A BANACH SPACE OF LOCALLY UNIVALENT FUNCTIONS

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1. INTRODUCTION

In this paper, we study a certain real Banach space $\mathscr{L}$ of functions

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

that are holomorphic and locally univalent ($f'(z) \neq 0$) in the open unit disk $D = \{z: |z| < 1\}$. We let S denote the set of functions f(z) that are holomorphic and univalent in D with an expansion of the form (1.1). The algebraic operations in $\mathscr{L}$ (defined in Section 2 below) are not the usual pointwise operations, and the algebraic structure of $\mathscr{L}$ is of particular interest in relation to S, because local univalence in D is preserved by the addition in $\mathscr{L}$. We also study a certain closed subspace of $\mathscr{L}$, denoted by $\mathscr{L}_1$. Our spaces $\mathscr{L}$ and $\mathscr{L}_1$ are natural generalizations of a space introduced by H. Hornich [11].

The main results in this paper pertain to the metric properties of the sets $S \cap \mathscr{L}$ and $S \cap \mathscr{L}_1$. The set $S \cap \mathscr{L}$ is not compact, and it is of first category in $\mathscr{L}$. We also show that there are no isolated univalent functions in $\mathscr{L}_1$. These results contrast sharply with theorems of H. Hornich [11] and G. Piranian [16]. Hornich [10] and Piranian [16] have studied topological properties of the set of univalent functions in the space $H(\phi)$ of functions f(z) holomorphic in D, equipped with pointwise operations and a metric $\rho(f, g) = \phi(f - g)$ induced by the functional

$$\phi(f) = \sup_n |f^{(n)}(0)/n!|^{1/n}.$$

We also show that $\mathscr{L}_1$ is separable and has infinite dimension. We show that $\mathscr{K}$, the set of univalent convex functions of the form (1.1), is a closed convex subset of $\mathscr{L}_1$. A complete characterization of the extreme points of $\mathscr{K}$ is given. We determine the dual space of continuous linear functionals on $\mathscr{L}_1$. We list examples and results that indicate the relationship of $\mathscr{L}$ (as a set of functions) to the Hardy spaces H^p and to the set of functions holomorphic on D and continuous on the closure of D.

2. THE LINEAR SPACES $\mathscr{L}$ AND $\mathscr{L}_1$

Let Λ denote the class of functions that are holomorphic in the unit disk, have nonvanishing derivative, and satisfy the normalization conditions $f(0) = 0$ and $f'(0) = 1$. When we refer to a class of functions, we shall mean the intersection of that class with Λ. For each f in Λ, define the increasing function

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$$\mathscr{I}(r; f) = \frac{1}{2\pi} \int_0^{2\pi} \left| \log |f'(re^{it})| \right| dt .$$

We define $\mathscr{L}$ to be the set of all f in Λ for which $\sup\{\mathscr{I}(r; f): 0 \leq r < 1\}$ is finite. We consider $\mathscr{L}$ as a real normed linear space with the operations

$$[f+g](z) = \int_0^z f'(\zeta) g'(\zeta) d\zeta \qquad (f, g \in \mathscr{L}),$$

$$[\alpha f](z) = \int_0^z (f'(\zeta))^\alpha d\zeta \qquad (f \in \mathscr{L}, \ \alpha \text{ real})$$

(square brackets will indicate the algebraic operations in $\mathscr{L}$) and with the norm

$$\|f\| = \sup_{r<1} \mathscr{I}(r; f) \qquad (f \in \mathscr{L}). \tag{2.1}$$

There are many equivalent conditions that describe $\mathscr{L}$, and we list three of these [17, pp. 37-40]:

(2.2a) $\left| \log |f'(z)| \right|$ has a harmonic majorant,

(2.2b) $\log |f'(z)|$ can be expressed as the difference of two positive harmonic functions,

(2.2c) there exists a function m(t) of bounded variation on $0 \leq t \leq 2\pi$ such that

$$\log |f'(z)| = \int_0^{2\pi} \Re\{(e^{it}+z)/(e^{it}-z)\} dm(t) \qquad (z \in D), \tag{2.3}$$

where $\int_0^{2\pi} dm(t) = 0$ and $\int_0^{2\pi} |dm(t)| = \|f\|$.

We shall use the notation

$$C(z, t) = (e^{it}+z)/(e^{it}-z)$$

and

$$P(z, t) = \Re\{C(z, t)\}$$

for the Cauchy and Poisson kernels. We shall refer to dm of (2.3) as the *measure for* f.

If $\mathscr{H} = \{f \in \Lambda: \sup |\arg f'(z)| < \infty, \ z \in D\}$, then $\mathscr{H}$ is a linear subspace of $\mathscr{L}$. With the norm

$$\|f\|_{\mathscr{H}} = \sup\{|\arg f'(z_1) - \arg f'(z_2)|: z_1, z_2 \in D\},$$

$\mathscr{H}$ is a Banach space [11]. This space was introduced by Hornich in his paper [11]. A function $f(z) \in \Lambda$ belongs to $\mathscr{H}$ if and only if $\Im \log f'(z)$ is in the space h_∞ of bounded harmonic functions in D. By the theorem of M. Riesz on conjugate functions

[18, p. 346], we see if $f \in \mathscr{H}$ and $F(z) = \log f'(z)$, then $\Re F(z)$ belongs to each class h_p, for $0 < p < \infty$. Hence F is in each Hardy space H^p $(0 < p < \infty)$. It is natural to consider the extensions of $\mathscr{H}$ that consist of the sets

$$\mathscr{L}^q = \{f \in \Lambda: \log|f'(z)| \in h_q\} \qquad (q > 0)$$

whose structure as linear spaces is the same as that of $\mathscr{H}$. We have restricted our attention to the space $\mathscr{L} = \mathscr{L}^1$, because this is the largest of the extensions $\mathscr{L}^q$ that is clearly related to the Hardy spaces H^p. For if $f \in \mathscr{L}$, then $\log f' \in H^p$ $(0 < p < 1)$ [13], but there exist functions $f \in \mathscr{L}^q$ $(0 < q < 1)$ such that $\log f' \notin H^p$ for all $p > 0$ [8].

We have the following characterization of functions in $\mathscr{H}$. If $f \in \mathscr{H}$ and

$$\pi K/2 \geq \sup|\arg f'(z)| \qquad (z \in D),$$

then the function $Q(z) = \exp\{K^{-1}\log f'(z)\}$ is holomorphic, has positive real part in D, and satisfies the condition $Q(0) = 1$. The function

$$\omega(z) = (Q(z) - 1)/(Q(z) + 1)$$

is a bounded holomorphic function satisfying the hypotheses of Schwarz's lemma. Thus f belongs to $\mathscr{H}$ if and only if there exist a positive number K and a holomorphic function $\omega(z)$ $(|\omega(z)| \leq 1,\ \omega(0) = 0)$ such that

$$f(z) = \int_0^z \left(\frac{1 + \omega(\zeta)}{1 - \omega(\zeta)}\right)^K d\zeta. \tag{2.4}$$

We shall define a certain subspace of $\mathscr{L}$ that contains $\mathscr{H}$. First, we recall that for each $f \in \mathscr{L}$, $\log f'$ is in H^p $(p < 1)$, the boundary function $\log f'(e^{it})$ exists a.e. and belongs to L^p $(p < 1)$ [21, p. 272], and $|\log f'(e^{it})| \in L^1[0, 2\pi]$ (by (2.2c) and [17, p. 34]). In fact, the measure dm of (2.3) can be written in the form

$$dm = \log|f'(e^{it})|\,\frac{dt}{2\pi} + ds,$$

where ds is a singular (real) measure. Since $\log|f'(e^{it})|$ is the radial limit a.e. of $\log|f'(re^{it})|$, an application of Fatou's lemma yields the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} \big|\log|f'(e^{it})|\big|\,dt \leq \|f\| \qquad (f \in \mathscr{L}). \tag{2.5}$$

We obtain the Jordan decomposition of dm by taking the Jordan decomposition $ds = ds^+ - ds^-$ and writing

$$dm = \left(\log^+|f'(e^{it})|\,\frac{dt}{2\pi} + ds^+\right) - (\log^+|f'(e^{it})|^{-1} + ds^-).$$

Thus equality holds in (2.5) if and only if dm is absolutely continuous with respect to Lebesgue measure. We define $\mathscr{L}_1$ to be the set of functions f in $\mathscr{L}$ for which

$$\|f\| = \frac{1}{2\pi}\int_0^{2\pi} |\log|f'(e^{it})||\,dt.$$

By means of the representation (2.3) and an application of Fubini's theorem, one can show that $\mathscr{L}_1$ is a subspace of $\mathscr{L}$.

3. THE STRUCTURE OF $\mathscr{L}$ AND $\mathscr{L}_1$

In this section, we discuss the topological properties of $\mathscr{L}$ and $\mathscr{L}_1$. First we note that the set of functions

$$q_n(z) = \int_0^z \exp \zeta^n d\zeta \qquad (n = 1, 2, \cdots)$$

is linearly independent in $\mathscr{L}_1$, since the equation

$$x = \sum_{n=1}^{k} [\alpha_n q_n](x) = \int_0^x \exp\left\{\sum_{n=1}^{k} \alpha_n t^n\right\} dt$$

does not hold for all $x \in (0, 1)$, except if $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$. Therefore $\mathscr{L}_1$ is infinite-dimensional.

THEOREM 3.1. *$\mathscr{L}_1$ is a Banach space.*

Proof. We have already observed that $\mathscr{L}_1$ is a linear subspace of $\mathscr{L}$. If $\{f_n\}$ is a Cauchy sequence in $\mathscr{L}_1$, then $\{\log|f_n'(e^{it})|\}$ is a Cauchy sequence in $L^1[0, 2\pi]$. This sequence converges in the L^1-norm to an L^1-function that we denote by $\log|f'(e^{it})|$. We define $f(z)$ by the equation

$$\log f'(z) = \frac{1}{2\pi}\int_0^{2\pi} C(z, t)\log|f'(e^{it})|\,dt.$$

Then the sequence $\{f_n\}$ converges to f in the $\mathscr{L}$-norm, and by Fubini's theorem, f is in $\mathscr{L}_1$.

THEOREM 3.2. *$\mathscr{L}_1$ is separable.*

Proof. If f belongs to $\mathscr{L}_1$, then the Cesàro means $\sigma_n(t)$ of the Fourier series for $\log|f'(e^{it})|$ converge in $L^1[0, 2\pi]$ to $\log|f'(e^{it})|$ [9, p. 16]. If we define the functions

$$\Sigma_n(z) = \int_0^{2\pi} C(z, t)\,\sigma_n(t)\,\frac{dt}{2\pi} \qquad (z \in D),$$

then the sequence of functions

$$f_n(z) = \int_0^z \exp\{\Sigma_n(\zeta)\}\,d\zeta \qquad (n = 1, 2, \cdots)$$

converges in the $\mathscr{L}_1$-norm to f. The f_n belong to $\mathscr{L}_1$, since the Σ_n are polynomials $(n = 1, 2, \cdots)$. Each σ_n can be uniformly approximated on $[0, 2\pi]$ by a trigonometric polynomial with complex rational coefficients. This completes the proof of the theorem.

To see that the larger space $\mathscr{L}$ is not separable, let f_1 and f_2 be defined by

$$\log f_j'(z) = C(z, t_j) \qquad (j = 1, 2),$$

where t_j is real $(j = 1, 2)$. If $t_1 \not\equiv t_2 \pmod{2\pi}$, then

$$\|[f_1 - f_2]\| = \int_0^{2\pi} |d(m_1 - m_2)(t)| = 2,$$

and the conclusion follows.

THEOREM 3.3. *$\mathscr{L}$ is complete.*

Proof. Let $\{f_k\}$ be a Cauchy sequence in $\mathscr{L}$. Then the representation

$$\log f_k'(z) = \frac{1}{2\pi}\int_0^{2\pi} C(z, t)\,dm_k(t) \qquad (z \in D)$$

shows that $\{\log f_k'(z)\}$ is a normal family in D. There is a subsequence of $\{\log f_k'(z)\}$ that converges locally uniformly in D to a holomorphic function G(z), with $G(0) = 1$. If we define the function

$$g(z) = \int_0^z \exp G(\zeta)\,d\zeta,$$

then this subsequence converges normally to $\log g'(z)$. Furthermore, g(z) is an element of $\mathscr{L}$, since the sequence $\{\|f_k\|\}$ is bounded. The inequality

$$\frac{1}{2\pi}\int_0^{2\pi} \left|\log|f_k'(re^{it})| - \log|f_n'(re^{it})|\right| dt < \varepsilon,$$

which holds for all $r < 1$ and for $k, n \geq N$, shows that the full sequence $\{f_k\}$ converges to g in the $\mathscr{L}$-topology. This completes the proof of Theorem 3.3.

In his study of the space $H(\phi)$, Hornich demonstrated that the space has an uncountable number of components and is not locally connected at any point [10]. The present authors [2, Theorem 5.1] have shown that the space $H(\phi)$ can not be normed. Of course, the linear spaces $\mathscr{L}$ and $\mathscr{L}_1$ have no such pathological properties.

4. SOME FUNCTION CLASSES RELATED TO $\mathscr{L}$

We let $\mathscr{A}$ denote the set of functions $f \in \Lambda$ that belong to the disk algebra of functions holomorphic on D and continuous on $\overline{D}$. We denote by $\mathscr{A}^*$ the subset of functions in $\mathscr{A}$ that are absolutely continuous on $|z| = 1$. In this section, we indicate how the classes of functions $\mathscr{L}$, $\mathscr{L}_1$, and $\mathscr{H}$ overlap with $\mathscr{A}$, $\mathscr{A}^*$, the Hardy spaces H^p, and certain classes of univalent functions. Some of our observations are known results, but we include them for the sake of completeness.

THEOREM 4.1. $\mathscr{A}^* \subset \mathscr{L}$.

Proof. If f(z) belongs to $\mathscr{A}^*$, then f'(z) is in H^1 [6, p. 360]. The H^1-function f'(z) has a canonical factorization

$$f'(z) = S(z)F(z),$$

where S(z) is singular and F(z) is an outer function. The inequalities

$$\frac{1}{2\pi}\int_0^{2\pi} |\log|S||\,\frac{d\theta}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{2\pi} P(z,t)\,d\mu(t)\right|\frac{d\theta}{2\pi} \leq 1$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} |\log|F||\,\frac{d\theta}{2\pi} = \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^{2\pi} P(z,t)\log|f'(e^{it})|\,dt\right|\frac{d\theta}{2\pi}$$

$$\leq \|\log|f'(e^{it})|\,\|_1 < \infty$$

show that f is in $\mathscr{L}$. The proof of Theorem 4.1 is complete.

If $f \in S$, then the functions $f_r(z) = f(rz)/r$ $(0 < r < 1)$ are univalent and holomorphic in the closed disk $\overline{D}$, and therefore they belong to the subspace $\mathscr{H}$. Thus $\mathscr{H}$ contains a set of functions that is dense in S in the topology of uniform convergence on compacta. However, an example constructed by Lohwater, Piranian, and Rudin [14] shows that neither S nor $\mathscr{A}$ is a subset of the space $\mathscr{L}$. Their example is a function $f(z) \in \Lambda$ that is both continuous and univalent in $\overline{D}$, but for which the radial limits $\lim_{r\to 1} f'(re^{i\theta})$ exist almost nowhere.

In view of the example of Lohwater, Piranian, and Rudin, it is natural to inquire whether $f(z) \in S$ implies that $\log|f'(z)| \in h_p$ for some $p < 1$. A negative answer is provided by the following example due to P. L Duren (private communication). Consider the functions $f_t(z) \in \Lambda$ $(t \in [0, 1])$ defined by the equation

$$f_t'(z) = \exp\{\alpha g_t(z)\}, \tag{4.1}$$

where $g_t(z) = \sum_{n=1}^{\infty} \phi_n(t) a_n z^n$, $\phi_n(t)$ is the nth Rademacher function, $a_n = 1$ if $n = 2^k$ $(k = 1, 2, \cdots)$ and $a_n = 0$ otherwise, and α is a complex number. It is elementary to verify that

$$|f_t''(z)/f_t'(z)| \leq |\alpha| \sum_{k=1}^{\infty} 2^k |z|^{2^k - 1} < 8|\alpha|/(1 - |z|^2),$$

and hence, by a result of Duren, Shapiro, and Shields ([5, Theorem 2]), $f_t(z)$ is univalent if $|\alpha| \leq (\sqrt{5} - 2)/4$. For almost every $t \in [0, 1]$, the function $g_t(z)$ has a radial limit almost nowhere ([3, Lemma]), and consequently, for almost every $t \in [0, 1]$, either the real or imaginary part of $g_t(z)$ cannot belong to h_p for any $p > 0$. Thus for $|\alpha| \leq (\sqrt{5} - 2)/4$ and appropriate choices of arg α, almost every choice of t gives a function $f_t(z)$ in (4.1) that belongs to S with the property that $\log|f_t'(z)| \notin h_p$ for all $p > 0$.

Some of the familiar subclasses of S are contained in $\mathscr{L}$.

THEOREM 4.2. *The close-to-convex functions constitute a subset of* $\mathscr{H}$, *and the spiral-like functions belong to* $\mathscr{L}$.

Proof. A function $f(z) = z + a_2 z^2 + \cdots$ holomorphic in D is said to be close-to-convex if there exist a convex function $h(z) \in \Lambda$ and a holomorphic function $P(z) = 1 + p_1 z + \cdots$ with positive real part in D such that $f'(z) = e^{i\alpha} h'(z) P(z)$ for some real α [12]. It is well known that convex functions satisfy the inequality

$$|\arg h'(z)| \leq 2 \arcsin |z| \qquad (z \in D)$$

[1]. Hence $|\arg f'(z)| \leq \alpha + 3\pi/2$, and f belongs to $\mathscr{H}$. In particular, $\mathscr{H}$ contains the class of starlike functions and the class of convex functions.

A function $f(z) = z + a_2 z^2 + \cdots$ holomorphic in D is said to be spiral-like if there exist a holomorphic function P(z) (P(0) = 1) with positive real part and a real number β ($|\beta| < \pi/2$) such that

$$zf'(z)/f(z) = e^{-i\beta}[(\cos\beta) P(z) + i \sin\beta] \qquad (z \in D)$$

[19]. By the Herglotz theorem [9, p. 67], there exists a nondecreasing function m(t) with variation 1 on $[0, 2\pi]$ such that

$$P(z) = \int_0^{2\pi} C(z, t)\, dm(t) \qquad (z \in D).$$

A straightforward calculation yields the formula

$$f(z) = z \exp\left\{-2\gamma \int_0^{2\pi} \log(1 - ze^{-it})\, dm(t)\right\},$$

where $\gamma = e^{-i\beta}\cos\beta$. Finally, we see that

$$\begin{aligned} \log|f'(z)| &= -2\cos^2\beta \int_0^{2\pi} \log|1 - ze^{-it}|\, dm(t) \\ &\quad - \sin 2\beta \int_0^{2\pi} \arg(1 - ze^{-it})\, dm(t) + \log|i\sin\beta + (\cos\beta) P(z)| . \end{aligned} \tag{4.2}$$

The function $i \sin\beta + (\cos\beta) P(z)$ has positive real part in D, and this implies that $\arg(i\sin\beta + (\cos\beta)P(z))$ belongs to h_∞ and $\log|i\sin\beta + (\cos\beta)P(z)|$ belongs to h_p for all $0 < p < \infty$. The second term in (4.2) is bounded. For the first term in (4.2), we use the elementary estimate

$$\int_0^{2\pi} \left| \int_0^{2\pi} \log|1 - ze^{-it}|\, dm(t)\right| d\theta \leq 4\pi \log 2 \qquad (z = re^{i\theta}). \tag{4.3}$$

It implies that $\sup_{r<1} \mathcal{I}(r; f) < \infty$ and f belongs to $\mathcal{L}$. This completes the proof of Theorem 4.2.

One sees easily that $\sup\{|\arg f'(z)| : z \in D\}$ may not be finite for spiral-like f(z); but with an argument similar to that above, we can show that $\arg f'(z) \in h_1$.

We observe that $\mathcal{H}$ is not a subset of H^∞ (consider $-\log(1 - z)$).

THEOREM 4.3. *For each* $f \in \mathcal{H}$, *there exists a* $p > 0$ *such that* $f \in H^p$.

Proof. Let f belong to $\mathcal{H}$. Then, by (2.4), f'(z) is subordinate to the function

$$G(z) = \left(\frac{1+z}{1-z}\right)^K$$

for some $K > 0$. It is easy to verify that G belongs to H^p for all p $(0 < p < 1/K)$. By subordination, $f'(z) \in H^p$ $(0 < p < 1/K)$. Our result follows from a theorem of Hardy and Littlewood [7, p. 415], which implies that if $f' \in H^p$ for some $p < 1$, then $f \in H^q$ for $q = p/(1 - p)$ (see also [4]).

Finally, we note that $\mathcal{L}$ contains functions that belong to none of the H^p-spaces. Indeed, the function

$$f(z) = \int_0^z \exp\{(1+\zeta)/(1-\zeta)\}\, d\zeta$$

belongs to $\mathcal{L}$, since $\log|f'(z)|$ is merely the Poisson kernel; but its growth along the segment $z = x$ $(0 < x < 1)$ shows that it cannot belong to any H^p-class, because if $\alpha > 0$ and $x > 0$, then

$$|f(x)|(1-x)^\alpha = (1-x)^\alpha \int_0^x \exp\{(1+t)/(1-t)\}\, dt$$

$$\geq (1-x)^\alpha \int_{x^2}^x \exp\{1/(1-t)\}\, dt \geq x(1-x)^{\alpha+1} \exp\{1/(1-x^2)\}.$$

5. METRIC PROPERTIES OF THE SET OF UNIVALENT FUNCTIONS IN $\mathcal{L}$

We let $S^* = S \cap \mathcal{L}$ and $S_1^* = S \cap \mathcal{L}_1$ denote the sets of normalized univalent functions in $\mathcal{L}$ and in $\mathcal{L}_1$, respectively.

THEOREM 5.1. S^* *is a closed subset of* $\mathcal{L}$.

Proof. Let $\{f_n\}$ be a sequence in S^* that converges in the topology of $\mathcal{L}$ to a function f(z). From the proof of Theorem 3.3, we see that some subsequence of $\{f_n(z)\}$ converges uniformly on compacta to f(z). Thus, by Hurwitz's theorem, f(z) is univalent.

Let S_n^* $(n > 1)$ denote the class of n-valent functions f(z) in $\mathcal{L}$. That is, let S_n^* consist of all functions in $\mathcal{L}$ that assume some value n times in D but assume no value more than n times.

THEOREM 5.2. S_n^* $(n > 1)$ *is a closed subset of* $\mathcal{L}$.

The proof of Theorem 5.2 is the same as the proof of Theorem 5.1, and we omit it.

Remark. The function $2^{-1}\{1 - \exp(2z/(z-1))\}$ belongs to $\mathscr{L}$, but it is not in S_n^* for any n.

In addition to proving that the set of univalent functions in $\mathscr{H}$ is a closed subset of $\mathscr{H}$ [11], Hornich notes that the ball

$$\{f \in \mathscr{H}\colon \|f\|_{\mathscr{H}} \leq \pi\}$$

contains only univalent functions. The latter fact is equivalent to the result that if $f \in \mathscr{H}$ has a representation (2.4) with $K \leq 1$, then $\Re f'(z) \geq 0$ in D, and hence f(z) is univalent [20].

THEOREM 5.3. S^* *is nowhere dense in* $\mathscr{L}$.

Proof. Let $f \in S^*$, and let $F(z) = \log f'(z)$; then a necessary condition for f(z) to be univalent is

$$(5.1) \qquad |F'(z)| = |f''(z)/f'(z)| \leq 6/(1 - |z|^2)$$

([5, Theorem 2]). We shall use (5.1) to show that every ε-ball centered at f contains nonunivalent functions.

Suppose $\varepsilon > 0$, and define the functions g(z) and G(z) by the condition

$$G(z) = \log g'(z) = \frac{\varepsilon\, 2z}{1 - z^2} = \frac{\varepsilon}{2}\frac{1+z}{1-z} - \frac{\varepsilon}{2}\frac{1-z}{1+z}\,.$$

The function g(z) belongs to $\mathscr{L}$, since $\Re G(z)$ is expressible as the difference of two positive harmonic functions: $\Re G(z) = U_1(z) - U_2(z)$. Furthermore, $\|g\| = U_1(0) + U_2(0) = \varepsilon$ and

$$(5.2) \qquad (1 - |z|)|G'(z)| = \frac{\varepsilon\, 2|1+z^2|}{|1-z|\,|1+z|}\,.$$

If we let $H(z) = F(z) + G(z)$ and $\log h'(z) = H(z)$, then $h \in \mathscr{L}$ and $\|[h - f]\| = \|g\| = \varepsilon$. The function h(z) is not univalent, since the relations $H'(z) = F'(z) + G'(z)$, (5.1), and (5.2) imply that H'(z) does not satisfy the necessary order condition (5.1). This completes the proof of the theorem.

THEOREM 5.4. S_1^* *has no isolated points.*

Proof. Let f(z) belong to S_1^*, and for each real number t, let $f_t(z) = e^{-it}f(ze^{it})$. Each function $f_t(z)$ belongs to S_1^*, and

$$\|[f - f_t]\| = \sup_{r<1}\int_0^{2\pi}\left|\log|f'(re^{i\theta})| - \log|f'(re^{i(\theta+t)})|\right|\frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi}\left|\log|f'(e^{i\theta})| - \log|f'(e^{i(\theta+t)})|\right|\frac{d\theta}{2\pi}\,,$$

since $|\log|f'(z)| - \log|f_t'(z)||$ is subharmonic in D. The function $\log|f'(e^{i\theta})|$ belongs to $L^1[0, 2\pi]$, and hence $\lim_{t\to 0} \|[f - f_t]\| = 0$. Thus every point $f \in S_1^*$ is an accumulation point of S_1^*.

G. Piranian [16] has constructed an isolated univalent function in the space $H(\phi)$. It would be interesting to determine whether there are any isolated schlicht functions in the whole space $\mathscr{L}$. It would also be of interest to determine when the metric topology on a linear space of holomorphic functions admits isolated univalent functions. This question was first raised by G. Piranian (oral communication to one of the authors).

THEOREM 5.5. S^* *is not compact.*

Proof. Let h(z) be any function in S that does not belong to $\mathscr{L}$, for example, a function defined by (4.1). Then the set

$$\{h_n: h_n(z) = h(r_n z)/r_n,\ r_n = n/(n+1),\ n = 1, 2, \cdots\}$$

is a sequence in $S_1^* \subset S^*$ that can have no convergent subsequence (in the $\mathscr{L}$-metric), since convergence in $\mathscr{L}$ implies local uniform convergence in D.

6. CONVEX UNIVALENT FUNCTIONS

Let $\mathscr{K}$ denote the set of convex functions in S. We have already seen that $\mathscr{K}$ is a subset of $\mathscr{H}$. In this section, we show that $\mathscr{K}$ is a convex set in the linear space $\mathscr{L}$, and we determine the extreme points of $\mathscr{K}$. We also show that $\mathscr{L}_1$ is essentially determined by $\mathscr{K}$ (Theorem 6.3).

It is well known that a function $f(z) \in \Lambda$ belongs to $\mathscr{K}$ if and only if

$$\Re\{1 + zf''(z)/f'(z)\} \geq 0 \qquad (z \in D). \tag{6.1}$$

Another familiar characterization of $\mathscr{K}$ follows from (6.1) and the Herglotz integral formula. Namely, a function $f(z) \in \Lambda$ belongs to $\mathscr{K}$ if and only if there is a nondecreasing function m(t) on $[0, 2\pi]$ with variation 1 such that

$$\log f'(z) = -2\int_0^{2\pi} \log(1 - ze^{-it})\,dm(t) \qquad (z \in D). \tag{6.2}$$

If we relax the restriction that the measure in (6.2) be positive by permitting m(t) to be any function of bounded variation, then (6.2) is the structural formula that characterizes V, the class of functions with bounded boundary rotation [15]. In most applications, one studies the subclasses V_α $(\alpha \geq 1)$ of functions f(z) in V with associated measures dm (in (6.2)) of total variation at most α. It is interesting to note that V is contained in the linear span of $\mathscr{K}$ in the subspace $\mathscr{H}$. For if m(t) is a function of bounded variation, then m(t) can be written in the form $m(t) = \alpha_1 m_1(t) - \alpha_2 m_2(t)$, where the $m_k(t)$ $(k = 1, 2)$ are nondecreasing with total variation 1 on $[0, 2\pi]$ and $\alpha_k \geq 0$ $(k = 1, 2)$. If we define $f_k(z)$ $(k = 1, 2)$ to be the convex functions corresponding to $m_k(t)$ $(k = 1, 2)$ in (6.2), then $f_k \in \mathscr{H}$ $(k = 1, 2)$, and $f = [[\alpha_1 f_1] - [\alpha_2 f_2]]$.

We also mention that (4.3) and (6.2) show that $\mathscr{K}$ is contained in the ball of radius log 16. Similar results hold for each of the classes V_α.

THEOREM 6.1. *$\mathscr{K}$ is a closed convex subset of $\mathscr{H}$.*

Proof. To prove that $\mathscr{K}$ is closed, we use the characterization (6.1) and the fact that sequential convergence in the $\mathscr{L}$-metric implies local uniform convergence in D. The convexity of $\mathscr{K}$ follows easily from (6.1) (or (6.2)) and the definitions of the algebraic operations in $\mathscr{L}$. We leave the details to the reader.

It is well known that the coefficient problem, the distortion theorems, and similar classical extremal problems for convex functions are solved by functions (6.2) for which m(t) is an appropriate step function. Our next theorem shows that the step functions m(t) also play a role as extremals in $\mathscr{K}$ as a subset of the linear space $\mathscr{L}$.

THEOREM 6.2. *A convex function* f(z) *is an extreme point of the convex set $\mathscr{K}$ in $\mathscr{L}$ if and only if the support of the associated measure* dm *in* (6.2) *consists of one point.*

Proof. Let $m(t) = 0$ if $0 \leq t < t_0$, and let $m(t) = 1$ if $t_0 < t \leq 2\pi$. If the corresponding function f(z) in (6.2) is not an extreme point of $\mathscr{K}$, then there exist functions f_1 and f_2 in $\mathscr{K}$ such that $f = [(1/2)[f_1 + f_2]]$. Thus, if dm_k (k = 1, 2) is the measure associated with f_k (k = 1, 2), then

$$-2\log(1 - ze^{-it_0}) = -\int_0^{2\pi} \log(1 - ze^{-it})\, d(m_1 + m_2).$$

From the power series expansions, we obtain the relations

$$\cos nt_0 = \int_0^{2\pi} \cos nt\, d\left(\frac{m_1 + m_2}{2}\right) \qquad (n = 0, \pm 1, \cdots)$$

and

$$\sin nt_0 = \int_0^{2\pi} \sin nt\, d\left(\frac{m_1 + m_2}{2}\right) \qquad (n = 0, \pm 1, \cdots).$$

Therefore $2^{-1}\, d(m_1 + m_2)$ is the point mass at $t = t_0$. It follows that $f_1 = f_2$, since both dm_1 and dm_2 must be the one-point mass at $t = t_0$.

To prove the converse, we assume that f(z) in (6.2) is an extreme point of $\mathscr{K}$. If dm is not a single point mass, then there are at least two points in the support of dm, say t_j ($j = 1, 2$; $0 \leq t_1 < t_2 < 2\pi$). Let t_0 ($t_1 < t_0 < t_2$) be a point of continuity for the monotone function m(t). We may assume that $m(t) = m(t-)$ throughout the interval $[0, 2\pi]$ and that $m(0) = 0$. We let $\alpha = m(t_0)$, and we define the functions

$$m_1(t) = \begin{cases} \alpha^{-1} m(t) & (0 \leq t \leq t_0), \\ 1 & (t_0 < t \leq 2\pi) \end{cases}$$

and

$$m_2(t) = \begin{cases} 0 & (0 \leq t \leq t_0), \\ (1 - \alpha)^{-1} m(t) & (t_0 < t \leq 2\pi). \end{cases}$$

Since dm has support at t_1 and t_2, we see that $0 < \alpha < 1$. If $f_j(z)$ $(j = 1, 2)$ is the convex function corresponding to $m_j(t)$ $(j = 1, 2)$ in (6.2), then

$$f = [[\alpha f_1] + [(1 - \alpha)f_2]]$$

is a convex combination of the f_j $(j = 1, 2)$. This contradiction shows that dm cannot have support at more than one point.

THEOREM 6.3. *The set* $\{[\alpha f]: f \in \mathscr{K},\ \alpha \text{ real}\}$ *is dense in* $\mathscr{L}_1$.

Proof. Let $g(z)$ belong to $\mathscr{L}_1$, and let $\varepsilon > 0$. Then, by Theorem 3.2, there exists a polynomial $\Sigma(z)$ such that $\|[g - \sigma]\| < \varepsilon$, where σ denotes the function

$$\sigma(z) = \int_0^z \exp(\Sigma(\zeta))\,d\zeta .$$

We let $M = \max_{|z| \le 1} |\Sigma'(z)|$, and we define the function

$$f(z) = [M^{-1}\sigma](z) = \int_0^z \exp(M^{-1}\Sigma(\zeta))\,d\zeta .$$

Then $\|[g - [Mf]]\| = \|[g - \sigma]\| < \varepsilon$, and f belongs to $\mathscr{K}$ since

$$\Re\{1 + zf''(z)/f'(z)\} = \Re\{1 + z\Sigma'(z)/M\} > 0 .$$

This completes the proof of the theorem.

COROLLARY 6.1. *Every continuous linear functional on* $\mathscr{L}_1$ *is determined by its values on* $\mathscr{K}$.

Proof. If $\Sigma(\zeta)$ is a polynomial and if $f(z) = \int_0^z \exp(\Sigma(\zeta))\,d\zeta$, then $[\alpha f] \in \mathscr{K}$ for all sufficiently small $\alpha > 0$. By Theorems 3.2 and 6.3, the equation $\alpha T(f) = T([\alpha f])$ determines T on a dense subset of $\mathscr{L}_1$.

7. THE DUAL OF $\mathscr{L}_1$

Corollary 6.1 gives some information about the space of continuous linear functionals on $\mathscr{L}_1$. We shall give a more complete description of the dual of $\mathscr{L}_1$ in Theorem 7.1.

Let M be the subspace of $L^1[0, 2\pi]$ corresponding to the measures of functions f in $\mathscr{L}_1$. Then M is the set of L^1-functions g that satisfy the condition $\int_0^{2\pi} g(e^{it})dt = 0$. Clearly, each g in M determines an f in $\mathscr{L}_1$.

THEOREM 7.1. *The dual of* $\mathscr{L}_1$ *can be identified with* $L^\infty[0, 2\pi]/M^\perp$, *where*

$$M^\perp = \left\{ h \in L^\infty : \int_0^{2\pi} h(e^{it})g(e^{it})\,dt = 0,\ g \in M \right\}.$$

Proof. The dual of L^1 is L^∞. Let T be a continuous linear functional on M. By the Hahn-Banach theorem, there exists a linear functional T_h on L^1 that is a norm-preserving extension of T. Furthermore, if $g \in M$, then

$$T(g) = \int_0^{2\pi} h(e^{it})g(e^{it})\frac{dt}{2\pi} = T_h(g),$$

where h belongs to L^∞. Therefore $M^* = L^\infty/M^\perp$, where M^* denotes the dual of M. Since the correspondence between $\mathscr{L}_1$ and M is an isometry, each continuous linear functional τ on $\mathscr{L}_1$ is defined by means of a function $h \in L^\infty$, and its action on $\mathscr{L}_1$ is given by the equation

$$\tau(f) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{it})\log|f'(e^{it})|\,dt \qquad (f \in \mathscr{L}_1).$$

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REFERENCES

1. L. Bieberbach, *Aufstellung und Beweis des Drehungssatzes für schlichte konforme Abbildungen.* Math. Z. 4 (1919), 295-305.

2. J. A. Cima and J. A. Pfaltzgraff, *The Hornich topology for meromorphic functions in the disk.* J. Reine Angew. Math. 235 (1969), 207-220.

3. P. L. Duren, *On the Bloch-Nevanlinna conjecture.* Colloq. Math. 20 (1969), 295-297.

4. P. L. Duren, B. W. Romberg, and A. L. Shields, *Linear functionals on* H^p *spaces with* $0 < p < 1$. J. Reine Angew. Math. 238 (1969), 32-60.

5. P. L. Duren, H. S. Shapiro, and A. L. Shields, *Singular measures and domains not of Smirnov type.* Duke Math. J. 33 (1966), 247-254.

6. G. M. Golusin, *Geometrische Funktionentheorie.* Deutscher Verl. Wissensch., Berlin, 1957.

7. G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals. II.* Math. Z. 34 (1932), 403-439.

8. ———, *Some properties of conjugate functions.* J. Reine Angew. Math. 167 (1932), 405-423.

9. K. Hoffman, *Banach spaces of analytic functions.* Prentice-Hall, Englewood Cliffs, N. J., 1962.

10. H. Hornich, *Zur Struktur der schlichten Funktionen.* Abh. Math. Sem. Univ. Hamburg 22 (1958), 38-49.

11. ———, *Ein Banachraum analytischer Funktionen in Zusammenhang mit den schlichten Funktionen.* Monatsh. Math. 73 (1969), 36-45.

12. W. Kaplan, *Close-to-convex schlicht functions.* Michigan Math. J. 1 (1952), 169-185.

13. J. E. Littlewood, *On a theorem of Kolmogoroff.* J. London Math. Soc. 1 (1926), 229-231.

14. A. J. Lohwater, G. Piranian, and W. Rudin, *The derivative of a schlicht function.* Math. Scand. 3 (1955), 103-106.

15. V. Paatero, *Über die konforme Abbildungen von Gebieten, deren Ränder von beschränkter Drehung sind.* Ann. Acad. Sci. Fenn. Ser. A 33, No. 9 (1931), 78 pp.

16. G. Piranian, *An isolated schlicht function.* Abh. Math. Sem. Univ. Hamburg 24 (1960), 236-238.

17. I. I. Privalov, *Randeigenschaften analytischer Funktionen.* Deutscher Verl. Wissensch., Berlin, 1956.

18. W. Rudin, *Real and complex analysis.* McGraw-Hill, New York, N. Y., 1966.

19. L. Špaček, *Contribution à la théorie des fonctions univalentes.* Časopis Pěst. Mat. 62 (1933), 12-19.

20. J. Wolff, *L'intégrale d'une fonction holomorphe et à partie réelle positive dans un demi-plan est univalente.* C. R. Acad. Sci. Paris 198 (1934), 1209-1210.

21. A. Zygmund, *Trigonometric series,* Vol. 1. Second edition. Cambridge University Press, Cambridge, 1959.

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