PARACOMPACTNESS OF LOCALLY COMPACT HAUSDORFF SPACES

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A topological space is *paracompact* if it is a Hausdorff space and if every open cover has a locally finite refinement that is also an open cover.

Let X be a locally compact Hausdorff space, let A = C(X) be the ring of all continuous real-valued functions on X, and let J(X) be the ideal in A consisting of all continuous functions having compact support.

THEOREM (R. Bkouche). *The space* X *is paracompact if and only if* J(X) *is a projective* A-*module.*

This theorem is a corollary of a deep result [1] of R. Bkouche. The authors heard of it through P. Samuel, who suggested that an elementary proof would be desirable.

Recall that if a space X is paracompact and $\{V_\beta\}$ is an open cover of X, then there exists a *partition of unity subordinate to* $\{V_\beta\}$; in other words, there exist continuous functions $f_\beta: X \to I = [0, 1]$ such that

i) for each β, $\operatorname{supp} f_\beta = \overline{\{x \in X: f_\beta(x) \neq 0\}} \subset V_\beta$;

ii) the family $\{\operatorname{supp} f_\beta\}$ is a locally finite cover of X;

iii) for each $x \in X$, $1 = \sum_\beta f_\beta(x)$.

An A-module M is projective [2, p. 132, Proposition 3.1] if and only if it has a projective basis, that is, if there exist elements $f_\beta \in M$ and A-homomorphisms $\phi_\beta: M \to A$ such that for each $g \in M$,

i) $\phi_\beta(g) = 0$ for almost all β,

ii) $g = \sum_\beta \phi_\beta(g) f_\beta$.

Also, in a locally compact Hausdorff space each compact subset K has a compact neighborhood in X, and for each such neighborhood V there exists a continuous separating function $s: X \to I$ that is 1 on K and 0 on X - V.

X *is paracompact* $\Rightarrow$ J *is projective.* Let $\{U_\alpha\}$ be a covering of X by open sets with compact closure. Since X is paracompact, there exists a locally finite refinement $\{V_\beta\}$ (of course, each $\overline{V}_\beta \subset \overline{U}_\beta$ is compact). If $\{f_\beta\}$ is a partition of unity subordinate to $\{V_\beta\}$, then each f_β has compact support, hence lies in J.

For each β, let s_β be a separating function that is 1 on the support of f_β and 0 on $X - V_\beta$. Define $\phi_\beta: J \to A$ by

$$\phi_\beta(g) = g s_\beta, \qquad \text{where } g \in J.$$

We claim that the f_β and ϕ_β give a projective basis of J.

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To prove that for each $g \in J$, all but a finite number of $\phi_\beta(g)$ are 0, note that supp g, being compact, meets only finitely many of the V_β; call these V_{β_i}. For each $x \in X$,

$$\phi_\beta(g)(x) = (gs_\beta)(x) = g(x)s_\beta(x),$$

and the last member is nonzero only if $x \in$ supp g. For such an x, however, $s_\beta(x) \neq 0$ only if β is among the β_i.

To prove that each $g \in J$ has the form $\sum \phi_\beta(g) f_\beta$, we note first that $f_\beta = f_\beta s_\beta$, for each β (because $s_\beta = 1$ on supp f_β). Hence

$$g = g\sum f_\beta = g\sum f_\beta s_\beta = \sum (gs_\beta) f_\beta = \sum \phi_\beta(g) f_\beta.$$

J is projective ⇒ X is paracompact. We show that X has a locally finite open cover $\{V_\beta\}$ for which each $\overline{V}_\beta$ is compact; from this it follows easily that X is paracompact.

Let $\{f_\beta \in J,\ \phi_\beta\colon J \to A\}$ be a projective basis for J, and for each β define V_β to be the interior of supp $\phi_\beta(f_\beta)$.

Each $\overline{V}_\beta =$ supp $\phi_\beta(f_\beta)$ is compact. Let U be a compact neighborhood of supp f_β, and let s be a separating function that is 1 on supp f_β and 0 on X - U. Then supp s is compact because it lies in U, and $f_\beta = sf_\beta$. Hence

$$\operatorname{supp} \phi_\beta(f_\beta) = \operatorname{supp} \phi_\beta(sf_\beta) = \operatorname{supp} s\phi_\beta(f_\beta),$$

and the last member is compact because it lies in supp s.

Next, $\{V_\beta\}$ is a cover of X. Let x be a point of X, and let s be a separating function that is 1 at x and 0 outside a compact neighborhood of x. Then $s \in J$, so that $s = \sum \phi_\beta(s) f_\beta$. Since each ϕ_β is an A-homomorphism, $\phi_\beta(s) f_\beta = \phi_\beta(f_\beta)s$, and

$$1 = s(x) = \sum [\phi_\beta(f_\beta)s](x) = \sum \phi_\beta(f_\beta)(x).$$

Thus $\phi_\beta(f_\beta)(x) \neq 0$ for some β, and $x \in V_\beta$.

Finally, $\{V_\beta\}$ is locally finite. Take $x \in X$ and s as before, and let $Y = s^{-1}(0, 1]$. Clearly, Y is an open neighborhood of x; we claim Y meets only finitely many of the V_β. Let B be the finite set of indices for which $\phi_\beta(s) \neq 0$. If $\beta \notin B$ and $y \in Y$, then

$$\phi_\beta(f_\beta)(y) \cdot s(y) = \phi_\beta(s)(y) \cdot f_\beta(y) = 0.$$

Since $s \neq 0$ on Y, it follows that $\phi_\beta(f_\beta) = 0$ on Y for all $\beta \notin B$. That is, Y does not meet V_β if $\beta \notin B$. Thus $\{V_\beta\}$ is locally finite, and the proof is complete.

COROLLARY. *Let X be a C^r- or C^∞-manifold (not necessarily separable or paracompact), let A be the ring of real-valued C^r- (or C^∞-) functions on X, and let J be the ideal in A of functions with compact support. Then X is paracompact if and only if J is a projective A-module.*

Proof. The partition of unity and the separating functions that appear in the proof of the main theorem may now be chosen to have the appropriate degree of differentiability.

REFERENCES

1. R. Bkouche, *Pureté, molesse et paracompacité.* C. R. Acad. Sci. Paris Sér. A 270 (1970), 1653-1655.

2. H. Cartan and S. Eilenberg, *Homological algebra.* Princeton University Press, Princeton, New Jersey, 1956.

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