THE HOMOTOPY GROUPS OF BPL AND PL/O. III

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1. INTRODUCTION

Let Γ_{k-1} denote the group of oriented differentiable structures on the (k - 1)-sphere. M. Kervaire and J. Milnor [8] have constructed an exact sequence

1.0 $$0 \to bP_k \to \Gamma_{k-1} \to \pi_{k-1}^s/\mathrm{im}(J),$$

where $bP_k \subseteq \Gamma_{k-1}$ is the subgroup of exotic spheres that bound π-manifolds, π_{k-1}^s is the stable (k - 1)-stem, and J is the classical J-homomorphism. Further, it is known that

$$bP_k = \begin{cases} \mathbb{Z}_{\theta_k} & \text{if } k \equiv 0 \pmod 4, \\ \mathbb{Z}_2 & \text{if } k \equiv 2 \pmod 4 \text{ and } k \neq 2^j - 2, \\ 0 & \text{if } k \equiv 1 \text{ or } k \equiv 3 \pmod 4, \end{cases}$$

where θ_k is a large integer, and that $\Gamma_{k-1} \to \pi_{k-1}^s/\mathrm{im}(J)$ is surjective if $k - 1 \neq 2^j - 2$ [8], [3].

In [5] and [6], we showed that the exact sequence 1.0 splits if $k \equiv 0$, 2, or 4 (mod 8). That is,

$$\Gamma_{4m-1} \simeq \mathbb{Z}_{\theta_{4m}} \oplus (\pi_{4m-1}^s/\mathrm{im}(J)) \quad \text{and} \quad \Gamma_{8m+1} \simeq \mathbb{Z}_2 \oplus (\pi_{8m+1}^s/\mathrm{im}(J)),$$

for all m. In this note, we outline a proof of the splitting of the sequence 1.0 for $k \equiv 2 \pmod 4$ but $k \neq 2^j - 2$.

THEOREM 1.1. *There exists an isomorphism*

$$\Gamma_{4m+1} \simeq \mathbb{Z}_2 \oplus (\pi_{4m+1}^s/\mathrm{im}(J))$$

if $4m + 2 \neq 2^j - 2$.

Theorem 1.1 includes the dimensions $k \equiv 2 \pmod 8$ dealt with in [6]. The proof given here is perhaps more elementary.

In the dimensions $2^j - 2$, it is known that $bP_{2^j-2} = 0$ if the element $(h_{j-1})^2$ survives to E_∞ in the Adams spectral sequence [3]. If $bP_{2^j-2} = 0$, there is no splitting problem in the exact sequence 1.0. M. Mahowald has shown that $(h_{j-1})^2$ does, in fact, survive to E_∞, if $j \leq 6$.

There exists an isomorphism $\Gamma_{k-1} \simeq \pi_{k-1}(PL/O)$ due to M. Hirsch and B. Mazur [7]. From the fibration $PL/O \to BO \to BPL$, it is clear that $\pi_k(BPL) \simeq \pi_{k-1}(PL/O)$ for $k \equiv 6 \pmod 8$, since $\pi_k(BO) = \pi_{k-1}(BO) = 0$. In [5] and

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[6], we also described the homotopy groups $\pi_k(BPL)$ for $k \equiv 0$, 2, or 4 (mod 8). This paper completes the determination of the groups $\pi_k(BPL)$ and $\pi_{k-1}(PL/O) \simeq \Gamma_{k-1}$ in terms of homotopy groups of spheres, at least if $k \neq 2^j - 2$ or $k \neq 2^j - 1$.

I am grateful to E. Brown for sending me an unpublished manuscript and to W. Browder for useful discussions. Theorem 1.1 is essentially a corollary of their work on the Kervaire invariant.

2. NOTATION

We write spaces in ordinary type, and spectra appear in script. Thus, K(G, n) denotes the Eilenberg-MacLane space, while $\mathscr{K}(G, n)$ denotes the spectrum $\{K(G, n+k)\}_k$. Let $\mathscr{S}^0 = \{S^k\}_k$ denote the sphere spectrum. The nth suspension of a spectrum $\mathscr{Y} = \{Y_k\}_k$ is the spectrum $S^n \wedge \mathscr{Y} = \{S^n \wedge Y_k\}_k$. We write $S^n \wedge \mathscr{S}^0 = \mathscr{S}^n$. All spectra $\mathscr{Y} = \{Y_k\}_k$ that we consider have the property that the space Y_k is (k - 1)-connected. It follows that

$$\pi_n(\mathscr{Y}) = \pi_{n+k}(Y_k) \quad \text{and} \quad H^n(\mathscr{Y}) = H^{n+k}(Y_k),$$

if k is large. All cohomology is with $\mathbb{Z}_2$-coefficients unless otherwise indicated.

Suppose $\mathscr{Y}$ is a spectrum. By $\mathscr{Y}^{(q)}$, we mean a spectrum for which there exists a map h: $\mathscr{Y} \to \mathscr{Y}^{(q)}$ such that $h_*: \pi_j(\mathscr{Y}) \xrightarrow{\simeq} \pi_j(\mathscr{Y}^{(q)})$ if $j \leq q$ and $\pi_j(\mathscr{Y}^{(q)}) = 0$ if $j > q$. One can think of $\mathscr{Y}^{(q)}$ as the qth stage of a Postnikov resolution of $\mathscr{Y}$.

Let $BO\langle v_{n+1}\rangle$ be the fibration over BO whose fibre is $K(\mathbb{Z}_2, n)$ and whose k-invariant is equal to the Wu class $v_{n+1} \in H^{n+1}(BO)$. Similarly, let $BSO\langle v_{n+1}\rangle$ and $BSPL\langle v_{n+1}\rangle$ be fibrations over BSO and BSPL whose fibre is $K(\mathbb{Z}_2, n)$ and whose k-invariant is equal to v_{n+1}.

Let $\mathscr{MO}\langle v_{n+1}\rangle$, $\mathscr{MSO}\langle v_{n+1}\rangle$, and $\mathscr{MSPL}\langle v_{n+1}\rangle$ be the Thom spectra of the universal bundles over $BO\langle v_{n+1}\rangle$, $BSO\langle v_{n+1}\rangle$, and $BSPL\langle v_{n+1}\rangle$, respectively. The homotopy group $\pi_q(\mathscr{MO}\langle v_{n+1}\rangle)$ is isomorphic to the group of cobordism classes of pairs consisting of a manifold N^q and a bundle map $\nu: \nu_N \to \gamma\langle v_{n+1}\rangle$, where ν_N is the stable normal bundle of N^q and $\gamma\langle v_{n+1}\rangle$ is the universal bundle over $BO\langle v_{n+1}\rangle$. Similarly, $\pi_q(\mathscr{MSO}\langle v_{n+1}\rangle)$ and $\pi_q(\mathscr{MSPL}\langle v_{n+1}\rangle)$ are the cobordism groups of such pairs (N^q, ν), where N^q is an oriented or an oriented PL manifold, respectively.

For a q-manifold N^q, the Wu class $v_{n+1}(N^q)$ of the normal bundle of N^q satisfies the formula

$$v_{n+1}(N^q) \cdot x = Sq^{n+1}(x),$$

if $x \in H^{q-n-1}(N^q)$. It follows that $v_{n+1}(N^q) = 0$ if $q \leq 2n+1$. In particular, the classifying map for the normal bundle $\nu: N^q \to BO$ always lifts to $\nu: N^q \to BO\langle v_{n+1}\rangle$, if $q \leq 2n+1$.

3. OUTLINE OF THE PROOF

The proof of Theorem 1.1 is similar to the proofs of the splittings of the sequence 1.0 given in [5] and [6].

Consider odd integers n with $2n \neq 2^j - 2$. We want to define a homomorphism $f\colon \Gamma_{2n-1} \to \mathbf{Z}_2$ such that the restriction of f to $bP_{2n} = \mathbf{Z}_2$ is the identity. It suffices to consider $\Sigma^{2n-1} \in \Gamma_{2n-1}$ whose order is a power of 2. We then choose a suitable oriented, smooth manifold M^{2n} with $\partial M^{2n} = \Sigma^{2n-1}$, together with a lifting of the stable normal bundle of M^{2n} to $BSO\langle v_{n+1}\rangle$. This induces a lifting of the normal bundle of the closed PL manifold

$$\hat{M}^{2n} = M^{2n} \bigcup_{\Sigma^{2n-1}} C\Sigma^{2n-1}$$

to $BSPL\langle v_{n+1}\rangle$.

E. Brown [4] has defined a homomorphism $\psi\colon \pi_{2n}(\mathcal{MO}\langle v_{n+1}\rangle) \to \mathbf{Z}_8$ for which the diagram

$$\begin{array}{ccc} \pi_{2n}(\mathcal{S}^0) & \xrightarrow{K} & \mathbf{Z}_2 \\ \downarrow i_* & & \downarrow j \\ \pi_{2n}(\mathcal{MO}\langle v_{n+1}\rangle) & \xrightarrow{\psi} & \mathbf{Z}_8 \end{array}$$

commutes, where K is the classical Kervaire invariant, $i\colon \mathcal{S}^0 \to \mathcal{MO}\langle v_{n+1}\rangle$ is the Thom cell, and j is the inclusion of $\mathbf{Z}_2$ in $\mathbf{Z}_8$. It is easy to see from Brown's definition that if (M^{2n}, ν) is an oriented manifold (n odd), then $\psi(M^{2n}, \nu) \in \mathbf{Z}_2 \subset \mathbf{Z}_8$. Further, Brown's homomorphism generalizes immediately to PL manifolds. Hence, we have a homomorphism $\psi\colon \pi_{2n}(\mathcal{MSPL}\langle v_{n+1}\rangle) \to \mathbf{Z}_2$, extending the classical Kervaire invariant. We set $f(\Sigma^{2n-1}) = \psi(\hat{M}^{2n}) \in \mathbf{Z}_2$, where Σ^{2n-1} and $\hat{M}^{2n}$ are as above.

In order that f be well-defined, it is necessary to restrict the choice of the manifold M^{2n}. This is done as follows. Using Browder's homotopy-theoretic description of $\mathcal{MSO}\langle v_{n+1}\rangle^{(2n)}$ [3], we construct in Section 4 a spectrum $\mathcal{X}$ and maps $\mathcal{S}^0 \xrightarrow{\alpha} \mathcal{X} \xrightarrow{\beta} \mathcal{MSO}\langle v_{n+1}\rangle^{(2n)}$ such that

3.1. $\pi_0(\mathcal{X}) = \mathbf{Z}$, with generator $\alpha\colon \mathcal{S}^0 \to \mathcal{X}$,

3.2. $\beta\alpha\colon \mathcal{S}^0 \to \mathcal{MSO}\langle v_{n+1}\rangle^{(2n)}$ is some odd multiple, say d, of the inclusion of the Thom cell,

3.3. $\pi_{2n-1}(\mathcal{X}) = 0$; hence, $\alpha_*\colon \pi_{2n-1}(\mathcal{S}^0) \to \pi_{2n-1}(\mathcal{X})$ is zero,

3.4. $\pi_{2n}(\mathcal{X}) = \mathbf{Z}_2$; moreover, $\pi_{2n}(\mathcal{X})$ is a direct summand of $\pi_{2n}(\mathcal{X}, \mathcal{S}^0)$.

By 3.4, there exists a homomorphism $\Phi\colon \pi_{2n}(\mathcal{X}, \mathcal{S}^0) \to \mathbf{Z}_2$ such that the composition

$$\pi_{2n}(\mathcal{X}) \to \pi_{2n}(\mathcal{X}, \mathcal{S}^0) \xrightarrow{\Phi} \mathbf{Z}_2$$

is an isomorphism. The following is an easy consequence of this and 3.3.

3.5. For each map $\sigma: \mathscr{S}^{2n-1} \to \mathscr{S}^0$, there exists a unique extension (up to homotopy) of σ, say $\bar{\sigma}: \mathscr{D}^{2n}, \mathscr{S}^{2n-1} \to \mathscr{X}, \mathscr{S}^0$, such that $\Phi(\bar{\sigma}) = 0$.

Now, for a fixed $\Sigma^{2n-1} \in \Gamma_{2n-1}$, choose a framing of Σ^{2n-1} in S^{k+2n-1} (k large). This defines a map $\sigma: S^{k+2n-1} \to S^k$ or $\sigma: \mathscr{S}^{2n-1} \to \mathscr{S}^0$. If $n \equiv 1 \pmod 4$, the framing is not unique. However, if we choose some homomorphism $e: \pi_{2n-1}(\mathscr{S}^0) \to \mathbf{Z}_2$ such that $\mathbf{Z}_2 \xrightarrow{J} \pi_{2n-1}(\mathscr{S}^0) \xrightarrow{e} \mathbf{Z}_2$ is the identity [2], and if we require in addition that $e(\sigma) = 0$, then σ is well-defined.

Choose an extension $\bar{\sigma}: \mathscr{D}^{2n}, \mathscr{S}^{2n-1} \to \mathscr{X}, \mathscr{S}^0$ of σ that satisfies the condition $\Phi(\bar{\sigma}) = 0$. Consider the composition

$$\beta\bar{\sigma}: \mathscr{D}^{2n}, \mathscr{S}^{2n-1} \to \mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle^{(2n)}, \mathscr{S}^0.$$

Using the Five Lemma, one can show that the map

$$h: \mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle \to \mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle^{(2n)}$$

induces an isomorphism

$$h_*: \pi_{2n}(\mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle, \mathscr{S}^0) \xrightarrow{\approx} \pi_{2n}(\mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle^{(2n)}, \mathscr{S}^0).$$

Thus $\beta\bar{\sigma}$ determines a homotopy class of maps $\mathscr{D}^{2n}, \mathscr{S}^{2n-1} \to \mathscr{M}\mathscr{S}\mathscr{O}\langle v_{v+1}\rangle, \mathscr{S}^0$. By 3.2, we have that

$$\beta\bar{\sigma} \mid \mathscr{S}^{2n-1} = d \cdot \sigma: \mathscr{S}^{2n-1} \to \mathscr{S}^0.$$

Applying a standard transverse regularity argument, we may assume that the inverse image of $BSO\langle v_{n+1}\rangle$, under the map $\mathscr{D}^{2n}, \mathscr{S}^{2n-1} \to \mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle, \mathscr{S}^0$, is a manifold M^{2n} with $\partial M^{2n} = d \cdot \Sigma^{2n-1}$. It is easy to see that the $BSPL\langle v_{n+1}\rangle$-bordism class of $\hat{M}^{2n}$ is uniquely determined by Σ^{2n-1} and the construction above. It follows that the formula $f(d \cdot \Sigma^{2n-1}) = \psi(\hat{M}^{2n})$ unambiguously defines $f(d \cdot \Sigma^{2n-1}) \in \mathbf{Z}_2$.

Since d is odd, each 2-torsion element of Γ_{n-1} can be written as $d \cdot \Sigma^{2n-1}$. Thus, we have $f: \Gamma_{n-1} \to \mathbf{Z}_2$. Clearly, f is a homomorphism. For, given $d \cdot \Sigma_i^{2n-1} = \partial M_i^{2n}$ (i = 1, 2) as above, we may geometrically construct the map $\mathscr{D}^{2n}, \mathscr{S}^{2n-1} \to \mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle, \mathscr{S}^0$ corresponding to $\Sigma^{2n-1} = \Sigma_1^{2n-1} + \Sigma_2^{2n-1}$ so that the manifold M^{2n}, with boundary $d \cdot \Sigma^{2n-1}$, is $M_1^{2n} \cup M_2^{2n}$. Then

$$f(d \cdot \Sigma^{2n-1}) = \psi(\hat{M}^{2n}) = \psi(\hat{M}_1^{2n}) + \psi(\hat{M}_2^{2n}) = f(d \cdot \Sigma_1^{2n-1}) + f(d \cdot \Sigma_2^{2n-1}).$$

Finally, suppose $\Sigma^{2n-1} \in bP_{2n} = \mathbf{Z}_2$ is nonzero. Since Σ^{2n-1} bounds a π-manifold, specifically the Kervaire manifold K^{2n}, the map $\sigma: \mathscr{S}^{2n-1} \to \mathscr{S}^0$, given by the framing of Σ^{2n-1}, extends to $\bar{\sigma}: \mathscr{D}^{2n} \to \mathscr{S}^0$, with $(\bar{\sigma})^{-1}(pt) = K^{2n}$. If we regard $\bar{\sigma}$ as a map $\bar{\sigma}: \mathscr{D}^{2n}, \mathscr{S}^{2n-1} \to \mathscr{X}, \mathscr{S}^0$, then $\bar{\sigma} \sim 0$; hence the condition of 3.5 clearly holds. Thus,

$$f(\Sigma^{2n-1}) = f(d \cdot \Sigma^{2n-1}) = \psi(d \cdot \hat{K}^{2n}) = 1 \in \mathbf{Z}_2,$$

and f splits the exact sequence 1.0, as desired.

4. CONSTRUCTION OF THE SPECTRUM $\mathscr{X}$

It follows from Browder's work [3] on the Kervaire invariant that $\mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle^{(2n)}$ is a (generalized) 2-stage Postnikov tower over $\mathscr{M}\mathscr{S}\mathscr{O}^{(2n)}$. Specifically, there is a diagram

$$\begin{array}{ccccc} \mathscr{K}(\mathbf{Z}_2, 2n) & & \mathscr{K}(V_*) & & \\ \downarrow i_1 & & \downarrow j_1 & & \\ \mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle^{(2n)} & \xrightarrow{\rho_1} & \mathscr{Y}' & \xrightarrow{\pi_1} & \mathscr{M}\mathscr{S}\mathscr{O}^{(2n)}, \end{array}$$

where (π_1, j_1) and (ρ_1, i_1) are fibrations, V_* is a graded $\mathbf{Z}_2$-vector space, and $\mathscr{K}(V_*)$ is the Eilenberg-MacLane spectrum with $\pi_*(\mathscr{K}(V_*)) = V_*$. Moreover, $V_j = 0$ if $j < n$.

Recall that $\mathscr{M}\mathscr{S}\mathscr{O}$ has the 2-primary homotopy type of a product of Eilenberg-MacLane spectra $\mathscr{K}(\mathbf{Z})$ and $\mathscr{K}(\mathbf{Z}_2)$ [9, p. 209 or p. 233]. In particular, there exist maps $g: \mathscr{K}(\mathbf{Z}, 0) \to \mathscr{M}\mathscr{S}\mathscr{O}^{(2n)}$ such that $g^*(U) = d \cdot \iota_0$, where $U \in H^0(\mathscr{M}\mathscr{S}\mathscr{O}^{(2n)}, \mathbf{Z})$ is the Thom class, $\iota_0 \in H^0(\mathscr{K}(\mathbf{Z}, 0), \mathbf{Z})$ is the fundamental class, and d is an odd integer.

Let

$$\bigvee_{j\geq 1} \mathscr{K}(\mathbf{Z}_2, 2^j - 1) \xrightarrow{j_2} \mathscr{X}' \xrightarrow{\pi_2} \mathscr{K}(\mathbf{Z}, 0)$$

be the fibration with k-invariant $\bigoplus \tau(\iota_{2^j-1}) = \bigoplus Sq^{2^j}(\iota_0) \in H^*(\mathscr{K}(\mathbf{Z}, 0))$, where τ is the transgression. Then there exist fibre maps $g': \mathscr{X}' \to \mathscr{Y}'$ covering g, that is, the diagram

$$\begin{array}{ccc} \bigvee_{j\geq 1} \mathscr{K}(\mathbf{Z}_2, 2^j - 1) & \xrightarrow{g'} & \mathscr{K}(V_*) \\ \downarrow j_2 & & \downarrow j_1 \\ \mathscr{X}' & \xrightarrow{g'} & \mathscr{Y}' \\ \downarrow \pi_2 & & \downarrow \pi_1 \\ \mathscr{K}(\mathbf{Z}, 0) & \xrightarrow{g} & \mathscr{M}\mathscr{S}\mathscr{O}^{(2n)} \end{array} \tag{4.1}$$

commutes. For, the fibration (π_1, j_1) is classified by a map $\gamma: \mathscr{M}\mathscr{S}\mathscr{O}^{(2n)} \to \mathscr{K}(V'_*)$, where $V'_i = V_{i-1}$, and such a map g' exists if the composition $\gamma g \pi_2$ is homotopic to zero. But $0 = \pi_2^*: H^i(\mathscr{K}(\mathbf{Z}, 0)) \to \mathscr{H}^i(X')$ if $i > 0$, because the elements $Sq^{2^j}(\iota_0)$ generate $\bigoplus_{i>0} H^i(\mathscr{K}(\mathbf{Z}, 0))$ as a module over the Steenrod algebra A. It follows that $\gamma g \pi_2 \sim 0$.

Now, define $\mathscr{K}(\mathbf{Z}_2, 2n) \xrightarrow{i_2} \mathscr{X} \xrightarrow{\rho_2} \mathscr{X}'$ to be the fibration over $\mathscr{X}'$ induced by $g': \mathscr{X}' \to \mathscr{Y}'$ and the fibration (ρ_1, i_1) over $\mathscr{Y}'$. Let $\beta: \mathscr{X} \to \mathscr{M}\mathscr{S}\mathscr{O}\langle v_{n+1}\rangle^{(2n)}$ be the fibre map covering g'.

Properties 3.1 and 3.2 of $\mathscr{X}$ are obvious. Further, $\pi_{2n-1}(\mathscr{X}) = \pi_{2n-1}(\mathscr{X}') = 0$, because n is odd and $n \neq 1$; hence $2n - 1 \neq 2^j - 1$. Thus, 3.3 holds.

To establish 3.4, we first note the following. If $i > 0$, then

$$j_2^*: H^i(\mathscr{X}') \xrightarrow{\simeq} \text{kernel}\left(H^i\Big(\bigvee_{j\geq 1} \mathscr{K}(\mathbb{Z}_2, 2^j - 1)\Big) \xrightarrow{\tau} H^{i+1}(\mathscr{K}(\mathbb{Z}, 0))\right).$$

Thus, it follows from results of J. F. Adams [1] that, as an A-module, $H^*(\mathscr{X}')$ is generated by $\pi_2^*(\iota_0) \in H^0(\mathscr{X}')$ and elements $h_i h_j \in H^{2^i+2^j-1}(\mathscr{X}')$, where $i \geq j \geq 0$, $i \neq j+1$, and $(i, j) \neq (0, 0)$. Moreover,

$$j_2^*(h_r h_s) = Sq^{2^r}(\iota_{2^s-1}) + \sum_{k<s} b_k(\iota_{2^k-1}) \in H^{2^r+2^s-1}\Big(\bigvee_{j\geq 1} \mathscr{K}(\mathbb{Z}_2, 2^j - 1)\Big),$$

where $b_k \in A$. The elements $h_i h_j$ correspond to relations

$$Sq^{2^i} Sq^{2^j} + \sum_{k<j} b_k Sq^{2^k} = 0$$

in the Steenrod algebra. Note that there is no generator $h_i h_j \in H^{2n}(\mathscr{X}')$, because n is odd; hence $2n \neq 2^i + 2^j - 1$. Thus all elements of $H^{2n}(\mathscr{X}')$ and of $H^{2n}(\mathscr{X}', \mathscr{S}^0)$ are decomposable over A.

PROPOSITION 4.2. *In the exact sequence*

$$0 \longrightarrow H^{2n}(\mathscr{X}') \xrightarrow{\rho_2^*} H^{2n}(\mathscr{X}) \xrightarrow{i_2^*} H^{2n}(\mathscr{K}(\mathbb{Z}_2, 2n)) \xrightarrow{\tau_2} H^{2n+1}(\mathscr{X}'),$$

we have the relation $\tau_2(\iota_{2n}) = \sum a_i x_i' \in H^{2n+1}(\mathscr{X}')$, *where* $x_i' \in H^*(X')$, $a_i \in A$, *and* $0 < \deg(a_i) < 2n+1$.

Before proving Proposition 4.2, we use it to establish 3.4. Proposition 4.2 says that the k-invariant of the fibration (ρ_2, i_2) is decomposable over A. It follows that $\pi_{2n}(\mathscr{X}) = \mathbb{Z}_2$. To show that $\pi_{2n}(\mathscr{X})$ is a direct summand of $\pi_{2n}(\mathscr{X}, \mathscr{S}^0)$, we distinguish two cases. First, if $\tau_2(\iota_{2n}) = \sum a_i x_i' = 0 \in H^{2n+1}(\mathscr{X}')$, then $\mathscr{X} = \mathscr{K}(\mathbb{Z}_2, 2n) \vee \mathscr{X}'$, and 3.4 clearly holds. On the other hand, if

$$\tau_2(\iota_{2n}) = \sum a_i x_i' \neq 0,$$

it follows that $\rho_2^*: H^j(\mathscr{X}') \xrightarrow{\simeq} H^j(\mathscr{X})$ is an isomorphism if $j \leq 2n$. Thus, there are elements $x_i \in H^{\deg(x_i')}(\mathscr{X}) = H^{\deg(x_i')}(\mathscr{X}, \mathscr{S}^0)$, corresponding to the $x_i' \in H^{\deg(x_i')}(\mathscr{X}')$, and $\rho_2^*\left(\sum a_i x_i'\right) = \sum a_i x_i = 0 \in H^{2n+1}(\mathscr{X})$. Further, all elements of $H^{2n}(\mathscr{X})$ and $H^{2n}(\mathscr{X}, \mathscr{S}^0)$ are decomposable over A, and, hence, every map $\sigma: \mathscr{S}^{2n} \to \mathscr{X}$, or $\bar{\sigma}: \mathscr{D}^{2n}, \mathscr{S}^{2n-1} \to \mathscr{X}, \mathscr{S}^0$, induces the zero map in cohomology.

Define a homomorphism $\Phi: \pi_{2n}(\mathscr{X}, \mathscr{S}^0) = \pi_{2n}(\mathscr{X}/\mathscr{S}^0) \to \mathbb{Z}_2$ as follows. For $\gamma: \mathscr{S}^{2n} \to \mathscr{X}/\mathscr{S}^0$, set

$$\Phi(\gamma) = \sum a_i x_i \in \text{image}\left(H^{2n}(\mathscr{S}^{2n}) \xrightarrow{\delta} H^{2n+1}((\mathscr{X}/\mathscr{S}^0) \cup_\gamma \mathscr{D}^{2n+1})\right) = \mathbb{Z}_2,$$

where δ is the coboundary. In other words, Φ is the primary functional operation associated to the relation $\sum a_i x_i = 0 \in H^{2n+1}(\mathscr{X}/\mathscr{S}^0)$. We need to show that $\Phi(\gamma) \neq 0$, if γ is the composition $\mathscr{S}^{2n} \xrightarrow{\iota} \mathscr{K}(\mathbb{Z}_2, 2n) \xrightarrow{i_2} \mathscr{X} \longrightarrow \mathscr{X}/\mathscr{S}^0$. But this is a basic property of the fibration (ρ_2, i_2). Specifically, there is a map $\bar{\rho}_2\colon \mathscr{X} \cup_{i_2\iota} \mathscr{D}^{2n+1} \to \mathscr{X}'$, extending $\rho_2\colon \mathscr{X} \to \mathscr{X}'$, which induces an isomorphism of homotopy groups through dimension $2n$ and an epimorphism in dimension $2n+1$. Thus, the cohomology map $\bar{\rho}_2^*$ is a monomorphism through dimension $2n+1$. Hence,

$$\sum a_i x_i \neq 0 \in H^{2n+1}(\mathscr{X} \underset{i_2\iota}{\cup} \mathscr{D}^{2n+1}) = H^{2n+1}((\mathscr{X}/\mathscr{S}^0) \underset{\gamma}{\cup} \mathscr{D}^{2n+1}),$$

or, equivalently, $\Phi(\gamma) \neq 0$.

Proof of Proposition 4.2. We use ideas of Browder [3]. Consider the commutative diagram 4.1. By naturality, $\tau_2(\iota_{2n}) = (g')^*(\tau_1(\iota_{2n}))$, where $\tau_1(\iota_{2n}) \in H^{2n+1}(\mathscr{Y}')$ is the k-invariant of the fibration (ρ_1, i_1). Then $j_2^*(\tau_2(\iota_{2n})) = (g')^* j_1^*(\tau_1(\iota_{2n}))$. Now, $\mathscr{K}(V_*)$ is $(n-1)$-connected; hence $j_1^*(\tau_1(\iota_{2n})) = \sum c_i y_i$, where $c_i \in A$, $y_i \in H^*(\mathscr{K}(V_*))$, and $\deg(y_i) \geq n$. In [3], Browder shows further that $j_1^*(\tau_1(\iota_{2n}))$ is decomposable over A. Hence, we may assume that $\deg(c_i) > 0$. We thus have the relation $j_2^*(\tau_2(\iota_{2n})) = \sum c_i z_i$, where $z_i = (g')^*(y_i)$, $\deg(z_i) \geq n$, and $\deg(c_i) > 0$.

Let $\tau_2(\iota_{2n}) = \sum a_{ij}(h_i h_j) \in H^{2n+1}(\mathscr{X}')$, where $a_{ij} \in A$. We must show that if $2n+1 = 2^r + 2^s - 1$ $(r \geq s)$, then the (constant) coefficient a_{rs} equals 0. Since $2n \neq 2^j - 2$, we may assume that $r > s$. Then

$$j_2^*(\tau_2(\iota_{2n})) = \sum a_{ij} j_2^*(h_i h_j) = \sum a_{ij}\left(Sq^{2^j}(\iota_{2^j-1}) + \sum_{k<j} b_k(\iota_{2^k-1})\right) = \sum c_i z_i,$$

where $\deg(z_i) \geq n > 2^s - 1$ and $\deg(c_i) > 0$. Equating the coefficients of ι_{2^s-1} in this last equality, we see that $a_{rs} Sq^{2^r}$ must be decomposable in A. Hence, $a_{rs} = 0$.

REFERENCES

1. J. F. Adams, *On the structure and applications of the Steenrod algebra.* Comment. Math. Helv. 32 (1958), 180-214.

2. ———, *On the groups* J(X). IV. Topology 5 (1966), 21-71.

3. W. Browder, *The Kervaire invariant of framed manifolds and its generalization.* Ann. of Math. (2) 90 (1969), 157-186.

4. E. H. Brown, Jr., *The Arf invariant of a manifold.* Conference on Algebraic Topology, Univ. of Ill. at Chicago Circle (1968), pp. 9-18.

5. G. Brumfiel, *On the homotopy groups of* BPL *and* PL/O. Ann. of Math. (2) 88 (1968), 291-311.

6. ———, *On the homotopy groups of* BPL *and* PL/O. II. Topology 8 (1969), 305-311.

7. M. Hirsch and B. Mazur, *Smoothings of piecewise linear manifolds*. Mimeographed Notes, Cambridge University, 1964.

8. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres*. I. Ann. of Math. (2) 77 (1963), 504-537.

9. R. E. Stong, *Notes on cobordism theory*. Princeton University Press, Princeton, N.J., 1968.

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