COBORDISM AND BUNDLES OVER SPHERES

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1. INTRODUCTION

P. E. Conner and E. E. Floyd [3] have determined when a (smooth, closed) manifold is cobordant modulo 2 to a bundle over the 1-sphere S^1. If w_i is the ith Stiefel-Whitney class of M^n and σ_n is the modulo-2 fundamental class, then M^n is cobordant to a bundle over S^1 if and only if the Stiefel-Whitney number $\langle w_n, \sigma_n \rangle$ vanishes. We study the analogous question for bundles over S^2. Let

$$C(M^n) = \begin{cases} \langle w_n, \sigma_n \rangle & \text{(n even)}, \\ \langle w_2 \cdot w_{n-2}, \sigma_n \rangle & \text{(n odd)}. \end{cases}$$

THEOREM 1. *If* M^n *is cobordant to a bundle over* S^2, *then* $C(M^n) = 0$. *If* $C(M^n) = 0$, *then* M^n *is cobordant to a connected bundle over* S^2 *with group* $U(1)$.

This answers in the negative the question of Conner [2, end of paper] whether the generator of the oriented cobordism group $\Omega_5 \cong Z_2$ can be represented by a bundle over S^2. However, we do not know the complete oriented analogue of Theorem 1.

According to Corollary 6.2 of [3], the square of a bundle over S^1 is cobordant to a bundle over S^2. We prove a generalization and an analogue of this result.

COROLLARY 1. *The product of two bundles over* S^1 *is cobordant to a bundle over* S^2.

THEOREM 2. *The square of a bundle over* S^2 *is cobordant to a bundle over* S^4.

In Section 2, we derive some necessary conditions for a manifold to be a bundle over S^k $(k > 0)$ by applying the theorem of E. H. Brown and F. P. Peterson [1, Section 1] on relations among characteristic classes to the fiber of the bundle. In Section 3, we construct some bundles over S^2, S^4, and S^8; and in Section 4, we show that we have enough bundles over S^2 to generate the kernel of the character $C: \mathfrak{N}_* \to Z_2$, where $\mathfrak{N}_*$ denotes the unoriented cobordism ring. In Section 5, we show how to make our bundles connected, and in Section 6 we prove Theorem 2.

An optimistic conjecture is that the necessary conditions derived from Proposition 2.1 are sufficient for a manifold to be cobordant to a bundle over S^k for $k = 1, 2, 4, 8$. (They are, for $k = 1, 2$.) For other values of k, see Theorem 8.1 of [3].

2. NECESSARY CONDITIONS

Let $k > 0$ and $n > k$, and let M^n be a bundle over S^k with projection $p: M^n \to S^k$ and fiber F. The tangent bundle of M^n decomposes as a direct sum

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$$\tau M^n = p^* \tau S^k \oplus \bar{\tau} F,$$

where $\bar{\tau} F$ is a bundle over M^n whose restriction to each fiber is τF. Because $w(\tau S^k) = 1$, it follows that

$$w_i(M^n) = 0 \qquad (n - k < i \leq n).$$

Let τ_M: $M^n \to BO$ classify the stable tangent bundle of M^n, and let the modulo-2 Steenrod algebra A act on the left of $H^*(BO)$ and $H^*(M^n)$ as in [1]. (All our homology is modulo 2.)

PROPOSITION 2.1. $\tau_M^* Sq^r(H^s(BO) Sq^t) = 0$, *if* $r > s + t - k$ *and* $2t > n - s - k$.

Proof. Let j: $F \to M$ be the inclusion of the fiber. Then

$$j^* w(M) = j^* w(\bar{\tau} F) = w(F),$$

and hence $j^* \tau_M^* = \tau_F^*$. Let $u \in H^s(BO) Sq^t$. Then $\tau_F^* u = 0$, by [1], because $2t > n - k - s = \dim F - s$. Thus $j^* \tau_M^* u = 0$, and we can pull back $\tau_M^* u$ to a class $v \in H^{s+t}(M, F)$. Now it suffices to show that $Sq^r v = 0$.

Let D_+^k and D_-^k denote the northern and southern hemispheres of S^k. Since the bundle is trivial over a hemisphere, we have the relations

$$H^*(M, F) \cong H^*(M, F \times D_+^k) \cong H^*(F \times D_-^k, F \times S^{k-1}) \cong H^*(F) \otimes H^*(D_-^k, S^{k-1}).$$

Let x_k generate $H^k(D_-^k, S^{k-1})$, and suppose that v is $y \otimes x_k$ under this isomorphism. Then

$$Sq^r v = Sq^r(y \otimes x_k) = (Sq^r y) \otimes x_k = 0,$$

because $r > s + t - k = \dim y$.

COROLLARY 2.2. *If* M^n *is a bundle over* S^2 *and* $n = 2t$ *or* $n = 2t + 1$, *then* $w_{2t} = 0$ *and* $w_{2t-1} = 0$.

Proof. Take $k = 2$ and $s = 0$ in Proposition 2.1. Then, because the ith Wu class v_i of M^n is $\tau_M^*(1) Sq^i$, we see by Proposition 2.1 that $Sq^{t-1} v_t = 0$ and $Sq^t v_t = 0$. Now

$$w_i = v_i + Sq^1 v_{i-1} + \cdots,$$

and by [1], $v_j = 0$ for $j > t$; therefore

$$w_{2t} = Sq^t v_t = 0, \qquad w_{2t-1} = Sq^{t-1} v_t = 0.$$

Remark 2.3. From Corollary 2.2 we deduce that if M^n is cobordant to a bundle over S^2, then certain Stiefel-Whitney numbers must vanish, namely those corresponding to w_n, $w_1 \cdot w_{n-1}$ for even n and to w_n, $w_1 \cdot w_{n-1}$, $w_1^2 \cdot w_{n-2}$, $w_2 \cdot w_{n-2}$ for odd n. Moreover, if we compute some relations among Stiefel-Whitney classes, using [1], we see that the conditions of Theorem 1 imply those above. In fact, if n is even, then $w_1 \cdot w_{n-1} = \binom{n-3}{2} w_n$; and if n is odd, then $w_n = 0$, $w_1 \cdot w_{n-1} = 0$, $w_1^2 \cdot w_{n-2} = 0$.

3. SOME BUNDLES OVER S^2, S^4, AND S^8

Let k be 2, 4, or 8, in this section. There exists a real k-plane bundle η over S^k such that $w(\eta) = 1 + x$, where x generates $H^k(S^k)$. (Take η to be the underlying real bundle of the complex, quaternion, or Cayley line bundles, according as $k = 2, 4$, or 8.)

Definition 3.1. Let $P_k^{m+k} = RP(\eta \oplus (m+1-k)R)$, for $m \geq k-1$.

Note that P_k^{m+k} is a bundle over S^k whose fiber is real projective m-space P^m.

PROPOSITION 3.2. *If* m *is even, then the cobordism class of* P_k^{m+k} *is an indecomposable element of* $\mathfrak{N}_*$.

Proof. The cohomology ring of P_k^{m+k} is a free $H^*(S^k)$-module with basis $1, c, \cdots, c^m$, where $c \in H^1(P_k^{m+k})$, and with the defining relation $c^{m+1} = xc^{m+1-k}$. The total Stiefel-Whitney class of P_k^{m+k} is

$$w(P_k^{m+k}) = (1+c)^{m+1} + x(1+c)^{m+1-k} = (1+c)^{m+1-k}(1+c^k+x)$$

(the latter because $(1+c)^k = 1+c^k$; see Section 2 of [3]).

To factor $w(P_k^{m+k})$ into linear factors, we let

$$1 + c^k + x = \prod_{i=1}^{k} (1+t_i).$$

Then it is sufficient to prove that the Stiefel-Whitney class

$$s_{m+k}(P_k^{m+k}) = (m+1-k)c^{m+k} + \sum_{i=1}^{k} t_i^{m+k}$$

is nonzero. If we expand $\sum_{i=1}^k t_i^{m+k}$ in the elementary symmetric functions of the t_i, we see that $\sum_{i=1}^k t_i^{m+k} = 0$. Because $c^{m+k} = xc^m \neq 0$, we see that

$$s_{m+k}(P_k^{m+k}) = (m+1-k)xc^m,$$

and this is nonzero for even m. Hence P_k^{m+k} is indecomposable for even m.

We now have some even-dimensional generators of $\mathfrak{N}_*$ that are bundles over S^k, and we define some odd-dimensional generators as follows. Consider the bundle $\eta \times \tau P^n$ over $S^k \times P^n$.

Definition 3.3. Let $P_k(m, n)$ be $RP(\eta \times \tau P^n \oplus (m+1-k)R)$, for $m \geq k-1$.

Note that $P_k(m, n)$ is a bundle over $S^k \times P^n$ with fiber P^{m+n}, and that the projection of this bundle composed with the projection on S^k makes $P_k(m, n)$ into a bundle over S^k.

PROPOSITION 3.4. $P_k(m, n)$ *is indecomposable if and only if* n *is even and*

$$\binom{m+k+n-1}{n} \equiv 1 \pmod 2.$$

Proof. We know that $H^*(P_k(m, n))$ is a free $H^*(S^k) \otimes H^*(P^m)$-module on $1, c, \cdots, c^{m+n}$, with defining relation

$$c^{m+n+1} = \sum_{i=1}^{m+n} c^{m+n+1-i} u_i,$$

where $u = 1 + u_1 + \cdots = (1 + x)(1 + \alpha)^{n+1}$, with $\alpha \in H^1(P^m)$. This relation may be written as

$$c^{m-k}(c^k + x)(c + \alpha)^{n+1} = 0. \tag{1}$$

Multiplying (1) by x, we find that

$$xc^m(c + \alpha)^{n+1} = 0, \tag{2}$$

because $x^2 = 0$; multiplying (1) by c^k and applying (2), we obtain the equation

$$c^{m+k}(c + \alpha)^{n+1} = 0. \tag{3}$$

Now $w(P_k(m, n)) = (1 + \alpha)^{n+1}(1 + c)^{m-k}(1 + c^k + x)(1 + c + \alpha)^{n+1}$, and hence, after factoring $1 + c^k + x$ into linear factors, we can write down $s_{m+k+2n}(P_k(m, n))$. The first term $(n + 1)\alpha^{m+k+2n}$ is zero, because $\alpha^{n+1} = 0$. The second term is zero, as we see by applying (3) to $c^{m+k+2n} = c^{m+k}(c + \alpha + \alpha)^{2n}$. The third term corresponding to $1 + c^k + x$ is zero, as in Proposition 3.2. Applying (3) to

$$(c + \alpha)^{m+k+n-1}(c + \alpha)^{n+1},$$

we see that

$$(c + \alpha)^{m+k+2n} = \binom{m + k + n - 1}{n} c^{m+k+n}\alpha^n = \binom{m + k + n - 1}{n} c^{m+n} x\alpha^n.$$

Because $c^{m+n}x\alpha^n \neq 0$ and the last term is $(n + 1)(c + \alpha)^{m+k+2n}$, the proposition follows.

4. SUFFICIENCY FOR S^2

The ring $\mathfrak{N}_*$ is a polynomial algebra over Z_2 with one generator in each dimension not of the form $2^s - 1$. Take $k = 2$, in Section 3. By Proposition 3.2, we have generators that are bundles over S^2 in all even dimensions greater than two. If n is odd and is not of the form $2^s - 1$, then n can be written uniquely in the form $2^p(2q + 1) - 1$ with $p > 0$ and $q > 0$. For $p > 1$ and $q > 0$, the n-manifold $P_2(2^p - 3, 2^p q)$ is indecomposable, by Proposition 3.4. Let $p = 1$, so that $n = 4q + 1$. For $q > 1$, the n-manifold $P_2(4q - 5, 2)$ is indecomposable, by Proposition 3.4. This covers all required odd dimensions except 5. Furthermore, the square of the five-dimensional generator is cobordant to a bundle over S^2 by [3, Corollary 6.2]. Thus, for each $x \in \mathfrak{N}_n$ $(n \geq 2)$, we can write x uniquely as

$$x = y + rx_2^i x_5^\varepsilon,$$

where y can be represented by a bundle over S^2 ($r \in Z_2$, $i = 0, 1, 2, \cdots$, $\varepsilon = 0$ or $\varepsilon = 1$, and $2i + 5\varepsilon = n$). Direct computation shows that $C(x_2^i x_5^\varepsilon) \neq 0$, and hence

$C(x) = r$. Thus, if $C(x) = 0$, then $x = y$. This proves Theorem 1 except for the connectivity assertion.

To prove Corollary 1, let M^m and N^n be bundles over S^1. Then

$$w_{m+n}(M^m \times N^n) = w_m(M^n) \otimes w_n(N^n) = 0 \otimes 0 = 0,$$

$$w_2 \cdot w_{m+n-2}(M^m \times N^n) = w_1 \cdot w_{m-1}(M^m) \otimes w_1 \cdot w_{n-1}(N^n).$$

If $m + n$ is even, the first formula proves the result. If $m + n$ is odd, then, say, m is odd. By Remark 2.3, $w_1 \cdot w_{m-1}(M^m) = 0$, and the result follows.

5. CONNECTED SUMS OF BUNDLES

Suppose M_1^n and M_2^n are bundles over a manifold B with fibers F_1 and F_2. Suppose there exist sections $s_i\colon B \to M_i^n$ $(i = 1, 2)$ such that the normal bundles ν_i of $s_i(B)$ in M_i^n $(i = 1, 2)$ are isomorphic.

LEMMA 5.1. *There exists a bundle* M *over* B *whose fiber is the connected sum* $F_1 \# F_2$ *and such that*

(i) M *is cobordant to the disjoint union* $M_1 + M_2$,

(ii) *if* ν_i *has a nonvanishing section, then there exists a section* $s\colon B \to M$ *such that the normal bundle of* $s(B)$ *in* M *is isomorphic to* ν_1 *and* ν_2.

Proof. Let $E(\nu_i)$ be the total space of ν_i, realized as a small open subbundle neighbourhood of $s_i(B)$ in M_i. Then $M_i - E(\nu_i)$ is a manifold whose boundary is the sphere bundle $S(\nu_i)$. Form M from

$$M_1 - E(\nu_1) \cup S(\nu_1) \times I \cup M_2 - E(\nu_2)$$

by identifying $\partial(M_1 - E(\nu_1))$ with $S(\nu_1) \times \{0\}$ and $\partial(M_2 - E(\nu_2))$ with $S(\nu_1) \times \{1\}$. Then M is a bundle over B with fiber $F_1 \# F_2$.

Let D be the closed-disc bundle of $E(\nu_1)$. We can form a manifold W from

$$M_1 \times I \cup M_2 \times I \cup D \times I$$

by identifying the closure of $E(\nu_1) \times \{1\}$ with $D \times \{0\}$ and the closure of $E(\nu_2) \times \{1\}$ with $D \times \{1\}$. Then

$$\partial W = M_1 \cup M_2 \cup M,$$

and this proves (i). If $E(\nu_1)$ has a nonvanishing section, we can move the section s_1 over into $M_1 - E(\nu_1)$ and thus obtain a section of M, as in (ii).

Consider now P_2^{m+2}. If $m > 1$, there exists a section $s\colon S^2 \to P_2^{m+2}$ with normal bundle $\nu = \eta \oplus (m - 2)R$. Similarly, if $m > 1$, there exists a section $s\colon S^2 \to P_2(m, n)$ with normal bundle $\nu = \eta \oplus (m - 2 + 2n)R$. However, if we regard $P_2(m, n)$ as a bundle over $S^2 \times P^n$ and take its restriction to $S^2 \times pt$, we obtain a bundle over S^2 that has a section with normal bundle

$$\nu = \eta \oplus (m - 2 + 2n)R \qquad (m \geq 1),$$

and this implies that the restriction $m > 1$ can be replaced by $m \geq 1$ for $P_2(m, n)$.

The remaining generator of the kernel of the character C is x_5^2. We now give an explicit representative manifold. Let ξ be the nontrivial line bundle over S^1. It is easily verified that $RP(\xi \times \tau P^2)$ represents x_5. Applying the method in the proof of Proposition 6.2, we see that $CP(\eta \oplus \tau CP^2)$ represents x_5^2. There exists a section of $s: S^2 \to CP(\eta \oplus \tau CP^2)$ with normal bundle $\nu = \eta \oplus 6R$.

A product of a generator with another manifold has a section over S^2 with normal bundle of the form $\eta \oplus kR$. When we start connecting these bundles (using Lemma 5.1), we see that the hypothesis of (ii) is always satisfied, except in dimension four, where the normal bundle of the section of P_2^4 is η. Hence, in dimensions greater than four, we can obtain a connected bundle over S^2. Since x_4 is the only nonzero element of ker C in dimension four and P_2^4 is connected, the proof is complete.

6. SQUARES OF BUNDLES OVER S^2

In this section, we prove Theorem 2 by complexifying the generators in Section 3. Let $\bar{\eta}$ be the quaternion line bundle over S^4, considered now as a complex bundle of dimension two.

Definition 6.1. Let CP_2^{m+2} be $CP(\bar{\eta} \oplus (m-1)C)$, for $m \geq 1$.

Then CP_2^{m+2} is a bundle over S^4 whose fiber is the complex projective m-space CP^m. It is a $(2m+4)$-manifold.

PROPOSITION 6.2. CP_2^{m+2} *is cobordant to* $P_2^{m+2} \times P_2^{m+2}$.

Proof. The cohomology and Stiefel-Whitney classes of CP_2^{m+2} are the same as those of P_2^{m+2}, except that all dimensions are doubled (see [3, Section 2]). But for each M, the nonzero Stiefel-Whitney numbers of $M \times M$ are obtainable from those of M by doubling dimensions. Hence CP_2^{m+2} and $P_2^{m+2} \times P_2^{m+2}$ have the same Stiefel-Whitney numbers, and they are cobordant.

Definition 6.3. Let $CP_2(m, n)$ be $CP(\bar{\eta} \times \tau CP^n \oplus (m-1)C)$ $(m \geq 1)$.

PROPOSITION 6.4. $CP_2(m, n)$ *is cobordant to* $P_2(m, n) \times P_2(m, n)$.

Proof. The argument of 6.2 again applies.

The kernel of $C: \mathfrak{N}_* \to Z_2$ is generated by $x_5^2, x_4, x_6, x_8, x_9, \cdots$, and hence the ideal generated by the squares of elements in the kernel of C is generated by $x_5^4, x_4^2, x_6^2, \cdots$. By [3], we can represent x_5^4 by a bundle over S^4, and by Propositions 6.2 and 6.4, we can represent the remaining generators by bundles over S^4. This proves Theorem 2.

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