AN ELEMENTARY PROOF OF THE FIXED-POINT THEOREM OF BROWDER AND KIRK

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1. INTRODUCTION

In [1] and [3], F. E. Browder and W. A. Kirk showed independently that a nonexpansive self-mapping of a nonempty, closed, convex set in a uniformly convex Banach space has a fixed point. Their proofs are similar, and both are based on Zorn's Lemma and other nonelementary theorems of functional analysis.

We shall give an elementary proof of this fixed-point theorem, using only the definition of uniform convexity and some basic theorems of topology and analysis.

2. NOTATION AND DEFINITIONS

Let B be a uniformly convex Banach space with norm $\| \;\|$ and zero element Θ. Let K be a nonempty, closed, bounded, convex subset of B, and suppose (without loss of generality) that $\Theta \in K$. Let d(X) denote the diameter of the set $X \subset B$, and set $a(X) = \inf_{x \in X} \|x\|$. Finally, let $I_1 = (0, 1]$ and $I_2 = (0, 2]$.

The following definition of uniform convexity is equivalent to the classical one [2].

Definition 1. The Banach space B is called *uniformly convex* if there exists an increasing, positive function $\delta: I_2 \to I_1$ such that the inequalities $\|x\| \leq r$, $\|y\| \leq r$, and $\|x - y\| \geq \varepsilon r$ imply that

$$\left\|\frac{x+y}{2}\right\| \leq (1 - \delta(\varepsilon))r \qquad (x, y \in B).$$

It is obvious that $\lim_{\varepsilon \to 0} \delta(\varepsilon) = 0$ and $\delta(2) = 1$. We denote the inverse of δ by η, and we observe that $\lim_{y \to 0} \eta(y) = 0$.

Definition 2. A transformation $F: K \to K$ is called *nonexpansive* if the inequality $\|Fx - Fy\| \leq \|x - y\|$ holds for all x and y in K. A transformation F is a *contraction* if there exists a constant k $(0 \leq k < 1)$ such that $\|Fx - Fy\| \leq k\|x - y\|$ for all $x, y \in K$.

3. THE THEOREM OF BROWDER AND KIRK

THEOREM. *Every nonexpansive mapping* $F: K \to K$ *has at least one fixed point.*

LEMMA. *If* u, v, w *are elements of* B *such that*

$$\|u - w\| \leq R, \qquad \|v - w\| \leq R, \qquad \text{and} \left\|w - \frac{u+v}{2}\right\| \geq r > 0,$$

Received February 10, 1969.

then $\|u - v\| \leq R\eta\left(\frac{R - r}{R}\right)$.

To prove the lemma, we rewrite the condition $\left\|w - \frac{u+v}{2}\right\| \geq r > 0$ in the form

$$\left\|\frac{(w - u) + (w - v)}{2}\right\| \geq \left(1 - \frac{R - r}{R}\right)R,$$

and the result follows immediately from Definition 1.

Proof of the theorem. For $\varepsilon \in I_1$, set $F_\varepsilon = (1 - \varepsilon)F$. Obviously, F_ε is a contraction of K, and by the well-known Banach Fixed-Point Theorem, there exists for every $\varepsilon \in I_1$ an $x_\varepsilon \in K$ such that $x_\varepsilon = F_\varepsilon x_\varepsilon$. Now we have the relations

$$\|x_\varepsilon - Fx_\varepsilon\| = \|F_\varepsilon x_\varepsilon - Fx_\varepsilon\| = \varepsilon \|Fx_\varepsilon\| \leq \varepsilon\, d(K);$$

hence $\inf_{x \in K} \|x - Fx\| = 0$.

Set $C_\varepsilon = \{x\colon \|x - Fx\| \leq \varepsilon\}$ and $D_\varepsilon = \{x \in C_\varepsilon\colon \|x\| \leq a + \varepsilon\}$, where $a = \lim_{\varepsilon \to 0} a(C_\varepsilon)$. It suffices to prove that the intersection of all sets C_ε is nonempty. If this were false, it would follow that $a > 0$, because each C_ε is closed.

Choose elements u_1, u_2 in C_ε. Observe that for $i = 1, 2$,

$$(1)\quad \left\|u_i - F\left(\frac{u_1 + u_2}{2}\right)\right\| \leq \|u_i - Fu_i\| + \left\|Fu_i - F\left(\frac{u_1 + u_2}{2}\right)\right\| \leq \varepsilon + \frac{1}{2}\|u_1 - u_2\|$$

and

$$(2)\quad \left\|u_i - \frac{u_1 + u_2}{2}\right\| < \varepsilon + \frac{1}{2}\|u_1 - u_2\|.$$

Moreover, in view of the relation

$$\|u_1 - u_2\| \leq \left\|u_1 - \frac{1}{2}\left(\frac{u_1 + u_2}{2} + F\left(\frac{u_1 + u_2}{2}\right)\right)\right\| + \left\|u_2 - \frac{1}{2}\left(\frac{u_1 + u_2}{2} + F\left(\frac{u_1 + u_2}{2}\right)\right)\right\|,$$

the inequality

$$(3)\quad \left\|u_i - \frac{1}{2}\left(\frac{u_1 + u_2}{2} + F\left(\frac{u_1 + u_2}{2}\right)\right)\right\| \geq \frac{1}{2}\|u_1 - u_2\|$$

holds for at least one of the values $i = 1, 2$. Now the lemma and inequalities (1), (2), and (3) imply the relations

$$\left\|\frac{u_1 + u_2}{2} - F\left(\frac{u_1 + u_2}{2}\right)\right\| \leq \left(\varepsilon + \frac{1}{2}\|u_1 - u_2\|\right)\eta\left(\frac{\varepsilon}{\varepsilon + \|u_1 - u_2\|/2}\right)$$

$$\leq \sup_{0 < \xi \leq d(K)/2} (\varepsilon + \xi)\,\eta\left(\frac{\varepsilon}{\varepsilon + \xi}\right)$$

$$\leq \max\left[\sup_{0<\xi\leq\sqrt{\varepsilon}-\varepsilon} (\varepsilon+\xi)\eta\left(\frac{\varepsilon}{\varepsilon+\xi}\right), \sup_{\sqrt{\varepsilon}-\varepsilon<\xi\leq d(K)/2} (\varepsilon+\xi)\eta\left(\frac{\varepsilon}{\varepsilon+\xi}\right)\right]$$

$$\leq \max\left[2\sqrt{\varepsilon}, \left(\frac{d(K)}{2}+\varepsilon\right)\eta(\sqrt{\varepsilon})\right].$$

Denoting the last term by $\phi(\varepsilon)$, we can say that if $u_1, u_2 \in C_\varepsilon$, then $(u_1+u_2)/2 \in C_{\phi(\varepsilon)}$. Obviously, $\lim_{\varepsilon\to 0} \phi(\varepsilon) = 0$.

Now let $u_1, u_2 \in D_\varepsilon$. We have the inequalities $\|u_1\| \leq a+\varepsilon$ and $\|u_2\| \leq a+\varepsilon$, and since $(u_1+u_2)/2 \in C_{\phi(\varepsilon)}$, the relation $\|(u_1+u_2)/2\| \geq a(C_{\phi(\varepsilon)})$ follows. Using once again our lemma, we obtain the inequality

$$d(D_\varepsilon) = \sup_{u_1, u_2 \in D_\varepsilon} \|u_1 - u_2\| \leq (a+\varepsilon)\eta\left(\frac{a+\varepsilon - a(C_{\phi(\varepsilon)})}{a+\varepsilon}\right),$$

and $\lim_{\varepsilon\to 0} d(D_\varepsilon) = 0$. By Cantor's theorem, the intersection of all D_ε is nonempty, and our result is proved.

REFERENCES

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