ZEROS OF PARTIAL SUMS OF POWER SERIES. II

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1. INTRODUCTION

Problem 7.7 in W. K. Hayman's *Research Problems in Function Theory* [3] is the following: Let $f(z) = \sum_0^\infty a_k z^k$ denote an analytic function whose power series has radius of convergence 1. Set

$$S_n(z) = S_n(z; f) = \sum_{k=0}^{n} a_k z^k \qquad (n = 1, 2, 3, \cdots),$$

and let $\rho_n(f)$ denote the largest of the moduli of the zeros of S_n (with the convention that $\rho_n(f) = \infty$ if $a_n = 0$). Let

$$\rho(f) = \liminf_{n \to \infty} \rho_n(f)$$

and

$$P = \sup_f \rho(f).$$

The problem is to determine the value of P. In [2], J. Clunie and P. Erdös showed that $\sqrt{2} < P < 2$. The present author [1] obtained the estimates $1.7 < P \leq 12^{1/4}$. Later, J. L. Frank [1] improved these bounds to $1.7818 < P < 1.82$.

In the present paper, I determine the exact value of P. The determination depends on certain algebraic relations between the coefficients of a power series and the zeros of its partial sums. These relations are most conveniently expressed in terms of the polynomials $B_n(z; z_0, \cdots, z_{n-1})$ defined by

$$(1.1) \qquad B_0(z) = 1, \qquad B_n(z; z_0, \cdots, z_{n-1}) = z^n - \sum_{k=0}^{n-1} z_k^{n-k} B_k(z; z_0, \cdots, z_{k-1}).$$

(Here $B_k(z; z_0, \cdots, z_{k-1})$ is to be interpreted as 1 when $k = 0$.)

Set

$$H_n = \max |B_n(0; z_0, \cdots, z_{n-1})| \qquad (n = 0, 1, 2, \cdots),$$

where the maximum is taken over all sequences $\{z_k\}_0^{n-1}$ whose terms lie on the unit circle. On the basis of the algebraic relations mentioned above, we obtain the following result.

THEOREM 1. $P = \sup_{1 \leq n < \infty} H_n^{1/n} = \lim_{n \to \infty} H_n^{1/n}$.

Received March 4, 1969.
This research was supported in part by NSF Grant GP-8225 and Basic Science Development Grant.

Michigan Math. J. 17 (1970).

Theorem 1 yields numerical lower bounds for P. To obtain numerical upper bounds, we need a slightly more complicated result. For $m = 1, 2, \cdots$ and $0 \leq u < 1$, let

$$T_m(u) = \max \sum_{k=m}^{\infty} u^k \left| B_k(0; w_0, \cdots, w_{m-1}, 0, \cdots, 0) \right|, \tag{1.2}$$

where the maximum is taken over all sequences $\{w_j\}_0^{m-1}$ whose terms lie on the unit circle. Let u_m denote the positive root of the equation $T_m(u) = 1$, and let $K_m = 1/u_m$.

THEOREM 2. $P = \inf_{1 \leq m < \infty} K_m = \lim_{m \to \infty} K_m$.

On the basis of Theorems 1 and 2, P can (at least in theory) be calculated as accurately as desired. One can easily verify that $K_1 = 2$. The upper bound $P \leq 12^{1/4}$ in [1] was obtained by proving that $K_2 \leq 12^{1/4}$. Since then, the estimates

$$1.838 < K_2 < 1.839 \quad \text{and} \quad 1.81 < K_3 < 1.82$$

have been obtained by machine computation.

2. ALGEBRAIC PRELIMINARIES

For $k = 0, 1, 2, \cdots$, let $\mathcal{S}^k$ denote the operator that transforms the analytic function $f(z) = \sum_{m=0}^{\infty} a_m z^m$ into

$$\mathcal{S}^k f(z) = \sum_{m=0}^{\infty} a_{m+k} z^m .$$

If we rewrite (1.1) in the form

$$z^m = \sum_{k=0}^{m} z_k^{m-k} B_k(z; z_0, \cdots, z_{k-1})$$

and substitute this into the power series for f, we obtain

$$\begin{aligned} f(z) &= \sum_{m=0}^{\infty} a_m \sum_{k=0}^{m} z_k^{m-k} B_k(z; z_0, \cdots, z_{k-1}) \\ &= \sum_{k=0}^{\infty} \left(\sum_{m=k}^{\infty} a_m z_k^{m-k} \right) B_k(z; z_0, \cdots, z_{k-1}) \qquad (2.1) \\ &= \sum_{k=0}^{\infty} \mathcal{S}^k f(z_k) B_k(z; z_0, \cdots, z_{k-1}), \end{aligned}$$

whenever the interchange in the order of summation can be justified. In particular, (2.1) holds if f is a polynomial.

We now establish some of the basic properties of the polynomials $B_n(z; z_0, \cdots, z_{n-1})$.

LEMMA 1. *For each positive integer* n,

$$B_n(\lambda z; \lambda z_0, \cdots, \lambda z_{n-1}) = \lambda^n B_n(z; z_0, \cdots, z_{n-1}), \tag{2.2}$$

$$B_n(z_0; z_0, \cdots, z_{n-1}) = 0, \tag{2.3}$$

$$\mathscr{S}^k B_n(z; z_0, \cdots, z_{n-1}) = B_{n-k}(z; z_k, \cdots, z_{n-1}) \quad (0 \leq k \leq n), \tag{2.4}$$

$$B_{n+1}(z; z_0, \cdots, z_n) = zB_n(z; z_1, \cdots, z_n) - z_0 B_n(z_0; z_1, \cdots, z_n), \tag{2.5}$$

$$B_{n+1}(0; z_0, \cdots, z_n) = -z_0 B_n(z_0; z_1, \cdots, z_n). \tag{2.6}$$

Proof. Properties (2.2) and (2.3) follow from (1.1) and mathematical induction. It is enough to establish (2.4) for the case $k = 1$, and this case follows in the same way. Property (2.5) is then a consequence of (2.4), and (2.6) follows from (2.5).

LEMMA 2. *The following identities hold:*

$$B_n(z; z_0, \cdots, z_{n-1}) = \sum_{k=0}^{n} B_{n-k}(w_k; z_k, \cdots, z_{n-1}) B_k(z; w_0, \cdots, w_{k-1}), \tag{2.7}$$

$$B_n(z; z_0, \cdots, z_{n-1}) = \sum_{k=0}^{n} B_{n-k}(0; z_k, \cdots, z_{n-1}) z^k, \tag{2.8}$$

$$z^n B_n(1/z; z_n, \cdots, z_1) = \sum_{k=0}^{n} B_k(0; z_k, \cdots, z_1) z^k. \tag{2.9}$$

Proof. We deduce from (2.1) and (2.4) that

$$\begin{aligned} B_n(z; z_0, \cdots, z_{n-1}) &= \sum_{k=0}^{n} \mathscr{S}^k B_n(w_k; z_0, \cdots, z_{n-1}) B_k(z; w_0, \cdots, w_{k-1}) \\ &= \sum_{k=0}^{n} B_{n-k}(w_k; z_k, \cdots, z_{n-1}) B_k(z; w_0, \cdots, w_{k-1}). \end{aligned}$$

To obtain (2.8) from (2.7), take $w_k = 0$ $(0 \leq k \leq n)$. An obvious manipulation of (2.8) yields (2.9).

Identity (2.9) deserves a remark. The right member is the nth partial sum of the power series

$$\sum_{k=0}^{\infty} B_k(0; z_k, \cdots, z_1) z^k, \tag{2.10}$$

and by (2.3), the left member has a zero at $z = 1/z_n$. If $z_n = 0$, then the coefficient of z^n is 0, and (by convention) ∞ is a zero of the nth partial sum. Therefore (2.9) allows us to construct a power series by specifying one zero of each partial sum.

Conversely, every power series with constant term 1 can be written in the form (2.10), where, for each $n > 0$, z_n is the reciprocal of a zero of the nth partial sum. A proof of this is contained in the following lemma.

LEMMA 3. *Let* $\sum_{k=0}^{\infty} a_k z^k$ *denote a formal power series with* $a_0 = 1$. *For each positive integer* n, *choose a complex number* z_n *such that*

$$\sum_{k=0}^{n} a_k z_n^{n-k} = 0 .$$

Then

$$a_n = B_n(0; z_n, z_{n-1}, \cdots, z_1) \qquad (n = 1, 2, 3, \cdots). \tag{2.11}$$

Proof. The proof is by induction on n; for $n = 1$, we have the relation $z_1 + a_1 = 0$. Therefore

$$a_1 = -z_1 = B_1(0; z_1).$$

Let m be such that (2.11) holds for $n = 1, 2, \cdots, m$. Then

$$0 = \sum_{k=0}^{m+1} a_k z_{m+1}^{m+1-k} = \sum_{k=0}^{m} B_k(0; z_k, \cdots, z_1) z_{m+1}^{m+1-k} + a_{m+1},$$

by the induction hypothesis. Therefore, provided $z_{m+1} \neq 0$,

$$a_{m+1} = -z_{m+1}^{m+1} \sum_{k=0}^{m} B_k(0; z_k, \cdots, z_1) z_{m+1}^{-k} = -z_{m+1} B_m(z_{m+1}; z_m, \cdots, z_1),$$

by (2.9). Using (2.6), we obtain the equation

$$a_{m+1} = B_{m+1}(0; z_{m+1}, \cdots, z_1). \tag{2.12}$$

If $z_{m+1} = 0$, the definition of z_{m+1} guarantees that $a_{m+1} = 0$. The validity of (2.12) in this case follows from (2.3), and this completes the proof.

LEMMA 4. *If* $0 \leq n_1 \leq n$, *then*

$$B_n(z; z_n, \cdots, z_1) = \sum_{k=0}^{n_1} B_k(0; z_k, \cdots, z_1) B_{n-k}(z; z_n, \cdots, z_{n_1+1}, 0, \cdots, 0).$$

Proof. It follows from (2.7) that

$$B_n(z; z_n, \cdots, z_1) = \sum_{k=0}^{n} B_{n-k}(w_k; z_{n-k}, \cdots, z_1) B_k(z; w_0, \cdots, w_{k-1}).$$

Let

$$w_k = \begin{cases} z_{n-k} & (0 \leq k < n - n_1), \\ 0 & (n - n_1 \leq k \leq n). \end{cases}$$

From (2.3) we obtain the identity

$$B_n(z; z_n, \cdots, z_1) = \sum_{k=n-n_1}^{n} B_{n-k}(0; z_{n-k}, \cdots, z_1) B_k(z; z_n, \cdots, z_{n_1+1}, 0, \cdots, 0).$$

The replacement of k by n - k in the summation yields the lemma.

3. LOWER BOUNDS

LEMMA 5. *If* $0 \leq n_1 \leq n$, *then* $H_n \geq H_{n_1} H_{n-n_1}$.

Proof. Choose points $z_1, \cdots, z_n$ on the unit circle so that

$$H_{n_1} = |B_{n_1}(0; z_{n_1}, \cdots, z_1)|$$

and

$$H_{n-n_1} = |B_{n-n_1}(0; z_n, \cdots, z_{n_1+1})|.$$

Then

$$H_n \geq \max_{|\lambda|=1} |B_n(0; \lambda z_n, \cdots, \lambda z_{n_1+1}, z_{n_1}, \cdots, z_1)|.$$

From Lemma 4 and equation (2.2) we obtain the identity

$$B_n(0; \lambda z_n, \cdots, \lambda z_{n_1+1}, z_{n_1}, \cdots, z_1)$$
$$= \sum_{k=0}^{n_1} B_k(0; z_k, \cdots, z_1) \lambda^{n-k} B_{n-k}(0; z_n, \cdots, z_{n_1+1}, 0, \cdots, 0).$$

Let $Q(\lambda)$ denote the polynomial obtained by dividing the two sides of this equation by λ^{n-n_1}. Then

$$H_n \geq \max_{|\lambda|=1} |Q(\lambda)| \geq |Q(0)| = |B_{n_1}(0; z_{n_1}, \cdots, z_1) B_{n-n_1}(0; z_n, \cdots, z_{n_1+1})|$$
$$= H_{n_1} H_{n-n_1}.$$

LEMMA 6. $\lim_{n\to\infty} H_n^{1/n} = \sup_{1 \leq n < \infty} H_n^{1/n}$.

Proof. Let $m \geq 1$ be fixed. For $n \geq m$, let $n = qm + d$ $(0 \leq d < m)$. Lemma 5 implies the inequalities

$$H_n \geq H_{qm} H_d \geq H_m^q H_1^d = H_m^q.$$

Therefore

$$H_n^{1/n} \geq H_m^{(n-d)/mn} = H_m^{1/m} H_m^{-d/mn}.$$

Letting $n \to \infty$, we obtain the relation

$$\liminf_{n\to\infty} H_n^{1/n} \geq H_m^{1/m}.$$

Therefore

$$\liminf_{n\to\infty} H_n^{1/n} \geq \sup_{1\leq m<\infty} H_m^{1/m},$$

and the lemma follows.

Using (1.1) and induction, we obtain the inequality $H_n \leq 2^{n-1}$ $(n > 0)$; this guarantees that $\sup H_n^{1/n} \leq 2$.

THEOREM 3. $P \geq H_m^{1/m}$ $(m = 1, 2, 3, \cdots)$.

Proof. Let $\{w_k\}_1^m$ and $\{\lambda_q\}_0^\infty$ be sequences whose terms lie on the unit circle. For each positive integer n, write $n = qm + j$ $(1 \leq j \leq m)$, and let $z_n = \lambda_q w_j$. The function

$$f(z) = \sum_{k=0}^{\infty} B_k(0; z_k, \cdots, z_1) z^k \tag{3.1}$$

has the property that $\rho_n(f) \geq 1$ $(n = 1, 2, 3, \cdots)$. If we choose $\{w_k\}_1^m$ so that

$$H_m = |B_m(0; w_m, \cdots, w_1)|,$$

the method used in the proof of Lemma 5 allows us to choose the sequence $\{\lambda_q\}_0^\infty$ in such a way that

$$|B_{mq}(0; z_{mq}, \cdots, z_1)| \geq H_m^q \quad (q = 1, 2, \cdots). \tag{3.2}$$

Let R denote the radius of convergence of the series (3.1). From (3.2) and the remark preceding Theorem 3, it follows that $H_m^{1/m} \leq R^{-1} \leq 2$. Consequently, the function $f_1(z) = f(Rz)$ has radius of convergence 1 and satisfies the condition

$$\rho_n(f_1) \geq R^{-1} \geq H_m^{1/m} \quad (n = 1, 2, 3, \cdots).$$

Therefore $P \geq \rho(f_1) \geq H_m^{1/m}$.

Let

$$H = \sup_{1\leq n<\infty} H_n^{1/n} = \lim_{n\to\infty} H_n^{1/n}.$$

It follows at once from Theorem 3 that $P \geq H$. To complete the proof of Theorem 1, we need the following result.

THEOREM 4. $P \leq H$.

Proof. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ have radius of convergence 1, and suppose that $a_0 = 1$ (if necessary, divide by the first nonvanishing term of the series; this division does not affect $\rho(f)$). For each positive integer n, let $1/z_n$ denote a zero of $S_n(z; f)$ of modulus $\rho_n(f)$. From Lemma 3 it follows that

$$f(z) = \sum_{k=0}^{\infty} B_k(0; z_k, \cdots, z_1) z^k,$$

and Lemma 4 implies that

$$a_n = \sum_{k=0}^{n_1} a_k B_{n-k}(0; z_n, \cdots, z_{n_1+1}, 0, \cdots, 0) \qquad (n > n_1).$$

Fix n_1, and let

$$x_{n_1} = \sup_{n_1 < n < \infty} |z_n| = \left(\inf_{n_1 < n < \infty} \rho_n(f) \right)^{-1}.$$

For all $n > n_1$, we have the inequalities

$$|a_n| \leq \sum_{k=0}^{n_1} |a_k| x_{n_1}^{n-k} H_{n-k} = (H_n^{1/n} x_{n_1})^n \sum_{k=0}^{n_1} |a_k| x_{n_1}^{-k} H_{n-k} H_n^{-1}$$

$$\leq (H_n^{1/n} x_{n_1})^n \sum_{k=0}^{n_1} |a_k| (x_{n_1} H_k^{1/k})^{-k}.$$

Taking nth roots and letting $n \to \infty$, we obtain the relations

$$1 = \limsup_{n \to \infty} |a_n|^{1/n} \leq x_{n_1} \limsup_{n \to \infty} H_n^{1/n} = x_{n_1} H.$$

Therefore

$$H \geq \frac{1}{x_{n_1}} = \inf_{n_1 < n < \infty} \rho_n(f).$$

Letting $n_1 \to \infty$, we obtain the inequality $H \geq \rho(f)$. Therefore $H \geq P$, which completes the proof.

We note that the supremum P is actually assumed; one can modify the construction of Theorem 3 in such a way as to produce an analytic function f with radius of convergence 1 and with the property that $\rho_n(f) \geq H$ $(n = 1, 2, 3, \cdots)$.

4. UPPER BOUNDS

In order to justify the definitions of u_m and K_m, we must show that the series (1.2) does in fact converge for $0 \leq u < 1$, and that $T_n(u)$ assumes the value 1.

LEMMA 7. *If* $0 < u \leq u + h < 1$, *then*

$$T_m(u+h) \geq T_m(u) \left(1 + \frac{mh}{u}\right) \qquad (m = 1, 2, 3, \cdots).$$

Proof. If $k \geq m$, then

$$(u+h)^k \geq u^k + ku^{k-1}h \geq u^k\left(1+\frac{mh}{u}\right).$$

Choose points $\{w_j\}_0^{m-1}$ on the unit circle such that

$$T_m(u) = \sum_{k=m}^{\infty} u^k \,|B_k(0; w_0, \cdots, w_{m-1}, 0, \cdots, 0)|\,.$$

Then

$$T_m(u+h) \geq \sum_{k=m}^{\infty} (u+h)^k \,|B_k(0; w_0, \cdots, w_{m-1}, 0, \cdots, 0)|$$

$$\geq \left(1+\frac{mh}{u}\right) \sum_{k=m}^{\infty} u^k \,|B_k(0; w_0, \cdots, w_{m-1}, 0, \cdots, 0)|$$

$$= \left(1+\frac{mh}{u}\right) T_m(u)\,.$$

LEMMA 8. *If* m *is a positive integer and* $0 \leq u < 1$, *then*

$$u^m H_m \leq T_m(u) \leq (uP)^m\left(1+\frac{u}{(P-1)(1-u)}\right). \tag{4.1}$$

Proof. The first part of the inequality follows from the definitions of H_m and T_m. To obtain the second part, we observe that

$$T_m(u) \leq u^m H_m + \sum_{k=m+1}^{\infty} u^k (\max|B_k(0; w_0, \cdots, w_{m-1}, 0, \cdots, 0)|)\,,$$

where each of the maxima is taken over sequences $\{w_j\}_0^{m-1}$ whose terms lie on the unit circle. Now

$$B_k(0; w_0, \cdots, w_{m-1}, 0, \cdots, 0) = -\sum_{j=0}^{m-1} w_j^{k-j} B_j(0; w_0, \cdots, w_{j-1})\,,$$

so that

$$\max|B_k(0; w_0, \cdots, w_{m-1}, 0, \cdots, 0)| \leq \sum_{j=0}^{m-1} H_j \leq \sum_{j=0}^{m-1} H^j$$

$$= \sum_{j=0}^{m-1} P^j < \frac{P^m}{P-1}.$$

Therefore, since $H_m \leq P^m$, we obtain the inequality

$$T_m(u) \leq u^m P^m + \frac{P^m}{P-1}\sum_{k=m+1}^{\infty} u^k = (uP)^m\left(1+\frac{u}{(P-1)(1-u)}\right),$$

which completes the proof. As a special case, we note that

$$T_m(1/P) \leq 1 + \frac{1}{(P-1)^2}. \tag{4.2}$$

It follows from Lemma 7 that the function T_m is strictly increasing. We now establish its continuity. Suppose $0 \leq u \leq u + h < 1$. Then

$$T_m(u+h) = \max \sum_{k=m}^{\infty} (u^k + [(u+h)^k - u^k]) \, |B_k(0; w_0, \cdots, w_{m-1}, 0, \cdots, 0)|$$

$$\leq T_m(u) + \max \sum_{k=m}^{\infty} ((u+h)^k - u^k) \, |B_k(0; w_0, \cdots, w_{m-1}, 0, \cdots, 0)|$$

$$\leq T_m(u) + \sum_{k=m}^{\infty} hk\,(u+h)^{k-1} \frac{P^m}{P-1} = T_m(u) + \frac{hP^m}{P-1} \sum_{k=m}^{\infty} k(u+h)^{k-1},$$

which allows us to conclude that T_m is continuous. If $m > 1$, then

$$H_m \geq H_{m-2}H_2 = 2H_{m-2} \geq 2H_1^{m-1} = 2.$$

Therefore it follows from (4.1) that $T_m(u) > 1$ if u is sufficiently close to 1. Consequently, if $m > 1$, the function T_m assumes the value 1 exactly once, which justifies our definition of u_m. For $m = 1$, it is easy to verify that

$$T_1(u) = \frac{u}{1-u}.$$

so that $u_1 = 1/2$.

Let $f(z) = 1 + \sum_{k=1}^{\infty} A_k z^k$ be analytic in $|z| < 1$, let m be a nonnegative integer, and let $\{z_k\}_0^{\infty}$ be a sequence of points in $|z| < 1$ such that $z_k = 0$ for $k \geq m$. In this case, there is no difficulty in justifying the expansion (2.1), and we have the identity

$$f(z) = \sum_{k=0}^{\infty} \mathcal{S}^k f(z_k) B_k(z; z_0, \cdots, z_{k-1})$$

$$= \sum_{k=0}^{m-1} \mathcal{S}^k f(z_k) B_k(z; z_0, \cdots, z_{k-1}) + \sum_{k=m}^{\infty} A_k B_k(z; z_0, \cdots, z_{m-1}, 0, \cdots, 0).$$

If in addition $\mathcal{S}^k f(z_k) = 0$ $(0 \leq k < m)$, then

$$f(0) = 1 = \sum_{k=m}^{\infty} A_k B_k(0; z_0, \cdots, z_{m-1}, 0, \cdots, 0). \tag{4.3}$$

Suppose that f has the further property that $|A_k| \leq 1$ for $k \geq m$. We can then use equation (4.3) to obtain a positive lower bound on the largest of the numbers

$|z_0|, \cdots, |z_{m-1}|$. This bound is contained in Theorem 5. Theorems 5 and 6 are direct extensions of Theorems 1 and 2 of [1]. Their proofs are quite similar to those in [1], and therefore we omit them.

THEOREM 5. *If* f *satisfies all the hypotheses above, then* $\max_{0 \leq k < m} |z_k| \geq u_m$.

THEOREM 6. $P \leq K_m$ $(m = 1, 2, 3, \cdots)$.

It remains to prove Theorem 2. Theorem 2 is an immediate consequence of the following result.

THEOREM 7. *For every positive integer* m,

$$P \leq K_m < P + \frac{3}{m}.$$

Proof. From (4.2), Lemma 7, and Theorem 6, we obtain the inequality

$$1 + \frac{1}{(P-1)^2} \geq T_m(1/P) \geq T_m(u_m)\left(1 + \frac{m}{u_m}(P^{-1} - u_m)\right) = 1 + \frac{m}{u_m P} - m.$$

Therefore

$$(P-1)^{-2} \geq m(K_m/P - 1),$$

and

$$K_m \leq P + (P-1)^{-2} P/m.$$

Using the lower bound $1.78 < P$, we obtain the inequality

$$K_m < P + \frac{3}{m},$$

which, in view of Theorem 6, completes the proof.

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