# EXTENSIONS TO THE DISK OF PROPERLY NESTED PLANE IMMERSIONS OF THE CIRCLE

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## 1. INTRODUCTION

Let $f: S \to E$ be an immersion of the oriented circle into the oriented plane. For each point Q in the complement of the image [f], the *winding number* $\omega(f, Q)$ is the topological degree of the map $t \to (f(t) - Q)$ of the circle into the punctured plane. The *tangent winding number* $\tau(f)$ is the degree of the velocity $t \to f'(t)$. If f is an interior boundary (that is, if f extends to a map of the disk D, and if this map is topologically equivalent to an analytic map), then f has *nonnegative circulation* (that is, $\omega(f, Q) \geq 0$ for all Q not in [f]). C. J. Titus showed in [4, p. 435] that the converse of this is false. If f extends to an immersion of D, then $\tau(f) = +1$. The immersion f is said to be *normal* if its image [f] lies in general position. Thus, a normal immersion f has but a finite number of selfintersections. If N is a selfintersection of [f], then $f^{-1}(N) = \{t_1, t_2\}$, $t_1 \neq t_2$, and the two vectors $f'(t_1)$ and $f'(t_2)$ are linearly independent. We call such a point a *node* of [f]. (It was called *Knotenpunkt* in [2], crossing point in [6], double point in [4], and vertex in [5].) The specification of a preferred outside starting point, as in [6, p. 281], orients each of the nodes. A normal immersion is *properly nested* if each of its nodes is a cut point of the graph (that is, decomposes [f] into two disjoint figures). See Figure 2, for example.

Titus [4] gave a simple combinatorial criterion under which a properly nested immersion of the circle extends to an immersion of the disk. The proof follows, for example, from [5, p. 60]. (A different, explicit proof, based on [1], will be included in a subsequent paper.) In this paper we offer a proof of the following theorem based on the remarkable work of S. J. Blank [3].

THEOREM 1. *A properly nested, normal immersion of the circle into the plane has at most one class of topologically equivalent extensions to an immersion of the disk.*

*Proof.* Recall [6, p. 281] that the tangent winding number of a normal immersion may be computed as the algebraic sum of the orientations of the nodes, plus the orientation of the outside starting point. Thus, unless [f] is a Jordan loop, $\tau(f) = 1$ implies that f has at least one negatively oriented node. Nonnegative circulation for f implies that the outside starting point and the first subsequent node must both be positive. If in addition f is properly nested, we can always find a first negative node N, preceded by a positive node M. We can isolate a region $\mathscr{R}$, enclosing the regions $\mathscr{L}$ and $\mathscr{S}$ such that $[f] \cap (\mathscr{R} \setminus \mathscr{L} \cup \mathscr{S})$ has the appearance indicated by the solid lines in Figure 3. We replace the simple arc XMNY of [f] with the arc XZY over the region $\mathscr{L}$. This new curve may be parametrized so as to produce another properly nested immersion g with $\tau(g) = 1$. Its image [g] has two fewer nodes that [f]. In Section 3, we shall apply the methods of S. Blank [3] to prove the following proposition.

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THEOREM 2. *Two properly nested, normal immersions* f *and* g *of the circle into the plane, related to each other as in the preceding paragraph, have the same number of extension classes.*

Now, if the extension class number of a properly nested immersion f is at least one, then by Theorem 2, g has at least one extension to an immersion of the disk. Therefore g has nonnegative circulation also, and we repeat the argument until the number of nodes is zero. But the extension class number of a closed Jordan curve with nonnegative circulation is unity. ■

## 2. REVIEW OF BLANK'S THEOREM

Let f: S → E be a normal immersion of the circle into the plane. Pick a base-point $s_0$ on S and an orientation. An edgepath loop from $f(s_0)$ in the image [f] = f(S) is called *primitive* if it circumscribes exactly one bounded complementary component of [f] in the positive (= counterclockwise) sense. Let the fundamental group of [f] be presented by a finite number of primitive generators. Let $W_f$ be the image of the positive generator under the induced map $f_{\#}: \pi_1(S, s_0) \to \pi_1([f], f(s_0))$. This word, $W_f = f_{\#}(1)$, on a finite number of symbols and their negatives, we call the *Blank word* of f. It depends on the choice of primitive generators for the free group of rank equal to the number of bounded complementary components of [f].

An association of two instances of the same letter, but of opposite sign is called a *pairing*. A family of properly nested pairings that exhaust all negatively occurring symbols we call a *grouping*. (Poénaru [3] has no special name for the entire set of pairings. For Blank, "grouping" and "pairing" meant essentially the same thing. See Figure 2, for example.) The number of different groupings of $W_f$ we shall call the *grouping number* $G(W_f)$.

LEMMA 1. *If* W' *is obtained from* W *under an inner automorphism of the group, then* G(W) = G(W').

*Proof.* Performing the substitutions $\alpha \to w_\alpha^{-1} \alpha w_\alpha$ in W, we obtain the word $W^*$, which may not be cyclicly reduced. For a given grouping of W, construct the obvious set of pairings that group $W^*$. In the process of reducing $W^*$ to W', some letters will be deleted. Clearly, we shall delete both members of a pairing of $W^*$, leaving a well-defined grouping of the fully reduced word W'.

Two different groupings of W differ in the positive partner of some negative letter. The nest of induced pairings in $W^*$ cannot vanish entirely under the cancelling deletions. Therefore some letter of W' is paired to different letters in the respective induced groupings of W'. Thus the process is injective. Since G(W) is a finite number, the process is bijective. ■

THEOREM 3 (S. J. Blank). *If* $\tau(f) = 1$, *then the number* E(f) *of topologically inequivalent extensions of* f *to an immersion of the disk is equal to the number* G(f) *of distinct groupings in a primitive presentation.*

The only proof of this theorem available in the literature at this writing is in the Seminar Bourbaki [3]. We shall give a brief description of Blank's original and rather formidable proof.

To compute a primitive presentation for $\pi_1([f])$, draw a *ray* (an embedded positive half-line) from the interior of each bounded complementary component to the unbounded one, oriented towards infinity. The collection of rays R is to be mutually disjoint, and each ray lies in general position with respect to f. We construct the word $W_f$ by starting at the base point and listing the symbol of the ray crossed by the curve [f], with the sign given according to whether the ray crosses to the right (a *positive crossing*) or to the left (a *negative crossing*).

Now, the word $W_f$ may not yet be cyclicly reduced.

LEMMA 2. *It is possible to reduce* $W_f$ *to its cyclic reduction* $\overline{W}_f$ *by a succession of cancellations of the form*

$$\alpha^{-\varepsilon} m \alpha^{+\varepsilon} \to m \quad \text{or} \quad m\alpha^{-\varepsilon}\alpha^{+\varepsilon} w \to mw \qquad (\varepsilon = \pm 1);$$

*each cancellation of this form corresponds to a relocation of the corresponding ray* $[\alpha]$.

*Proof.* The original rays were drawn disjoint and so that every bounded complementary component has a ray starting in it. Let X and Y be the crossings corresponding to some cancellation on the letter $\alpha$. None of the situations in the left of Figure 1 can arise without the crossing, by another ray, of the segment of [f] between X and Y. Therefore that segment must be simple, and we can relocate the ray as shown on the right. The $\alpha$-cancellation corresponding to X and Y, possibly with further $\alpha$-cancellations due to situations such as at 6, constitutes the only change in the word. ■

The pairings in a grouping of $W_f$, where f extends to an immersion $F: D \to E$ of the disk, also have their geometric meaning. The preimage $F^{-1}([f] \cup [R])$ of the curve and rays consists of S plus a number of disjoint simple arcs, some of which begin and end on the boundary $\partial D$ (and which we shall call *secants*), and some (called

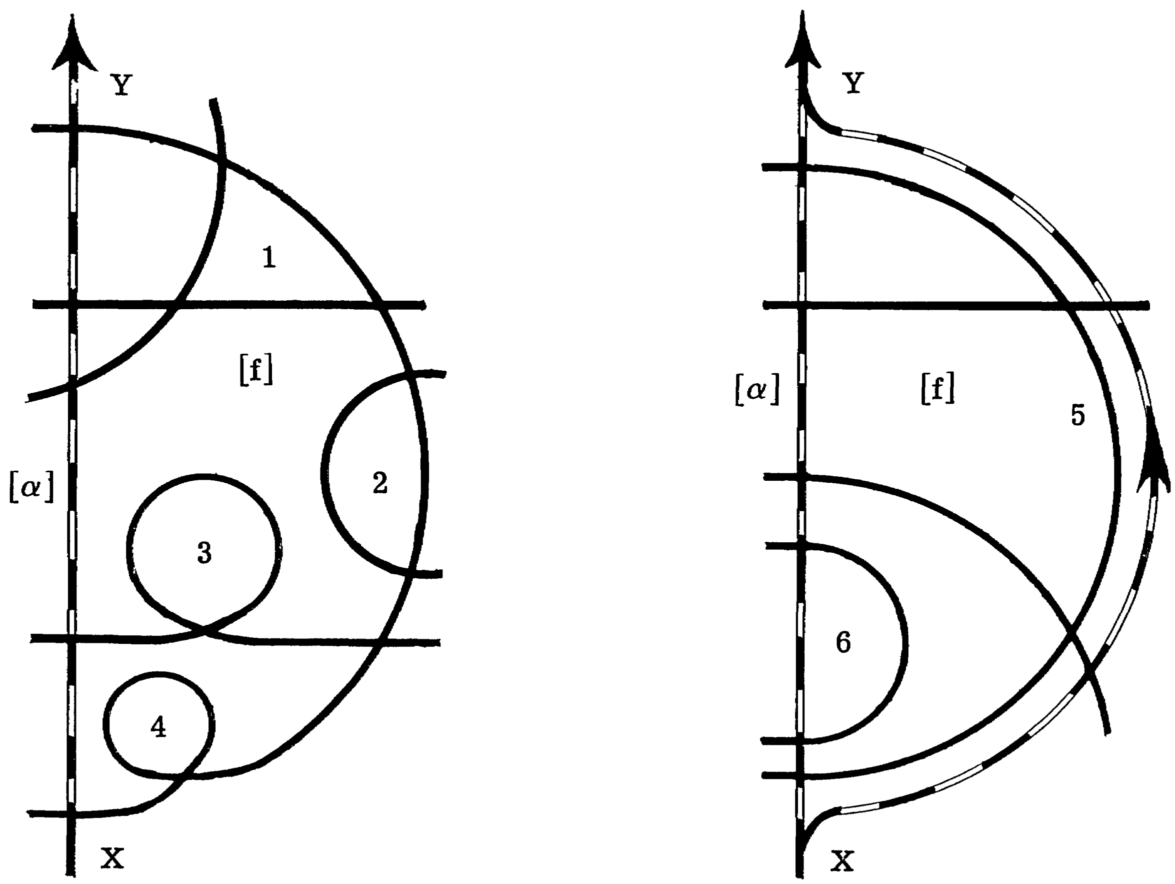


Figure 1.

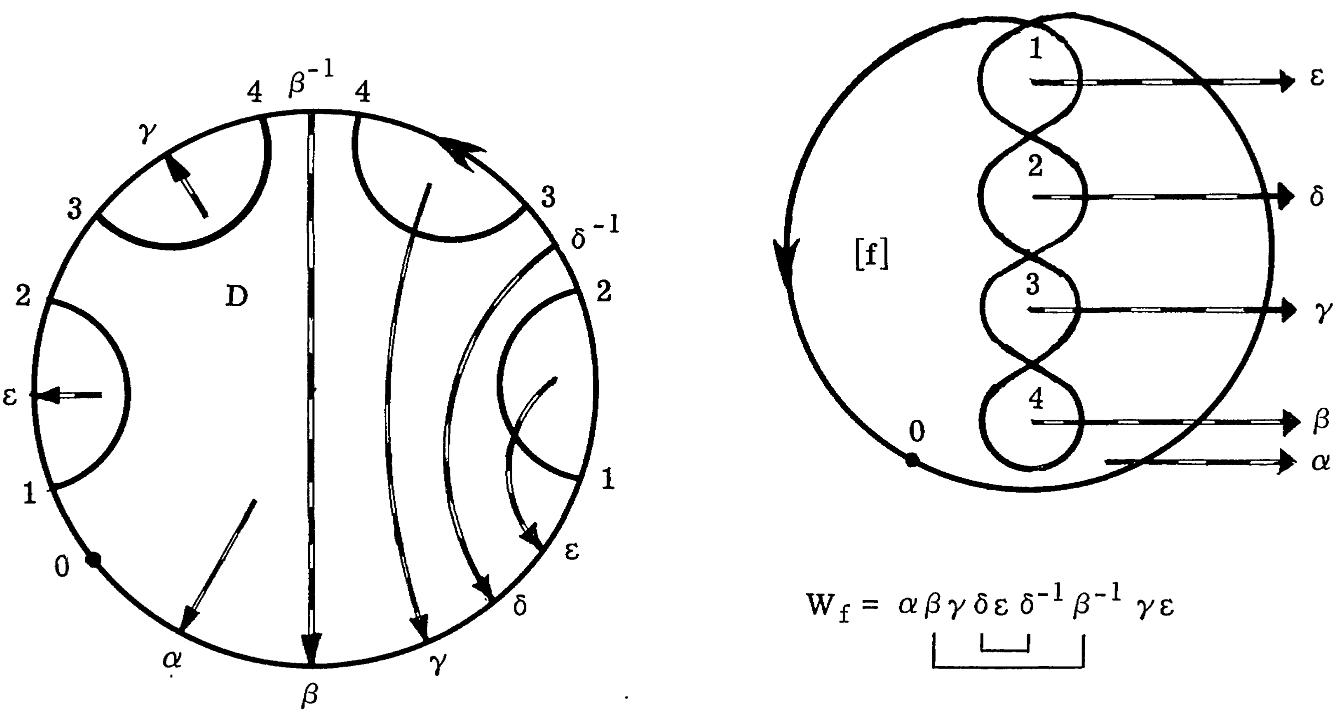


Figure 2.

*tails*) that begin in the interior of D and terminate on the boundary. (For an example, see Figure 2.) It follows that F and R decompose D into a two-dimensional cell-complex K whose one-skeleton $K^1$ consists of S and the secants. In our example, F is univalent on each of the two cells of K.

On the other hand, given a reduced word $W_f$, we can (by marking off the letters on S) draw a set of secants prescribed by a grouping; these secants decompose D into K. Since the rays cross [f] transversally at simple points, we can initially extend f to an immersion $f^*$ of a collaring (tubular neighborhood) of $K^1$. The difficult part of the theorem is to show that $f^*$ can be extended to an embedding of each open two-cell D' of the two-skeleton of $K^2$, provided $\tau(f) = 1$ and the word is reduced. There may be several pairings on the same letter, so that $f^*$ is not univalent on the boundary of a cell. However, it is true that the extension F is univalent on each secant and on $\partial D' \cap S$. Moreover,

$$F(\partial D' \cap S) \cap F(\partial D' \setminus S) = \emptyset .$$

## 3. PROOF OF THEOREM 2

We shall prove Theorem 2 by showing that the relocation of the segment XMNY to XZY as shown in Figure 3 preserves the grouping number, that is, G(f) = G(g). Since the arc XZY is a simple detour of f with support on XMNY, the new immersion g is regularly homotopic to f [1, Section 4, p. 274]. Hence, by [6, p. 279, Theorem 1], $\tau(g) = 1$ as well. Therefore, by Blank's Theorem, E(f) = E(g).

We must first draw a suitable set of rays. In Figure 3, the points A and B are evidently bounded by [f]. Since the node N is the first negatively oriented one, and the curve begins outside and positively, $\omega(f, D) \geq 0$. Hence $\omega(f, C) \geq 1$, and therefore C is also bounded. In the region $\mathscr{R}$ there may be selfintersections of [f] other than N and M. But, because the curve is properly nested, we may consider these to

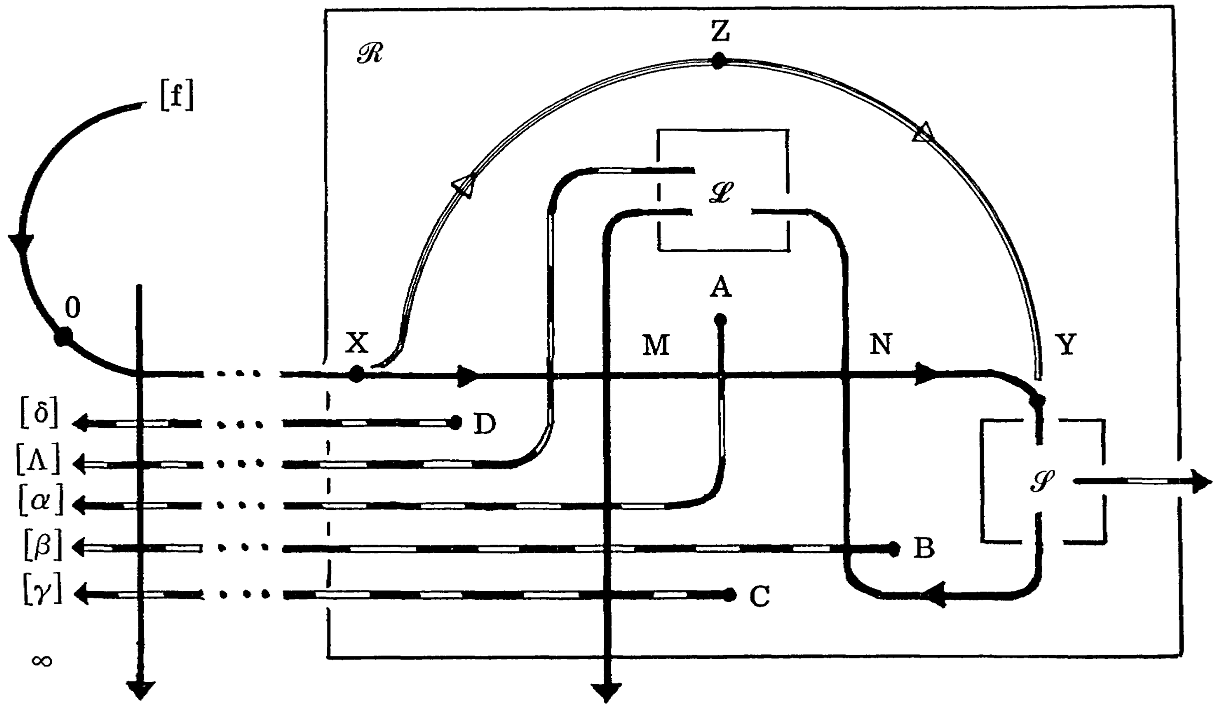


Figure 3.

occur in the regions $\mathscr{L}$ and $\mathscr{S}$. In these regions, draw rays originating in bounded complementary components of [f], so that each set $\{\sigma_j\}$ and $\{\lambda_j\}$ issues from the regions $\mathscr{S}$ and $\mathscr{L}$ in a parallel bundle. The bundle $\{\sigma_j\}$ is simply to leave the region $\mathscr{R}$ and continue to the unbounded component. The bundle $\{\lambda_j\}$, however, is to follow parallel along the lower side of the segment of [f] from N back to the starting point. The same bundle also conducts the rays $\alpha$, $\beta$, and $\gamma$ from A, B, and C to the unbounded component. (This is so because the starting point is positive.) If the point D is already in the unbounded component, we may stop. Otherwise, another ray $\delta$ is drawn from D and included in the later bundle of rays.

Since N is the first negatively oriented node, there are only positive crossings of this ray bundle outside the region $\mathscr{R}$, and they all read $\delta\Lambda\alpha\beta\gamma$, where $\Lambda = \lambda_1, \cdots, \lambda_r$.

All other necessary rays $\{\eta_n\}$ are drawn so as to be invisible in the picture. If for convenience we read $W_f$ starting from just below B, then

$$W_f = \beta^{-1} L\alpha\beta\gamma V\Lambda\alpha S,$$

where

$$\Lambda = \lambda_1 \lambda_2 \cdots \lambda_r, \qquad L = L(\lambda_j^{\pm 1}), \qquad V = V(\delta\Lambda\alpha\beta\gamma, \sigma_i^{\pm 1}, \eta_j^{\pm 1}), \qquad S = S(\sigma_j^{\pm 1}).$$

A priori, $W_f$ need not be a reduced word. The subwords L, S, V, and $\Lambda$ may be empty. However, V is reduced in the $\alpha$, $\beta$, $\gamma$, $\delta$, $\lambda_j$, since they appear only positively. We may make the necessary relocations of $\lambda_j$ entirely in region $\mathscr{L}$. In the proof of Lemma 2, we saw that a cancellation requires that at least one part of [f] is simple between the two relocation points. Hence the $\sigma_j$ can be relocated either entirely inside the region $\mathscr{S}$ or entirely outside the region $\mathscr{R}$. Finally, the $\eta_i$ may be relocated outside the region $\mathscr{R}$. Hence we may assume that L, V, S are reduced, and therefore $W_f$ is reduced.

The case where D is unbounded (so that no ray $\delta$ is needed and V does not contain any $\alpha$, $\beta$, $\gamma$, $\lambda_j$) is easy to dispose of.

After performing the detour from X to Y, we read

$$W^* = \beta^{-1} L\alpha\beta\gamma V00S \qquad \text{(0 denotes blanks)}.$$

Since B is now in the unbounded component and A shares a bounded component with C, the ray $\beta$ and one of the rays $\alpha$, $\gamma$ have become superfluous. Deleting $\alpha$ and $\beta$, we read

$$W^{**} = 0L00\gamma V00S.$$

As we noted above, no cancellation on a $\sigma_j$ can occur between the residual V and S. Therefore $W^{**}$ is also reduced, and we may use it as $W_g$, where g is the detoured immersion of the circle.

Each grouping of $W_f$ allows only one choice for the pairing of the $\beta^{-1}$. Hence the word L is positive, and no letter in it is paired. Consequently, the grouping goes over to a grouping of $W^{**}$, with the sole deletion of the $\beta$-pairing. On the other hand, each grouping of $W^{**} = W_g$ extends to a grouping of $W_f$, by the unambiguous insertion of the single $\beta$-pairing.

The case where D is not in the infinite component (so that $\delta$ appears) is a trifle more complicated. This time,

$$W^{**} = 0L00\gamma U00S, \qquad \text{where } U = V(\delta\Lambda 00\gamma,\ \eta_i^{\pm 1},\ \sigma_j^{\pm 1}).$$

LEMMA 3. $W^{**}$ *is a reduced word.*

*Proof.* Since V was reduced, so is U. Again, as we have seen, no $\sigma$-cancellation is possible between U and S. Even if S should be blank, no initial $\lambda_j^{-1}$ can cancel in U, because each $\lambda_j^{-1}$ occurs in a subword of the form $\delta\Lambda\gamma$ in U. ■

In each grouping of $W_f$, the initial $\beta^{-1}$ must pair. The purpose of drawing the rays as we did was to insure that the positive partner of the $\beta^{-1}$ is entirely determined by the pairing that involves L.

Let K be the subdivision of the disk D induced by the extension F of the grouping. Let D' be the two-cell to the right of the only $\beta$-segment of $K^1$.

If there is no pairing of a letter in L to the outside of L, then $F(\partial D' \cap S)$ includes both the ascending and the descending arcs of [f] visible in the picture. But by Blank's theorem, $F(\partial D' \cap S) \cap F(\partial D' \setminus S) = \emptyset$. Therefore the only $\beta^{-1}$ of $W_f$ must pair with the first $\beta^{+1}$.

If there is a pairing from L to the outside, consider the left-most $\lambda^{-1}$ in L that pairs out from L. Since the $\beta^{-1}$ pairs in V, so must this $\lambda^{-1}$. We claim that if it pairs with some occurrence of $\lambda$ in a subword $\delta\Lambda\alpha\beta\gamma$ of V, then the $\beta^{-1}$ must pair with the $\beta$ of that same subword. Again, consider $F(\partial D')$. As it emerges from the region $\mathscr{L}$, it moves west along [$\lambda$] until it turns south on [f] somewhere outside of the picture. The curve must turn east along [$\beta$] as soon as it reaches [$\beta$], in order to prevent $F(\partial D' \cap S)$ from intersecting $F(\partial D' \setminus S)$.

We conclude that in no grouping of $W_f$ can the final $\Lambda$ be a part of such a pairing. Its deletion on passing to $W^*$ kills no pairing. To reach $W^{**} = W_g$, we delete the only $\beta$-pairing there is; the others remain intact. This groups $W_g$. For each grouping of $W_g$ there is only one grouping of $W_f$. Hence G(f) = G(g). This completes the proof of Theorem 2. ■

## 4. CONCLUSION

It follows from the final remark in [3] that Theorem 1 extends to the case where $f: S \to V$ is a properly nested, normal immersion of the circle into a two-manifold other than the sphere or the projective plane. For in that case, the universal covering space $\tilde{V}$ of $V$ is the plane. The lift $\tilde{f}: S \to \tilde{V}$ of $f$ also is normal and properly nested. If $E(f) \geq 1$, then $[f]$ is null homotopic in $V$, and each extension $F: D \to V$ of $f$ lifts to an extension $\tilde{F}: D \to \tilde{V}$ of $\tilde{f}$. Thus $E(\tilde{f}) \geq 1$, and by Theorem 1, $E(\tilde{f}) = 1$. Suppose $F$ and $G$ both extend $f$; then there is a homeomorphism $H: D \to D$ such that $\tilde{F} = \tilde{G} \circ H$. If $p: \tilde{V} \to V$ is the covering projection, then $F = p \circ \tilde{F} = p \circ \tilde{G} \circ H = G \circ H$. Consequently, $E(f) = 1$.

It remains an open problem to give a topological classification of the extensions $F: M \to V$ of an immersion $f: S \to V$ to an arbitrary compact 2-manifold $M$ whose boundary is the circle $S$.

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