THE HARDY CLASS OF SOME UNIVALENT FUNCTIONS AND THEIR DERIVATIVES

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1. INTRODUCTION

If $f(z) = \sum_0^\infty a_n z^n$ is a function analytic for $|z| < 1$, then $f(z)$ is said to belong to H^λ $(\lambda > 0)$ if

$$M_\lambda(f, r) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda d\theta\right)^{1/\lambda} \leq K \qquad (0 \leq r < 1),$$

where K is a constant depending on $f(z)$. We denote by H^∞ the class of analytic functions bounded for $|z| < 1$.

In this section, we list some known theorems and lemmas for reference.

THEOREM A. *If* $f(z) \in H^\lambda$ $(0 < \lambda < 1)$, *then*

$$a_n = o(n^{1/\lambda - 1}).$$

THEOREM B. *If* $f(z)$ *is univalent, then* $f(z) \in H^\lambda$, *for all* $\lambda < 1/2$.

THEOREM C. *If* $f(z)$ *is univalent, then*

$$|a_n|/n \to \alpha |a_1|$$

as $n \to \infty$, *where* $0 \leq \alpha \leq 1$.

Theorem A is in [2], Theorem C in [6, p. 104], and Theorem B is, for example, in [9, p. 214].

The Koebe function $z(1 - z)^{-2} = \sum_1^\infty nz^n$ shows that there exist univalent functions that are not in $H^{1/2}$. We have, in fact, the following result.

THEOREM D. *If* $f(z)$ *is univalent, then*

$$\lim_{r \to 1} \int_{-\pi}^{\pi} |f(re^{i\theta})|^{1/2} d\theta \Big/ \log \frac{1}{1 - r} = 2|a_1|^{1/2} \alpha^{1/2},$$

where α *is as in Theorem* C.

Theorem D is an immediate consequence of Theorems I and VI and Lemma I in [5].

According to Theorem D, univalent functions whose coefficients satisfy the relation $|a_n|/n \not\to 0$ are necessarily excluded from $H^{1/2}$. It is less obvious that there

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exist univalent functions not in $H^{1/2}$ for which, nevertheless, $a_n/n \to 0$. We construct an example of such a function in Section 5.

In the following sections, we study the classes H^λ to which f(z) and f'(z) belong for three of the most familiar types of univalent functions, namely, convex functions, starlike functions, and close-to-convex functions. Though we have confined attention to f(z) and f'(z), it is likely that analogous results for fractional derivatives of f(z) could be obtained by similar methods. In the case of close-to-convex functions, we are able to discover the properties that determine the value of $\lim_{n\to\infty} |a_n|/n$.

We conclude this section with some further theorems and lemmas.

THEOREM E. *If* $f'(z) \in H^\lambda$ $(0 < \lambda \leq 1)$, *then* $f(z) \in H^{\lambda/(1-\lambda)}$.

THEOREM F. *If* f(z) *is univalent and maps* $|z| < 1$ *onto a domain* D, *then the boundary of* D *is a closed rectifiable curve if and only if* $f'(z) \in H^1$.

LEMMA A. $(1 - z)^{-1} \in H^\lambda$ *if and only if* $\lambda < 1$.

LEMMA B. *If* P(z) *is analytic and* $\Re P(z) > 0$ *for* $|z| < 1$, *then* $P(z) \in H^\lambda$ *for all* $\lambda < 1$.

Theorem E is a special case of [3, Theorem 12], with $H^{\lambda/(1-\lambda)}$ interpreted as H^∞ when $\lambda = 1$; Theorem F is in [12]. The lemmas are well known.

The definition and analytic characterization of convex and starlike functions are given in [12].

2. CONVEX FUNCTIONS

If f(z) is analytic for $|z| < 1$, then it is univalent and convex if and only if $f'(0) \neq 0$, $zf''(z)/f'(z)$ is analytic, and

$$\Re\left[1 + z\frac{f''(z)}{f'(z)}\right] > 0.$$

We then have the representation

$$1 + \frac{zf''(z)}{f'(z)} = \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\mu(t), \tag{1}$$

where $\mu(t)$ is nondecreasing, $\mu(\pi) - \mu(-\pi) = 1$, and we can suppose $\mu(t)$ to be normalized so that

$$\frac{1}{2}[\mu(t + 0) + \mu(t - 0)] = \mu(t), \qquad \int_{-\pi}^{\pi} \mu(t)\, dt = 0. \tag{2}$$

The normalization determines $\mu(t)$ uniquely; we shall call $\mu(t)$ *the measure associated with* f(z). Conversely, for each function $\mu(t)$ satisfying the conditions above, solution of the differential equation (1) yields a convex function f(z), uniquely determined up to translation, magnification, and rotation. From (1), we obtain the relations

(3) $$\log \frac{f'(z)}{f'(0)} = i \int_{-\pi}^{\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} \left[\mu(t) - \frac{t}{2\pi} \right] dt$$

and

(4) $$\log \frac{f'(z)}{f'(0)} = \int_{-\pi}^{\pi} \log(1 - ze^{-it})^{-2} \, d\mu(t) .$$

For $0 \le \alpha < 1$, we shall denote by K_α the class of convex functions satisfying the inequality

(5) $$\Re \left[1 + \frac{zf''(z)}{f'(z)} \right] > \alpha ,$$

in other words, K_α is the class of convex functions of order α. Then, for $0 \le \alpha < \beta < 1$, we have the inclusions $K_\beta \subseteq K_\alpha \subseteq K_0$ (we have preferred to adopt this definition rather than the usual one in which the supremum is taken over the α on the right side of (5), the various classes then being disjoint). The following lemma follows immediately from our definition.

LEMMA 1. *$f(z) \in K_\alpha$ if and only if $f'(z) = [g'(z)]^{1-\alpha}$, for some $g(z) \in K_0$.*

We also need the following lemma.

LEMMA 2. *If $f(z) \in K_0$ and $\mu(t)$ is the measure associated with $f(z)$, then $f(z)$ is unbounded if and only if there exists θ_0 such that $\mu(\theta_0 + 0) - \mu(\theta_0 - 0) \ge 1/2$.*

For a proof, see [10].

LEMMA 3. *If $f(z) \in K_0$, then $f'(z) \in H^\lambda$ for all $\lambda < 1/2$.*

Proof. By (4), $f'(z)/f'(0)$ is subordinate to $(1 - z)^{-2}$, and hence [9, p. 165] we have the inequality

$$M_\lambda(f', r) \le |f'(0)| \, M_\lambda((1 - z)^{-2}, r) .$$

By Lemma A, the right-hand side is bounded for all $\lambda < 1/2$.

Taking imaginary parts in (3) and letting $r \to 1$, we find [13, p. 97, (6.11)] that, for all θ, $\lim_{r \to 1} \arg f'(z) = V(\theta)$ exists and that

(6) $$V(\theta) = 2\pi\mu(\theta) - \theta + \arg f'(0) .$$

Equation (6) provides an expression for $\mu(\theta)$ intrinsically in terms of $f(z)$.

Now suppose that $f(z) \in K_\alpha$, so that by Lemma 1,

(7) $$f'(z) = [g'(z)]^{1-\alpha}$$

for some $g(z) \in K_0$, and let $\nu(t)$ be the measure associated with $g(z)$. Then $\arg f'(z) = (1 - \alpha) \arg g'(z)$. Hence, by (6), we have the following result.

THEOREM 1. *If $f(z) \in K_\alpha$ and $\mu(t)$ is the measure associated with $f(z)$, then*

(8) $$\mu(\theta) = (1 - \alpha)\nu(\theta) + \frac{\alpha\theta}{2\pi} ,$$

where $\nu(t)$ *is the measure associated with some function in* K_0. *Conversely, a measure* $\mu(t)$ *of the form* (8), *where* $\nu(t)$ *is the measure associated with a function in* K_0, *is the measure associated with a function in* K_α.

If f(z) is unbounded, then, by Lemma 2, there exists θ_0 such that $\mu(\theta_0 + 0) - \mu(\theta_0 - 0) \geq 1/2$. If, in addition, $f(z) \in K_{1/2}$, then, by (8), $\mu(\theta_0 + 0) - \mu(\theta_0 - 0) \leq 1/2$. It now follows from (8) that $\nu(\theta_0 + 0) - \nu(\theta_0 - 0) = 1$; hence $\nu(t)$ is a step-function with exactly one jump (at θ_0). By (7) and (4), we therefore have the relations

$$\log \frac{f'(z)}{f'(0)} = \frac{1}{2} \log \frac{g'(z)}{g'(0)} = \frac{1}{2} \int_{-\pi}^{\pi} \log (1 - ze^{-it})^{-2} \, d\nu(t) = \log (1 - ze^{-i\theta_0})^{-1} .$$

Straightforward computations now yield the following result.

COROLLARY. *If* $f(z) \in K_{1/2}$ *is unbounded, then*

$$f(z) = a + b \log (1 - ze^{i\tau})$$

for some complex constants a, b *and some real constant* τ.

This corollary is also a special case of part (ii) of Theorem 4 below.

Though we require only its corollary, we have included the next theorem for the sake of completeness. The conclusion of the theorem is perhaps geometrically obvious, but the proof is entirely analytic.

THEOREM 2. *Let* $\mu(t)$ *be the measure associated with* $f(z) \in K_0$, *and suppose that* $0 < \beta < \alpha < \pi$ *and* $\mu(\alpha) - \mu(-\alpha) < 1/2$. *Then*

$$\int_{-\beta}^{\beta} |f'(re^{i\theta})| \, d\theta = O(1) \qquad (r \to 1).$$

COROLLARY. *If* $\mu(t)$ *is continuous at* $t = 0$, *then there exists* $\eta > 0$ *such that*

$$\int_{-\eta}^{\eta} |f'(re^{i\theta})| \, d\theta = O(1) \qquad (r \to 1).$$

Remarks. (i) Since $|zf'(z)|$ is an increasing function of r for $f(z) \in K_0$ [8, p. 483] (reference to [8] here and elsewhere is to a property of starlike functions in S_0 equivalent by (14)), the theorem implies that $\int_{-\beta}^{\beta} |f'(e^{i\theta})| \, d\theta$ is finite. In other words, the part of the boundary of the convex domain given by $w = f(e^{i\theta})$ $(-\beta \leq \theta \leq \beta)$ is rectifiable. The corollary likewise implies a property of local rectifiability.

(ii) The theorem becomes false if either the number 1/2 is increased or the inequality $\beta < \alpha$ is replaced by $\beta \leq \alpha$, as consideration of maps onto strip domains shows.

Proof of the theorem. Taking real parts of (4), we obtain the equation

$$|f'(z)| = |f'(0)| \exp \int_{-\pi}^{\pi} \log |1 - ze^{-it}|^{-2} \, d\mu(t),$$

and for $z = re^{i\theta}$ $(-\beta \leq \theta \leq \beta)$, we have

$$\int_{-\pi}^{\pi} \log |1 - ze^{-it}|^{-2} \, d\mu(t) = \int_{-\alpha}^{\alpha} \log |1 - ze^{-it}|^{-2} \, d\mu(t) + O(1),$$

where the term $O(1)$ is uniform with respect to θ. Hence

$$\int_{-\beta}^{\beta} |f'(re^{i\theta})| \, d\theta = O(1) \cdot \int_{-\beta}^{\beta} d\theta \exp \int_{-\alpha}^{\alpha} \log |1 - re^{i(\theta - t)}|^{-2} \, d\mu(t).$$

Let $\mu(\alpha) - \mu(-\alpha) = \gamma$. If $\gamma = 0$, then on the right-hand side, the inner integral vanishes, and the proof is complete. If $\gamma \neq 0$, we define $\nu(t) = \mu(t)/\gamma$. By the inequality between the geometric and arithmetic mean in weighted integral form [4, p. 156], we have the inequality

$$\exp \int_{-\alpha}^{\alpha} \log |1 - ze^{-it}|^{-2} \, d\mu(t) = \exp \int_{-\alpha}^{\alpha} \log |1 - ze^{-it}|^{-2\gamma} \, d\nu(t)$$

$$\leq \int_{-\alpha}^{\alpha} |1 - ze^{-it}|^{-2\gamma} \, d\nu(t);$$

thus

$$\int_{-\beta}^{\beta} |f'(z)| \, d\theta \leq O(1) \cdot \int_{-\pi}^{\pi} d\theta \int_{-\alpha}^{\alpha} |1 - ze^{-it}|^{-2\gamma} \, d\nu(t).$$

Interchanging the order of integration and using Lemma A, together with the fact that $2\gamma < 1$, we find that the right-hand side is bounded as $r \to 1$.

THEOREM 3. *If* $f(z) \in K_0$ *is not of the form* $a + bz(1 - ze^{i\tau})^{-1}$ *for some complex* a, b *and real* τ, *then there exists* $\delta = \delta(f) > 0$ *such that* $f'(z) \in H^{1/2+\delta}$.

Proof. Let $\mu(t)$ be the measure associated with $f(z)$. If $f(z)$ is bounded, then, since the boundary of a bounded convex domain is a closed rectifiable curve, Theorem F implies that $f(z) \in H^1$. We may therefore suppose that $f(z)$ is unbounded so that, by Lemma 2, the maximum jump α of $\mu(t)$ satisfies the inequality $\alpha \geq 1/2$. If $f(z)$ is not of the form stated in the theorem, then we also have $\alpha < 1$. We may suppose without loss of generality that this maximum jump occurs at $t = 0$. Let $s(t)$ denote the step function

$$s(t) = \begin{cases} -\alpha/2 & (-\pi \leq t < 0), \\ 0 & (t = 0), \\ \alpha/2 & (0 < t \leq \pi), \end{cases}$$

and define $\nu(t)$ by $\mu(t) = s(t) + (1 - \alpha)\nu(t)$. Then $\nu(t)$ is a nondecreasing function, $\nu(\pi) - \nu(-\pi) = 1$, and the normalization conditions (2) are satisfied. Therefore $\nu(t)$ is the measure associated with a convex function $g(z)$, say, and $\nu(t)$ is continuous at $t = 0$. By (4), we obtain the decomposition

$$\log \frac{f'(z)}{f'(0)} = \int_{-\pi}^{\pi} \log(1 - ze^{-it})^{-2} d\mu(t)$$

$$= \alpha \log(1 - z)^{-2} + (1 - \alpha) \int_{-\pi}^{\pi} \log(1 - ze^{-it})^{-2} d\nu(t),$$

or

$$f'(z) = c(1 - z)^{-2\alpha} [g'(z)]^{1-\alpha}, \tag{9}$$

where c is a constant. We now choose η such that the conclusion of the corollary to Theorem 2 applies to $g(z)$, that is, such that

$$\int_{-\eta}^{\eta} |g'(z)|\, d\theta = O(1), \tag{10}$$

and we further choose

$$\delta < \frac{1}{2}\frac{1 - \alpha}{1 + \alpha}.$$

Then, with $q = (1 - \alpha)^{-1}(1/2 + \delta)^{-1}$ and $p = q/(q - 1)$, Hölder's inequality applied to (9) yields the relations

$$\begin{aligned} &\int_{-\eta}^{\eta} |f'(z)|^{1/2+\delta}\, d\theta \\ &\leq |c|^{1/2+\delta} \left(\int_{-\eta}^{\eta} |1 - z|^{-2p\alpha(1/2+\delta)} d\theta \right)^{1/p} \left(\int_{-\eta}^{\eta} |g'(z)|\, d\theta \right)^{1/q} = O(1). \end{aligned} \tag{11}$$

Finally, by (9), we find that

$$\begin{aligned} &\int_{-\pi}^{-\eta} |f'(z)|^{1/2+\delta}\, d\theta + \int_{\eta}^{\pi} |f'(z)|^{1/2+\delta}\, d\theta \\ &\leq |c| \cdot \max_{\eta \leq |\theta| \leq \pi} |1 - re^{i\theta}|^{-2\alpha(1/2+\delta)} \int_{-\pi}^{\pi} |g'(z)|^{(1-\alpha)(1/2+\delta)} d\theta = O(1) \end{aligned} \tag{12}$$

(the last equation follows by Lemma 3, since

$$(1 - \alpha)(1/2 + \delta) < (1 - \alpha)/(1 + \alpha) < 1/2).$$

Addition of (11) and (12) now yields the desired result.

Application of Lemma 1 and Theorem E gives the following extension of both Theorem 3 and the corollary of Theorem 1.

THEOREM 4. *If* $f(z) \in K_\alpha$ *is not of the form*

$$f(z) = a + b(1 - ze^{i\tau})^{2\alpha-1} \qquad (\alpha \neq 1/2),$$

$$f(z) = a + b\log(1 - ze^{i\tau}) \qquad (\alpha = 1/2),$$

for some complex a, b *and real* τ, *then the following two statements hold.*

(i) *There exists* $\delta = \delta(f) > 0$ *such that* $f'(z) \in H^{\frac{1}{2(1-\alpha)}+\delta}$

(ii) *If* $0 \leq \alpha \leq 1/2$, *then there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $f(z) \in H^{\frac{1}{1-2\alpha}+\varepsilon}$ ($f(z) \in H^\infty$ *if* $\alpha = 1/2$). *If* $\alpha > 1/2$, *then* $f(z) \in H^\infty$ *without exception.*

3. STARLIKE FUNCTIONS

If f(z) is analytic for $|z| < 1$ with $f(0) = 0$, then it is univalent and starlike if and only if $f'(0) \neq 0$, $zf'(z)/f(z)$ is analytic, and

$$\Re\left[\frac{zf'(z)}{f(z)}\right] > 0. \tag{13}$$

For $0 \leq \alpha < 1$, we denote by S_α the class of starlike functions with the property that

$$\Re\left[\frac{zf'(z)}{f(z)}\right] > \alpha .$$

It follows from the definitions that

$$g(z) \in K_\alpha \iff f(z) = zg'(z) \in S_\alpha . \tag{14}$$

In view of (14), the first part of the following theorem is the direct analogue of the first part of Theorem 4.

THEOREM 5. *If* $f(z) \in S_\alpha$ *is not of the form*

$$f(z) = az(1 - ze^{i\tau})^{2\alpha-2},$$

then

(i) *there exists* $\delta = \delta(f) > 0$ *such that* $f(z)/z \in H^{\frac{1}{2(1-\alpha)}+\delta}$;

(ii) *there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $f'(z) \in H^{\frac{1}{3-2\alpha}+\varepsilon}$

Proof of (ii). By (13), we have the relation

$$f'(z) = \frac{f(z)}{z} P(z),$$

where $\Re P(z) > 0$, and by part (i), $f(z)/z \in H^{\frac{1}{2(1-\alpha)}+\delta}$ for an appropriate $\delta > 0$. Writing

$$\frac{1}{3-2\alpha} + \varepsilon = \lambda$$

(where $\varepsilon < \delta$ is to be chosen later) and using Hölder's inequality with conjugate indices p and q, where p is defined by the condition

$$p\lambda = \frac{1}{2(1-\alpha)} + \delta$$

(note that $p > 1$), we obtain the inequality

$$M_\lambda^\lambda(f', r) \leq M_{p\lambda}^\lambda\left(\frac{f}{z}, r\right) M_{q\lambda}^\lambda(P, r) .$$

On the right-hand side, the first expression is bounded because of the choice of p. By Lemma B, the second expression is bounded, provided $q\lambda < 1$. A calculation shows that with $\varepsilon < \delta$, we have the relations

$$q\lambda = 1 - 4(1-\alpha)^2\delta + (3-2\alpha)\varepsilon + O(\delta^2) < 1 ,$$

provided that first δ and then ε are chosen sufficiently small.

4. CLOSE-TO-CONVEX FUNCTIONS

A function f(z) analytic for $|z| < 1$ is called close-to-convex [7] if

$$f'(z) = \frac{g(z)}{z} P(z), \tag{15}$$

where g(z) is starlike and $\Re P(z) > 0$. Close-to-convex functions are univalent, and the class of close-to-convex functions contains the class of starlike functions. There is no loss of generality in assuming $P(0) = e^{i\alpha}$, where α is real and $\cos\alpha > 0$. Defining

$$Q(z) = P(z)\sec\alpha - i\tan\alpha ,$$

we then have the relations $\Re Q(z) > 0$, $Q(0) = 1$, and

$$f'(z) = \frac{g(z)}{z}[Q(z)\cos\alpha + i\sin\alpha] .$$

THEOREM 6. *If* f(z) *is close-to-convex and* f'(z) *is not of the form* $a(1 - ze^{i\tau})^{-2} P(z)$, *where* $\Re P(z) > 0$, *then*

(i) *there exists* $\delta = \delta(f) > 0$ *such that* $f'(z) \in H^{1/3+\delta}$;

(ii) *there exists* $\varepsilon = \varepsilon(f) > 0$ *such that* $f(z) \in H^{1/2+\varepsilon}$.

Proof. Statement (ii) follows from (i) by Theorem E. To prove (i), we use a technique essentially the same as that used in the proof of part (ii) of the previous theorem. By (15), we have that f'(z) is of the form

$$f'(z) = \frac{g(z)}{z} P(z),$$

where $g(z)/z \in H^{1/2+\varepsilon}$ for a suitable $\varepsilon > 0$, by Theorem 5. Applying Hölder's inequality with conjugate indices p and q, where $p = (1/2 + \varepsilon)/(1/3 + \delta)$ and $\delta < \varepsilon$, we find that

$$\int_{-\pi}^{\pi} |f'(z)|^{1/3+\delta} d\theta \leq M_{p(1/3+\delta)}^{1/3+\delta}\left(\frac{g}{z}, r\right) M_{q(1/3+\delta)}^{1/3+\delta}(P, r).$$

The first expression on the right-hand side is bounded, by the choice of p. By Lemma B, the second expression is bounded provided $q(1/3 + \delta) < 1$, that is, provided we further choose δ such that $\delta < 4\varepsilon/(9 + 6\varepsilon)$.

5. THE VALUE OF $\lim |a_n|/n$

If f(z) is close-to-convex but is now of the form

$$f'(z) = a(1 - ze^{i\tau})^{-2} P(z), \tag{16}$$

then, again using Hölder's inequality and Lemmas A and B, we find that $f'(z) \in H^\lambda$ for all $\lambda < 1/3$. The fact that $f(z) \notin H^{1/2}$ (and *a fortiori* $f'(z) \notin H^{1/3}$), in general, is illustrated not only by the Koebe function $f(z) = az(1 - ze^{i\tau})^{-2}$ but, for example, by an f(z) defined by $f'(z) = (1 - z)^{-2}(1 + z^2)(1 - z^2)^{-1}$. In these and other obvious cases, however, $a_n/n \not\to 0$, and thus it seems natural to consider close-to-convex functions f(z) of the type defined by (16) and ask

a) is there an f(z) such that $a_n/n \to 0$ but $f(z) \notin H^{1/2}$?

b) what property of P(z) characterizes the value of $\lim |a_n|/n$?

An affirmative answer to the first question is contained in the following theorem.

THEOREM 7. *Let* $P(z) = (1 - z)^{-1}\left[\log 2 + \log \frac{1}{1 - z}\right]^{-1}$, *where the logarithm takes its principal value. Then* $\Re P(z) > 0$ *for* $|z| < 1$ *and, for a close-to-convex function* $f(z) = \sum a_n z^n$ *defined by the relation*

$$f'(z) = (1 - z)^{-2} P(z),$$

we have that $f(z) \notin H^{1/2}$ *although* $a_n/n \to 0$.

Proof. With $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$, we have the equation

$$\Re[1/P(e^{i\theta})] = (1 - \cos\theta)\log|\operatorname{cosec}\theta/2| - \sin\theta \cdot \arg(1 - e^{i\theta}).$$

The first term on the right is nonnegative. Since $\sin\theta$ and $\arg(1 - e^{i\theta})$ have opposite sign, the second term is also nonnegative. It follows from the minimum principle for harmonic functions that $\Re[1/P(z)] > 0$, and hence $\Re P(z) > 0$. Next, consider f(x), where x is real and positive. Integrating by parts, we find that

$$f(x) = \frac{1}{2}(1-x)^{-2}\left[\log 2 + \log\frac{1}{1-x}\right]^{-1} - \frac{1}{2}(\log 2)^{-1}$$

$$+ \int_0^x \frac{1}{2}(1-x)^{-3}\left[\log 2 + \log\frac{1}{1-x}\right]^{-2} dx$$

$$\geq \frac{1}{2}(1-x)^{-2}\left[\log 2 + \log\frac{1}{1-x}\right]^{-1} - \frac{1}{2}(\log 2)^{-1} \sim \frac{1}{2}(1-x)^{-2}\left[\log\frac{1}{1-x}\right]^{-1}$$

as $x \to 1$. Thus $\int_{-r}^{r} |f(x)|^{1/2}\,dx \to \infty$ as $r \to 1$. The Fejér-Riesz inequality [1]

$$\int_{-r}^{r} |f(x)|^{1/2}\,dx \leq A \int_{-\pi}^{\pi} |f(re^{i\theta})|^{1/2}\,d\theta$$

(where A is a constant) now implies that $f(z) \notin H^{1/2}$.

Finally, denoting by $P(r, \theta)$ the Poisson kernel $(1 - r^2)(1 - 2r\cos\theta + r^2)^{-1}$, writing $|\log 2 + \log 1/(1 - z)|^{-1} = w(z)$, and applying the Cauchy integral formula with $r = 1 - 1/n$, we find that

$$\left|\frac{a_n}{n}\right| = \left|\frac{1}{2\pi i}\int_{|z|=r} \frac{f'(z)}{n^2 z^n}\,dz\right| \leq \frac{r^{-n}}{2\pi}\int_{-\pi}^{\pi} P(r, \theta)w(re^{i\theta})\,d\theta\,.$$

Since $r^{-n} = O(1)$, it now remains to prove only that the latter integral tends to zero as $r \to 1$. Defining $w(1) = 0$, we find that $w(z)$ is continuous for $|z| \leq 1$; hence $w(re^{i\theta}) \to w(e^{i\theta})$ uniformly. By familiar properties of $P(r, \theta)$, we have

$$\lim_{r\to 1}\int_{-\pi}^{\pi} P(r, \theta)w(re^{i\theta})\,d\theta = \lim_{r\to 1}\int_{-\pi}^{\pi} P(r, \theta)w(e^{i\theta})\,d\theta = w(1) = 0\,.$$

The answer to question (b) is the following.

THEOREM 8. *If the close-to-convex function* $f(z) = \sum a_n z^n$ *is given by the condition that*

$$f'(z) = a(1 - ze^{-i\tau})^{-2}P(z),$$

where a *is complex,* τ *is real, and*

$$P(z) = Q(z)\cos\alpha + i\sin\alpha, \qquad Q(z) = \int_{-\pi}^{\pi}\frac{1 + ze^{-it}}{1 - ze^{-it}}\,d\mu(t)\,,$$

then

$$|a_n|/n \to |a|\cos\alpha\,[\mu(\tau + 0) - \mu(\tau - 0)]\,.$$

In particular, $|a_n|/n \to 0$ *if and only if* $\mu(t)$ *is continuous at* τ.

Proof. Let $Q(z) = \sum_0^\infty b_n z^n$ $(b_0 = 1)$, $\xi = ze^{-i\tau}$, and define $s_n = \sum_0^n b_k e^{ki\tau}$, $r_n = \sum_0^n s_k$. Then

$$f'(e^{i\tau}\xi) = a(1-\xi)^{-2} P(\xi e^{i\tau}) = a(1-\xi)^{-2}[Q(\xi e^{i\tau})\cos\alpha + i\sin\alpha]$$

$$= a\cos\alpha\,(1-\xi)^{-1}\sum_0^\infty s_n\xi^n + ai\sin\alpha\,(1-\xi)^{-2}$$

$$= a\cos\alpha\sum_0^\infty t_n\xi^n + ai\sin\alpha\sum_0^\infty (n+1)\xi^n.$$

Comparing coefficients of ξ^n, we find that

$$\left|\frac{a_n}{n}\right| = |a|\cos\alpha\left|\frac{t_{n-1}}{n^2}\right| + o(1)\,. \tag{17}$$

But $s_n/n \to 2[\mu(\lambda+0) - \mu(\lambda-0)]$ [13, p. 107, (9.3)], or

$$s_n = 2n[\mu(\lambda+0) - \mu(\lambda-0)] + n\varepsilon_n,$$

where $\varepsilon_n \to 0$. Summing this equation over n and dividing by n^2, we obtain the relation

$$\frac{t_n}{n^2} = (1+1/n)[\mu(\lambda+0) - \mu(\lambda-0)] + n^{-2}\sum_0^n k\varepsilon_k\,. \tag{18}$$

The last term tends to zero as $n \to \infty$, and the combination of (17) and (18) gives the desired result.

Added in proof (July 30, 1970). W. E. Kirwan has pointed out to the authors that if $f(z) \in S_0$, if $\beta \geq 0$ is the maximum discontinuity of the measure $\mu(t)$ associated with $f(z)$, and if $M(r) = \max_\theta |f(re^{i\theta})|$, then an application of the order relation

$$M(r) = O[(1-r)^{-2\beta-\varepsilon}]$$

(see [11, Theorem 1]) and of the inequality

$$M_\lambda^\lambda(f, r) \leq \lambda\int_0^r \rho^{-1} M^\lambda(\rho)\,d\rho$$

(see [9], for example) immediately shows that $f(z) \in H^\lambda$, for each $\lambda < 1/2\beta$. An application of Hölder's inequality then shows that $f'(z) \in H^\lambda$ for each $\lambda < 1/(1+2\beta)$. This remark permits the formulation of more precise forms of Theorems 4 and 5.

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