $$h(x) = \frac{1}{m_2(B)} \int_B |v(x+y)|^{1/2} g(x+y)^{1/2}\, dm_2(y)$$

for $x \in \overline{H} - \{\infty\}$, where $B \subset R^2$ is the open disk of radius a about the origin. Then, arguing as in [7] or [26], we see that the variational condition (63) implies that

(65) $$\int_\beta h\, dm_1 \geq \operatorname{cap}_{3/2}(R, g)$$

for each polygonal curve $\beta \subset R$ that separates C_0 and C_1 and whose distance from $C_0 \cup C_1$ exceeds a. Extend h to R^3 so that it is symmetric in L. Then it follows from (65) and the argument in [26] that

(66) $$\int_\gamma h\, dm_1 \geq \operatorname{cap}_{3/2}(R, g)$$

for all $\gamma \in \Gamma$. Hence the function

$$h_1 = (\operatorname{cap}_{3/2}(R, g))^{-1} \chi_F h$$

is an admissible density for Γ, where F is the set of points in $C(\overline{D})$ whose distance from $\overline{D} \cup L$ is at least b, and from (62), (64), and Minkowski's inequality we obtain the bounds

$$M_3(\Gamma) \leq (\operatorname{cap}_{3/2}(R, g))^{-3} \int_F h^3\, dm_3 \leq \pi \left(\frac{b}{b-a}\right) (\operatorname{cap}_{3/2}(R, g))^{-2} .$$

If we now let $a \to 0$, we see that

(67) $$M_3(\Gamma) \leq \pi (\operatorname{cap}_{3/2}(R, g))^{-2} .$$

Finally, arguing as in Section 3.4 in [11], we can easily show that (67) holds with Γ replaced by Γ', where Γ' is the subfamily of all rectifiable $\gamma \in \Gamma_e$ lying at distance at least b from $\overline{D}$ instead of $\overline{D} \cup L$. Then, since the subfamily of nonrectifiable $\gamma \in \Gamma_e$ has modulus zero, it follows from Lemma 2.3 in [27] that

$$M_3(\Gamma_e) = \lim_{b \to 0} M_3(\Gamma') \leq \pi (\operatorname{cap}_{3/2}(R, g))^{-2} .$$

This completes the proof of Theorem 7.

Added December 14, 1970. The author has recently learned of two articles by C. Bandle, *Konstruktion isoperimetrischer Ungleichungen der mathematischen Physik aus solchen der Geometrie* and *Einige Extremaleigenschaften von Kreissektoren und Halbkugeln,* in which inequalities similar to some of the consequences of Theorem 1 are derived. These papers will appear shortly in the Commentarii Mathematici Helvetici.

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University of Michigan
Ann Arbor, Michigan 48104