ON ARC AND BALL PAIRS AND QUASI-TRANSLATIONS OF THE 3-SPHERE

Shin'ichi Kinoshita and Carl D. Sikkema

INTRODUCTION

All manifolds considered in this paper are not only orientable but already oriented. Let a be an arc in a 3-ball B such that only the end point p(a) (but not the initial point) of a is on the boundary of B and a is locally tame in B except at the end point. Such an *arc and ball pair* will be denoted by (a, B). Two such pairs (a_1, B_1) and (a_2, B_2) are called *equivalent* if there exists an orientation-preserving homeomorphism of B_1 onto B_2 that carries a_1 onto a_2. In this paper, we do not always distinguish between a pair (a, B) and its equivalence class.

Next, we introduce a binary operation # for two arc and ball pairs (a_1, B_1) and (a_2, B_2), and we call it the *composition*. Under the composition #, the set of all equivalence classes of arc and ball pairs forms a semigroup whose identity is the trivial pair (e, B). Further, if $(a_1, B_1) \# (a_2, B_2) = (e, B)$, then

$$(a_1, B_1) = (a_2, B_2) = (e, B)$$

(Theorem 1).

The purpose of introducing such arc and ball pairs (a, B) is to apply them to the construction of quasi-translations (defined below) of the 3-sphere S^3. Let H be an infinite cyclic covering transformation group acting on 3-space E^3, where each element of H is orientation-preserving and the decomposition space E^3/H is a Hausdorff space (an orientable 3-manifold). A generator of H has been called a *quasi-translation of* $S^3 = E^3 \cup (p)$. In other words, h is an orientation-preserving autohomeomorphism of S^3 such that $h(p) = p$ for some point $p \in S^3$, and for every compact subset $C \subset S^3 - (p)$, we have the relation $h^n(C) \cap C = \emptyset$ for all but a finite number of integers n (Sperner's condition). (See [10]. For equivalent definitions see [9].)

In this paper, we construct a quasi-translation h(a, B) of S^3 for each arc and ball pair (a, B), and among other results, we prove that h(a, B) is topologically equivalent to a standard translation if and only if (a, B) is the trivial pair (e, B) (Theorem 5). Applying this, we answer the first problem in [8] in the negative.

In a later paper [13], it will be proved that (i) the composition # is noncommutative in general (hence, by Theorem 2 in this paper, the correspondence $(a, B) \to h(a, B)$ is not one-to-one) and (ii) there exist uncountably many mutually topologically inequivalent quasi-translations of S^3.

It is known that every quasi-translation of the 2-sphere S^2 is topologically equivalent to a standard translation of S^2 (see B. v. Kerékjártó [6], [7] and E. Sperner [14]). The existence of a nontrivial quasi-translation of S^3 was first proved in [9].

Received August 9, 1969.
The authors of this paper are partially supported by NSF Grant GP-8417.

Michigan Math. J. 17 (1970).

Since all manifolds in this paper are oriented, an arc has both an initial point and an end point, and the boundary of a 3-cell has the orientation inherited from the orientation of the 3-cell.

Let A be a subset of a topological space X. We denote the topological closure, interior, and boundary of A in X by Cl A, Int A, and Bdry A, respectively. The boundary and the interior of a manifold M will be denoted by ∂M and M°, respectively.

I. ON ARC AND BALL PAIRS (a, B)

1. In the introduction, we already defined an arc and ball pair (a, B).

Let S be a 2-sphere in a 3-ball B such that only one point p(S) of S is on the boundary ∂B of B and S is locally tame in B except at the point p(S). Then the equivalence between such pairs (S_1, B_1) and (S_2, B_2) is defined as before.

There exists a natural one-to-one correspondence of the equivalence classes of (a, B) onto those of (S, B) with the property that the image of the equivalence class of (a, B) is the equivalence class that contains the pair (S(a), B), where S(a) is the boundary of a nice thickening of a in B. Note that this correspondence is surjective, by [9]. Hence the equivalence or nonequivalence between (a, B) and (S, B) may be introduced.

There is another way to express the correspondence between (a, B) and (S, B) (see [12]). Consider a pair (a, B), and let q(a) be the initial point of a. The arc a is locally tame at q(a) in B. Hence we may attach a small, tame 2-sphere S to a at q(a) in such a way that $S \cup a$ is also locally tame at q(a) in B. Here, by attaching we mean that $S \cap a = q(a)$. Let us consider the decomposition space of B, where the only nondegenerate element of the decomposition $d\colon B \to d(B)$ is the arc a. Then d(B) is a 3-ball and (d(S), d(B)) is a sphere and ball pair. This mapping of the equivalence classes of (a, B) into those of (d(S), d(B)) is the natural one, as described above.

2. Let (a_1, B_1) and (a_2, B_2) be arc and ball pairs. Suppose that both B_1 and B_2 are tamely imbedded in the 3-sphere S^3. Then, if (a_1, B_1) is equivalent to (a_2, B_2), there exists an orientation-preserving autohomeomorphism of S^3 that carries a_1 onto a_2. In other words, we have the following result.

PROPOSITION 1. *Let* (a_1, B_1) *and* (a_2, B_2) *be arc and ball pairs. Suppose that both* B_1 *and* B_2 *are tamely imbedded in* S^3. *If there exists no orientation-preserving autohomeomorphism of* S^3 *that carries* a_1 *onto* a_2, *the pairs* (a_1, B_1) *and* (a_2, B_2) *are inequivalent.*

Hence there are at least as many different equivalence classes of arc and ball pairs as there are of arcs in S^3, each of which is locally tame except at the end point. Further, even if a_1 is equivalent to a_2 in S^3, (a_1, B_1) and (a_2, B_2) need not be equivalent. This can easily be seen, for instance, by considering mildly wild arcs in S^3 (see [4]).

3. Let (a_1, B_1) and (a_2, B_2) be arc and ball pairs. Then the *composition* of (a_1, B_1) and (a_2, B_2), which we denote by $(a_1, B_1) \# (a_2, B_2)$, is defined as follows: In $(B_2)^{\circ}$, take a small tame 3-ball B' such that $a_2 \cap B' = q(a_2)$ and $a_2 \cup B'$ is locally tame at $q(a_2)$. Let g be an orientation-preserving homeomorphism of B_1 onto B' such that $g(p(a_1)) = q(a_2)$. Consider the decomposition space of B_2, where the

only nondegenerate element of the decomposition d: $B_2 \to d(B_2)$ is the arc a_2. Then $d(B_2)$ is a 3-ball, and $(dg(a_1), d(B_2))$ is an arc and ball pair. We define

$$(a_1, B_1) \# (a_2, B_2) = (dg(a_1), d(B_2)).$$

The composition is well defined for equivalence classes of arc and ball pairs.

An alternate definition of the composition # is the following: Let (S_2, B_2) be a sphere and ball pair equivalent to (a_2, B_2). Let g' be an orientation-preserving homeomorphism of B_1 into B_2 such that $g'(\partial B_1) = S_2$ and $g'(p(a_1)) = p(S_2)$. We define

$$(a_1, B_1) \# (a_2, B_2) = (g'(a_1), B_2).$$

It can easily be seen that the binary operation # is associative and the pair (e, B), where e is a tame arc in B, serves as the identity for the operation #. (We call the pair (e, B) *trivial.*) Thus the family of all equivalence classes of (a, B) with the operation # forms a semigroup with identity.

4. Let (a, B) be an arc and ball pair, and let p = p(a) be the end point of a that is on ∂B. A *standard neighborhood of* p *in* B is a 3-cell V in B that satisfies the following conditions:

(i) V is a closed neighborhood of p in B,

(ii) V is tame in B,

(iii) $\partial V \cap \partial B$ is a 2-cell, and

(iv) the set $a \cap$ (Bdry V) consists of a finite number of points at each of which the arc a pierces the 2-cell Bdry V.

The family of all standard neighborhoods of p in B is a basis for the neighborhoods of the point p in B.

5. In this section, we shall prove the following result.

THEOREM 1. *Let* (a_1, B_1) *and* (a_2, B_2) *be arc and ball pairs, and let* $(a, B) = (a_1, B_1) \# (a_2, B_2)$. *Then* (a, B) *is trivial if and only if both* (a_1, B_1) *and* (a_2, B_2) *are trivial.*

We begin with the following proposition.

PROPOSITION 2. *Let* B *be the unit* 3-*ball and* S *the unit* 2-*sphere centered at the origin* (0, 0, 0) *in* 3-*space. Let* I *be the segment joining the points* (-1, 0, 0) *and* (1, 0, 0). *Let* A *be a tame* 2-*dimensional annulus in* B *such that*

(i) $\partial A \subset S$,

(ii) $A^{\circ} \subset B^{\circ}$,

(iii) *each component of* ∂A *separates the point* (-1, 0, 0) *from* (1, 0, 0) *in* S, *and*

(iv) $A \cap I = \emptyset$.

Then the closure of the complementary domain of A *in* B *that contains* I *is a* 3-*ball, and the closure of the other complementary domain is a solid torus.*

The proposition can be proved easily by applying C. H. Edwards' theorem about concentric tori [2]. (See also [11].)

Proof of Theorem 1. We assume that (a, B) is trivial. Using the notation in the definition of composition, we denote by S the boundary of $dg(B_1)$, which is a locally tame 2-sphere in B except at p(a). Let V_n be a sequence of standard neighborhoods of p(a) such that

(i) $\bigcap_{n=1}^{\infty} V_n = p(a)$,

(ii) Int $V_n \supset V_{n+1}$ for each $n \geq 1$, and

(iii) $a \cap (\text{Bdry } V_n)$ consists of a single point p_n.

Further, we assume that in the 3-cell $\text{Cl}(V_n - V_{n+1})$, the subarc $p_n p_{n+1}$ of the arc a represents a trivial knot.

By the usual cutting and matching method, we can deform S (by an isotopy of B that keeps the arc a fixed) in such a way that $(\text{Bdry } V_n) \cap S$ consists of a finite number of simple closed curves, each of which separates $\partial(\text{Bdry } V_n)$ from the point p_n in Bdry V_n.

Again by the cutting and matching method and by applying Proposition 2, we can deform S (by an isotopy of B that keeps the arc a fixed) in such a way that $(\text{Bdry } V_n) \cap S$ consists of only one simple closed curve, which separates $\partial(\text{Bdry } V_n)$ from the point p_n in Bdry V_n. (This type of modification, which throws an annulus away via a solid torus, was used in [11].)

Applying Proposition 2 again, we see that S is a trivial 2-sphere in B and the boundary of a nice thickening of the arc a in B. It follows that both (a_1, B_1) and (a_2, B_2) are trivial.

The converse is clear and, hence, the proof is complete.

II. ON QUASI-TRANSLATIONS h(a, B)

6. First, we construct a quasi-translation h(a, B), for each arc and ball pair (a, B).

Let S^3 be the one-point compactification of E^3, that is, $S^3 = E^3 \cup (\infty)$. Put $p = (2, 0, 0)$ and $q = (-2, 0, 0)$. For each positive integer n, let S_n be the 2-sphere of radius $1/n$ centered at p, and for each negative integer n, let S_n be the 2-sphere of radius $1/(-n)$ centered at q. Put

$$S_0 = \{(x, y, z) \mid x = 0\} \cup (\infty);$$

then S_0 is also a 2-sphere.

For each integer n, let C_n be the 3-ball that is the closure of the complementary domain of S_n containing the point q. Put $p_n = S_n \cap I$, where

$$I = \{(x, y, z) \mid -2 \leq x \leq 2, \ y = 0, \ z = 0\}.$$

Further, we use vectorial notation and let A_n be the 3-ball with radius $|p_n - p_{n-1}|/2$ centered at $(p_n + p_{n-1})/2$. Hence, $A_n \subset \text{Cl}(C_n - C_{n-1})$, for each integer n.

Now let (a, B) be an arc and ball pair, p(a) the end point of a, and q(a) the initial point. Let g be an orientation-preserving homeomorphism of B onto C_0 such that

(i) $g(a^\circ) \subset (A_0)^\circ$,

(ii) $g(p(a)) = p_0$ and $g(q(a)) = p_{-1}$, and

(iii) $g(a) \cup C_{-1}$ is locally tame at p_{-1} in C_0.

Let t be a dilation of S^3, whose repulsive point is p and whose attractive point is q, such that $t(S_n) = S_{n-1}$, $t(A_n) = A_{n-1}$, and $t(L) = L$, where

$$L = \{(x, y, z) \mid y = 0,\ z = 0\} \cup (\infty).$$

Put $\alpha = \bigcup_{n=-\infty}^{+\infty} t^n(g(a)) \cup (p) \cup (q)$. Then it is easy to see that every subarc of the arc α is locally peripherally unknotted at its initial point ([5]); hence, by [1], the decomposition space of S^3, where α is the only nondegenerate element of the decomposition $d: S^3 \to d(S^3)$, is a 3-sphere. For every point $x \in d(S^3)$, let $h(x) = dtd^{-1}(x)$. Note that $t(\alpha) = \alpha$ and, hence, $hd(\alpha) = d(\alpha)$. The mapping h is an autohomeomorphism of the 3-sphere $d(S^3)$ with the fixed point $d(\alpha)$, and further it can easily be seen that h is orientation-preserving and satisfies Sperner's condition. Hence, h is a quasi-translation of the 3-sphere $d(S^3)$ that is associated with the prescribed pair (a, B), and we denote it by $h(a, B)$.

The correspondence $(a, B) \to h(a, B)$ maps equivalence classes of arc and ball pairs into equivalence classes of quasi-translations of S^3. It is not known whether every quasi-translation of S^3 is topologically equivalent to one in the collection $\{h(a, B), h(a, B)^{-1}\}$.

7. In this section, we prove the following theorem.

THEOREM 2. $h((a_1, B_1) \# (a_2, B_2))$ *is topologically equivalent to* $h((a_2, B_2) \# (a_1, B_1))$.

Proof. We use the same notation as in Section 6. For each positive integer $m > 1$, let $S_{m/2}$ be the 2-sphere of radius $2/m$ centered at p, and for each negative integer $m < -1$, let $S_{m/2}$ be the 2-sphere of radius $2/(-m)$ centered at q. Moreover, $S_{1/2}$ is the 2-sphere of radius $3/2$ centered at p, and $S_{(-1)/2}$ is the 2-sphere of radius $3/2$ centered at q. The 2-sphere S_0 is the same as the one in Section 6. The 3-ball $C_{m/2}$ and the point $p_{m/2}$ are defined as in Section 6. A 3-ball $A'_{m/2}$ is defined similarly to A_n in Section 6; hence

$$A'_{m/2} \subset \mathrm{Cl}(C_{m/2} - C_{(m-1)/2}).$$

The dilation t is defined as before; in other words, $t(S_{m/2}) = S_{(m/2)-1}$ and $t(A'_{m/2}) = A'_{(m/2)-1}$ for every integer m, and $t(L) = L$.

There exists an orientation-preserving homeomorphism g_1 of B_1 onto $C_{(-1)/2}$ that carries a_1 onto an arc from p_{-1} to $p_{(-1)/2}$ in $A'_{(-1)/2}$, and g_1 satisfies conditions similar to (i), (ii), (iii) in Section 6. Similarly, there exists an orientation-preserving homeomorphism g_2 of B_2 onto C_0 that carries a_2 onto an arc from $p_{(-1)/2}$ to p_0 in A'_0, and g_2 satisfies conditions similar to (i), (ii), (iii) in Section 6.

Let us consider the decomposition $d_1: S^3 \to d_1(S^3)$, where the only nondegenerate elements are $t^n(g_2(a_2))$, for every integer n. Then $d_1(S^3)$ is a 3-sphere, and

$$d_1\left(\bigcup_{n=-\infty}^{+\infty} t^n(g_1(a_1)) \cup (p) \cup (q)\right)$$

is an arc α_1 from $d_1(q)$ to $d_1(p)$. Note that $t_1 = d_1 t d_1^{-1}$ is a dilation of $d_1(S^3)$ whose attractive point is $d_1(q)$ and whose repulsive point is $d_1(p)$ such that $t_1(\alpha_1) = \alpha_1$. From this, we can construct a quasi-translation $d_1' t_1 d_1'^{-1}$ on $d_1' d_1(S^3)$ that is in fact $h((a_1, B_1) \# (a_2, B_2))$.

Similarly, considering the decomposition space $d_2(S^3)$ of S^3, where the only nondegenerate elements of the decomposition are $t^n(g_1(a_1))$, for every integer n, we have a quasi-translation $d_2' t_2 d_2'^{-1}$ on $d_2' d_2(S^3)$ with $t_2 = d_2 t d_2^{-1}$. In fact, $d_2' t_2 d_2'^{-1}$ is topologically equivalent to $h((a_2, B_2) \# (a_1, B_1))$.

From the construction, it follows that $d_1' d_1(x) = d_2' d_2(x)$, for every $x \in S^3$. Further, we have the relations

$$d_1' t_1 d_1'^{-1} = d_1' d_1 t d_1^{-1} d_1'^{-1} = d_2' d_2 t d_2^{-1} d_2'^{-1} = d_2' t_2 d_2'^{-1}.$$

This means that the effects of $h((a_1, B_1) \# (a_2, B_2))$ and $h((a_2, B_2) \# (a_1, B_1))$ are the same on $d_1' d_1(S^3) = d_2' d_2(S^3)$. Thus, the proof is complete.

8. Let h be a quasi-translation of S^3 with $h(p) = p$. Then a simple closed curve c in S^3 is called a *translation curve* if $h(c) = c$. Hence, c must contain the point p.

Let c be a translation curve, and choose $r \in c$ $(r \neq p)$. The subarc rp of c that contains $h(r)$ will be denoted by $c_+(r)$, and the other (inversely oriented) subarc rp of c, which contains $h^{-1}(r)$, will be denoted by $c_-(r)$. A translation curve c is called *positively characteristic* if $c_-(r)$ is tame in S^3 for some r, and similarly, c is called *negatively characteristic* if $c_+(r)$ is tame in S^3 for some r. But, clearly, if c is a positively characteristic translation curve, then $c_-(r)$ is tame in S^3 for every $r \in c$ $(r \neq p)$. Similarly, if c is a negatively characteristic translation curve, then $c_+(r)$ is tame in S^3 for every $r \in c$ $(r \neq p)$. Hence, a positively (or negatively) characteristic translation curve is a simple closed curve that is locally tame in S^3 except at the point p.

A translation curve c that is either positively or negatively characteristic is called a *characteristic translation curve*. It is not known whether every quasi-translation has at least one characteristic translation curve. This is the second problem in [8].

With each positively characteristic translation curve c, we shall associate an arc and ball pair (a, B). Let $r \in c$ $(r \neq p)$ and $s \in c_+(r)$ $(s \neq p, s \neq r)$. Since $c_-(r)$ is a tame arc in S^3, there exists a 3-cell B' that is a nice thickening of $c_-(r) - (p)$ in $S^3 - (p)$ plus the point p, that is,

(i) B' is tame in S^3,

(ii) $c_-(r) \subset (B')^o \cup (p)$,

(iii) $c_+(s) \cap B' = (p)$, and

(iv) the pair $(c_-(r), B')$ is trivial.

Now let B be the closure of the complement of B' in S^3, which is a tame 3-cell in S^3. Then $c_+(s)$ is a locally tame arc in B except at p, and the arc $c_+(s)$ and the 3-ball B form a pair $(c_+(s), B)$. We associate the pair $(c_+(s), B)$ with the positively characteristic translation curve c and denote it by (c_+, B_c). It can easily be seen that the equivalence class of the pair $(c_+(s), B)$ does not depend on the choice of the points r and s.

For every quasi-translation h, the collection of all positively characteristic translation curves $\{c\}$ of h is a topological invariant of h. Hence we have the following result.

PROPOSITION 3. *The collection of arc and ball pairs* $\{(c_+, B_c)\}$ *associated with all possible positively characteristic translation curves* $\{c\}$ *of a quasi-translation* h *is a topological invariant of* h.

9. THEOREM 3. *For every quasi-translation* h(a, B), *there exists at least one positively characteristic translation curve.*

Proof. We use the notation of Section 6. Let

$$J = \{(x, y, z) \mid x \leq -2 \text{ or } x \geq 2,\ y = 0,\ z = 0\} \cup (\infty),$$

that is, $J = \mathrm{Cl}(L - I)$. Since the arc α has the property that every subarc of α is locally peripherally unknotted at its initial point, it is easy to see that d(J) is a positively characteristic translation curve of h(a, B), which acts on $d(S^3)$.

THEOREM 4. *Suppose that* c *is a positively characteristic translation curve of a quasi-translation* h(a, B). *Let* (c_+, B_c) *be the corresponding arc and ball pair. Then there exist arc and ball pairs* (a_1, B_1) *and* (a_2, B_2) *such that*

$$(c_+, B_c) = (a_1, B_1) \# (a, B) \# (a_2, B_2).$$

Proof. We use the notation of Section 6. First, we prove that c is the image of an arc c' from p to q, that is, d(c') = c. Let r be a point of $c - d(\alpha)$. Consider the arc $\beta = rh(r)$ on c. Then we have that $c = \bigcup_{n=-\infty}^{+\infty} h^n(\beta) \cup d(\alpha)$. Now $d^{-1}(\beta)$ is an arc in $S^3 - \alpha$. Since t is a dilation, it follows that

$$c' = \bigcup_{n=-\infty}^{+\infty} t^n(d^{-1}(\beta)) \cup (p) \cup (q)$$

is the required arc. The definition of $c'_+(r)$ and $c'_-(r)$, for $r \in c'$ $(r \neq p,\ r \neq q)$, is similar to that of $c_+(r)$ and $c_-(r)$, respectively. We choose $r \in c'$ in such a way that $c'_-(r) \cap C_1 = \emptyset$, and we choose $s \in c'$ in such a way that $c'_+(s) \subset C_{-2}$.

Let B' be a nice thickening of $d(c'_-(r))$, as defined in Section 8, and assume that $B' \cap d(C_0) = d(\alpha)$. Let B" be the closure of the complement of B'. Then (c_+, B_c) can be chosen to be the pair $(d(c'_+(s)), B'')$.

Put $a_1 = d(c'_+(s))$ and $B_1 = d(C_{-1})$. Let (a_2, B_2) be an arc and ball pair equivalent to the sphere and ball pair $(d(S_0), B'')$. Since $(d(S_{-1}), d(C_0))$ is equivalent to $(g(a), C_0)$ and hence to (a, B) and since $(d(S_0), B'')$ is equivalent to (a_2, B_2), it follows from the alternate definition of the composition of arc and ball pairs that

$$(c_+, B_c) = (a_1, B_1) \# (a, B) \# (a_2, B_2).$$

10. Applying Theorem 4, we have the following characterization of a standard translation in the class $\{h(a, B)\}$.

THEOREM 5. *A quasi-translation* h(a, B) *is topologically equivalent to a standard translation if and only if the pair* (a, B) *is trivial.*

Proof. If (a, B) is trivial, then clearly h(a, B) is a standard translation. Now suppose that h(a, B) is topologically equivalent to a standard translation. Then there exists a positively characteristic translation curve c such that (c_+, B_c) is trivial. By Theorem 4, we have the relation

$$(c_+, B_c) = (a_1, B_1) \# (a, B) \# (a_2, B_2)$$

for some (a_1, B_1) and (a_2, B_2). Since (c_+, B_c) is trivial, all the arc and ball pairs (a, B), (a_1, B_1), and (a_2, B_2) are trivial, by Theorem 1. Hence the proof is complete.

Remark. We point out the relation between Theorem 5 and the result of [9], which was written by one of the authors of this paper. In [9], it was proved that if a is one of the Fox-Artin arcs ([3]) and B a suitably chosen 3-cell, then, from the pair (a, B), we can construct a quasi-translation, which is actually h(a, B) of this paper. Moreover, it was proved that h(a, B) is not topologically equivalent to a standard translation. By the same technique, we can prove that for every pair (a, B), the induced quasi-translation h(a, B) is not topologically equivalent to a standard translation, provided a is wild in S^3 whenever B is tamely imbedded in S^3. Theorem 5 is clearly stronger than the statement above, because there exists an arc and ball pair (a, B) such that if B is tamely imbedded in S^3, then a is also tame in S^3, but the pair (a, B) is nontrivial. This pair can easily be constructed from a Wilder arc or a mildly wild arc [4].

11. The first problem in [8] was not correctly stated. It should read as follows: Let h be a quasi-translation of S^3. Suppose that there exist at least one positively characteristic translation curve and at least one negatively characteristic translation curve. Is h topologically equivalent to a standard translation? The answer is negative, by the following theorem.

THEOREM 6. *There exists a quasi-translation of* S^3 *that has a positively and negatively characteristic translation curve but that is not topologically equivalent to a standard translation.*

Proof. We use the notation of Section 6. Let (a, B) be a nontrivial arc and ball pair that is constructed from a Wilder arc ([4]). Let h(a, B) be the quasi-translation associated with (a, B). Hence, by Theorem 5, h(a, B) is not topologically equivalent to a standard translation. It is easy to see that for every $\varepsilon > 0$, there exists a tame 2-sphere S^2 in the 3-sphere S^3 such that

(i) S^2 is contained in the ε-neighborhood of α,

(ii) S^2 separates α from the point ∞ in S^3, and

(iii) $S^2 \cap J$ consists of two points, where J is defined as in the proof of Theorem 3.

Then the translation curve d(J) is locally peripherally unknotted at $d(\alpha)$ in $d(S^3)$. Thus d(J) is not only a positively characteristic translation curve, but also a negatively characteristic translation curve.

Problem. Suppose that a quasi-translation h of S^3 has at least one tame translation curve. Is h topologically equivalent to a standard translation?

REFERENCES

1. P. H. Doyle, *A sufficient condition that an arc in* S^n *be cellular.* Pacific J. Math. 14 (1964), 501-503.

2. C. H. Edwards, Jr., *Concentric tori in the* 3-*sphere.* Bull. Amer. Math. Soc. 67 (1961), 220-222.

3. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space.* Ann. of Math. (2) 49 (1948), 979-990.

4. R. H. Fox and O. G. Harrold, Jr., *The Wilder arcs.* Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), pp. 184-187. Prentice-Hall, Englewood Cliffs, N. J., 1962.

5. O. G. Harrold, Jr., *Locally peripherally unknotted surfaces in* E^3. Ann. of Math. (2) 69 (1959), 276-290.

6. B. v. Kerékjártó, *On a geometrical theory of continuous groups.* Ann. of Math. (2) 27 (1925), 105-117.

7. ———, *Über die fixpunktfreien Abbildungen der Ebene.* Acta Sci. Math. (Szeged) 6 (1934), 226-234.

8. S. Kinoshita, *On quasi-translations in* 3-*space.* Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), pp. 223-226. Prentice-Hall, Englewood Cliffs, N. J., 1962.

9. ———, *On quasi-translations in* 3-*space.* Fund. Math. 56 (1964), 69-79.

10. ———, *Notes on covering transformation groups.* Proc. Amer. Math. Soc. 19 (1968), 421-424.

11. H. Schubert, *Knoten und Vollringe.* Acta Math. 90 (1953), 131-286.

12. C. D. Sikkema, *A duality between certain spheres and arcs in* S^3. Trans. Amer. Math. Soc. 122 (1966), 399-415.

13. C. D. Sikkema, S. Kinoshita, and S. J. Lomonaco, Jr., *Penetration indices of arc and ball pairs and uncountably many quasi-translations of the* 3-*sphere* (to appear).

14. E. Sperner, *Über die fixpunktfreien Abbildungen der Ebene.* Abh. Math. Sem. Univ. Hamburg 10 (1934), 1-48.

The Florida State University
Tallahassee, Florida 32306