ON THE INTEGRAL COHOMOLOGY GROUPS OF THE CLASSIFYING SPACE FOR BSO

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1. INTRODUCTION

E. Thomas has shown [5] that all torsion in $H^*(B\,\mathrm{Spin}; Z)$ is of order 2; a result corresponding to the long-known result for BSO. In the study of spaces realizing the image of the stable J-homomorphism [2], the cohomology of the classifying space BBSO for BSO regarded as an H-space is of interest. J. D. Stasheff [3] has found torsion of order 2^n for each n in $H^*(BBSO; Z)$. In this paper, we shall present results for exterior algebras analogous to the results of Thomas for polynomial algebras in [4] and [5], in order to show that Stasheff has found essentially all of the higher torsion in $H^*(BBSO; Z)$.

If X is a graded set and $\lambda: X \to N = \{0, 1, 2, \cdots\}$ is a function, then $A[\lambda]$ is the graded, abelian group generated by the elements of X subject to the conditions $\lambda(x)x = 0$.

THEOREM 1. $H^*(BBSO; Z)$ *and* $(E \otimes A[\lambda]) \oplus T$ *are isomorphic as groups, where*

(1) E *is the graded, anticommutative,* Z*-exterior algebra on one class* P_{4i+1} *of each degree* $4i + 1 \geq 5$;

(2) $\lambda: \{a_{8n} \mid n \geq 1\} \to N$ *is defined by*

$$\lambda a_{8n} = \text{the greatest integer } 2^k \text{ that divides } 4n,$$

and degree $(a_{8n}) = 8n$;

(3) $2T = 0$.

2. PROOF OF THEOREM 1

Suppose X is a space each of whose integral cohomology groups is finitely generated. Let

$$\rho: H^*(\ ; Z) \to H^*(\ ; Z_2) \quad \text{and} \quad \rho': H^*(\ ; Z) \to H^*(\ ; Q)$$

be universal coefficient maps. Let $\{u_1, u_2, \cdots\} \subset H^*(X; Z)$, where the u_n are all of odd degree. We shall prove the following analogue of Theorem 4.2 in [5].

PROPOSITION 1. *Suppose the cohomology groups of* X *satisfy the following three conditions:*

(1) $H^*(X; Z)$ *has no odd torsion.*

(2) $H^*(X; Z_2) = Z_2\, E[w_1, w_2, \cdots; x_1, x_2, \cdots; y_1, y_2, \cdots;$

$$z_0, z_1, z_2, z_4, \cdots, z_{2^i}, \cdots],$$

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where the differentials of the Bockstein spectral sequence satisfy the conditions

(a) $d_1 w_n = x_n$ *for all* n $(d_1 = Sq^1)$,

(b) $d_k z_n = 0$ *for all* k *and for all* $n = 2^j$,

(c) $d_{k+1}(z_0 z_1 z_2 \cdots z_{2^k - 1}) = z_{2^k} +$ (sum of decomposables) *for* $k \geq 0$,

(d) $d_k y_n = 0$ *for all* n *and all* k.

(3) $\rho u_n = w_n x_n$.

Then there exist subsets $V = \{v_1, v_2, \cdots\}$ *and* $R = \{r_2, r_4, \cdots\}$ *of* $H^*(X; Z)$ *satisfying the following three conditions:*

(4) *Let* $n = \sum_{i=1}^{k} \nu(i)$, *where* $2 \leq \nu(1) < \cdots < \nu(k)$ *and where each* $\nu(i)$ *is a power of* 2. *Then* order$(r_n) = 2\,\nu(1)$;

(5) $\rho(r_n) = z_{\nu(1)} \cdots z_{\nu(k)} + t_n + \rho(a_n)$, *where* t_n *is a product of decomposable elements and* $\nu(1) a_n = 0$;

(6) $\rho(v_n) = y_n$.

Furthermore, if we define $\lambda: R \to N$ *by* $\lambda(r_n) = \nu(1) =$ order(r_n), *then, as groups,*

$$H^*(X; Z) = (ZE[u_1, u_2, \cdots, v_1, v_2, \cdots] \otimes A[\lambda]) \oplus T,$$

where $2T = 0$.

We shall prove Proposition 1 in Section 4. Let us apply it now to the proof of Theorem 1.

Proof of Theorem 1. We shall show that BBSO satisfies the conditions of Proposition 1.

There is a homotopy commutative diagram

$$\begin{array}{ccccc}
* & \longrightarrow & K(Z_2, 1) & \longrightarrow & K(Z_2, 1), \\
\downarrow & & \downarrow \kappa_1 & & \downarrow \kappa_2 \\
SU & \xrightarrow{\iota_1} & BBSO & \xrightarrow{\rho_1} & B\,Spin \\
\downarrow & & \downarrow \pi_1 & & \downarrow \pi_2 \\
SU & \xrightarrow{\iota_2} & BBO & \xrightarrow{\rho_2} & BSO
\end{array}$$

where the rows and columns are fibrations and all maps are the obvious ones. This diagram is well known, and it is easily deduced from [1].

By 4.3 of [2], $H^*(BBSO; Z_2) = Z_2 E[e_3, e_4, e_5, \cdots]$, where degree$(e_n) = n$, e_n is ρ_1^* applied to the nth Stiefel-Whitney class if $n \neq 2^k + 1$, and $e_n = Sq^I e_3$, where $I = (2^{k-1}, \cdots, 4, 2)$, if $n = 2^k + 1$.

Now take $\{e_n \mid n$ is even and n is not a power of $2\}$ as the w_i of Proposition 1. Take $\{e_n \mid n$ is odd and n is not 1 plus a power of $2\}$ for the x_i. Take $\{e_n \mid n = 2^k\}$ as the z_i and $\{e_n \mid n = 2^k + 1\}$ as the y_i.

It is clear that X satisfies hypotheses (1) and (2) of Proposition 1. Since (2a) hold in B Spin, it holds in BBSO. Stasheff has shown that

$$d_n(e_{\mu(1)} \cdots e_{\mu(k)}) = e_{2\mu(k)} + t_{2\mu(k)},$$

where $(\mu(1), \cdots, \mu(k)) = (3, 4, 8, \cdots, 2^{n-1})$ and t_n is decomposable [3]. Thus, if we replace the e_m by $e_m + t_m$ for $m = 2^k$, the new e_m satisfy (2b) and (2c). Condition (2d) holds in BBSO because it holds in B Spin. Finally, condition (3) is a simple consequence of the diagram at the beginning of this proof together with [1, pp. 17-23]. ∎

3. SOME RESULTS ON BOCKSTEIN EXACT COUPLES

A Bockstein exact couple is an exact triangle

$$\begin{array}{ccc} A & \xrightarrow{2} & A, \\ & \delta \nwarrow \quad \swarrow \rho & \\ & E & \end{array}$$

where A and E are either finitely generated, abelian groups or locally finitely generated, graded, abelian groups. In the latter case, ρ and δ have degree 0 and 1, respectively. In either case, we stipulate in addition that A has no odd torsion and that $2E = 0$. Sometimes we denote the exact couple simply by (A, E).

Let α: $(A_1, E) \to (A_2, E)$ be a map of Bockstein exact couples such that α: $E \to E$ is the identity. The following two lemmas are easily proved.

LEMMA. *If* $a \in A_2$ *and* $2^n a = 0$, *then* $a \in \mathrm{Im}(\alpha)$.

LEMMA. *If* $a \in A_1$, $a \neq 0$, *and* $\alpha a = 0$, *then there exist sequences* $\{a_n\}$ *and* $\{a'_n\}$ *in* A_1 *and* $\{e_n\}$ *in* E *that satisfy all of the conditions:*

(1) $a = a_1$,

(2) $a_n = 2a'_n$,

(3) $a'_n - \delta_1 e_n = a_{n+1} \neq 0$,

(4) $\alpha a'_n = \delta_2 e_n$.

If G is an abelian group or a graded abelian group, then T(G) is the torsion subgroup. From our two lemmas, we obtain the following proposition.

PROPOSITION 2. *If* α: $A_1/T(A_1) \to A_2/T(A_2)$ *is an epimorphism, then* α *is an isomorphism of Bockstein exact couples.*

Proof. The first lemma tells us that α: $A_1 \to A_2$ is epic. The second tells us that α is monic if A_1 is (locally) finitely generated, because

$$a = a_1 = 2a_2 = 4a_3 = \cdots . \quad ∎$$

Let E be the Z_2-vector space with basis

$$\{c(i, j) \mid i \geq 0 \text{ and } j \geq 0\} \cup \{e(i, j) \mid i \geq 1 \text{ and } j \geq 0\}.$$

Define a sequence of maps

$$d_1: E \to E, \quad d_2: \mathrm{Ker}(d_1) \to \mathrm{Ker}(d_1), \quad \cdots, \quad d_{n+1}: \mathrm{Ker}(d_n) \to \mathrm{Ker}(d_n), \quad \cdots,$$

by

$$d_k c(i, j) = 0, \qquad d_k e(i, j) = \begin{cases} 0 & (k \neq i), \\ c(i, j) & (k = i). \end{cases}$$

Define the mapping λ: $\{c(i, j) | \; i \geq 0 \text{ and } j \geq 0\} \rightarrow N$ by

$$\lambda(c(i, j)) = \begin{cases} 2^i & \text{if } i \geq 1, \\ 0 & \text{if } i = 0. \end{cases}$$

Define ρ: $A[\lambda] \rightarrow E$ and δ: $E \rightarrow A[\lambda]$ by $\rho c(i, j) = c(i, j)$ and

$$\delta e(i, j) = 2^{i-1} c(i, j), \qquad \delta c(i, j) = 0.$$

Then $(A[\lambda], E)$ is a Bockstein exact couple.

PROPOSITION 3. *If* (A', E) *is a Bockstein exact couple that gives rise to the spectral sequence* (E, d) *above, then* (A', E) *is isomorphic to* $(A[\lambda], E)$.

Proof. The statement that $d_n' e(i, j) = c(i, j)$ means that there are elements $a'(i, j) \in A'$ of order 2^i such that

$$\rho' a'(i, j) = c(i, j) + \rho' x(i, j),$$

where $2^{i-1} x(i, j) = 0$ and $\delta' e(i, j) = 2^{i-1} a'(i, j)$ if $i \geq 1$.

Since all differentials of the $c(0, j)$ are 0, we know that $\delta' c(0, j) \in 2^n A$ for all n. Thus $\delta' c(0, j) = 0$ and $c(0, j) = \rho'(a'(0, j))$, where $a'(0, j)$ is of infinite order. The $a'(i, j)$ can be chosen so that they generate A'.

Define α: $A[\lambda] \rightarrow A'$ by $\alpha c(i, j) = a'(i, j)$ and α: $E \rightarrow E$ by

$$\alpha c(i, j) = c(i, j) + \rho' x(i, j), \qquad \alpha e(i, j) = e(i, j).$$

Then α: $(A[\lambda], E) \rightarrow (A', E)$ is a map of exact couples that is an isomorphism on E and an epimorphism of $A[\lambda]/T(A[\lambda]) \rightarrow A'/T(A')$. By Proposition 2, α is an isomorphism. ■

4. PROOF OF PROPOSITION 1

If R is a commutative ring with identity and Y is a set of graded indeterminates, then $RE[Y]$ is the graded, anticommutative, exterior R-algebra with elements of Y as generators. We now state the exterior analogue of Theorem 1 in [4] and use it to prove Proposition 1. The proof of our Lemma follows the proof of Proposition 1.

LEMMA. *Let* $A = Z_2 E[w_1, \cdots, w_m; x_1, \cdots, x_m; y_1, \cdots, y_n]$ *with derivation* d *of degree* 1 *such that* $dw_k = x_k$ *and* $dy_k = 0$. *Let*

$$E = Z_2 E[w_1 x_1, \cdots, w_m x_m; y_1 \cdots y_n],$$

and let S *be the vector space spanned by all monomials of the form*

$$w_{\alpha(1)} x_{\alpha(1)} \cdots w_{\alpha(a)} x_{\alpha(a)} w_{\beta(1)} \cdots w_{\beta(b)} x_{\gamma(1)} \cdots x_{\gamma(c)} y_{\varepsilon(1)} \cdots y_{\varepsilon(e)},$$

where α, β, γ, *and* ε *satisfy all of the following conditions:*

(1) *α, β, γ, and ε are strictly increasing,*

(2) $\gamma(1) > \beta(1)$,

(3) *the images of α, β, and γ are pairwise disjoint,*

(4) a, c, e ≥ 0, *and* b ≥ 1.

Then $A = E \oplus S \oplus d[S]$ *as a vector space. Also,* $d \mid S$ *is monic and* $\operatorname{Ker}(d) = E \oplus d[S]$.

Proof of Proposition 1. If we think of the w_n and z_0 as being the w_n of our lemma, of the x_n and z_1 as being the x_n, and of the y_n and z_k for $k = 2^j \geq 2$ as being the y_n, it is evident that

$$H^*(X; Z_2) = Z_2 E[z_0 z_1;\ w_1 x_1, w_2 x_2, \cdots;\ y_1, y_2, \cdots;\ z_2, z_4, \cdots] \oplus S \oplus d_1[S].$$

Now write down a basis of S corresponding to the e(1, j) of Proposition 3 and a basis of d[S] corresponding to the c(1, j). By our hypotheses, we can now take as e(n, j) all monomials

$$z_0 z_{\nu(1)} \cdots z_{\nu(k)} w_{\alpha(1)} x_{\alpha(1)} \cdots w_{\alpha(a)} x_{\alpha(a)} y_{\beta(1)} \cdots y_{\beta(b)}$$

such that $n = \nu(1) + \cdots + \nu(j)$ and $z_{2\nu(j)}$ does not occur. We take as c(n, j) all monomials

$$z_{\nu(1)} \cdots z_{\nu(k)} w_{\beta(1)} x_{\alpha(1)} \cdots w_{\alpha(a)} x_{\alpha(a)} y_{\beta(1)} \cdots y_{\beta(b)}$$

such that $\nu(1)$ is the largest 2^j that divides n. As c(0, j), we take the set $\{u_1, u_2, \cdots;\ v_1, v_2, \cdots\}$, where the v_i are formal classes mapping to the y_i. The resulting exact couple gives the Bockstein spectral sequence for X, and Proposition 3 completes the proof of Proposition 1. ■

Proof of the lemma. We know that

$$H(A, d) = Z_2 E[w_1 x_1, \cdots, w_m x_m;\ y_1, \cdots, y_n].$$

From the exact sequences

$$0 \to \operatorname{Ker}(d) \to A \to \operatorname{Im}(d) \to 0 \quad \text{and} \quad 0 \to \operatorname{Im}(d) \to \operatorname{Ker}(d) \to H(A, d) \to 0$$

we see that Ker (d) has dimension $2^{2(m+n)-1} + 2^{m+n-1}$ and Im (d) has dimension $2^{2(n+m)-1} - 2^{m+n-1}$. The number of quadruples $(\alpha, \beta, \gamma, \varepsilon)$ satisfying (1) to (4) is $2^{2(m+n)-1} - 2^{m+n-1}$. Thus we need only show that d[S] generates d[A] as a vector space. Henceforth, we shall write y_ε instead of $y_{\varepsilon(1)} \cdots y_{\varepsilon(e)}$.

Now A has a vector space basis consisting of all $w_{\theta(1)} \cdots w_{\theta(p)} x_{\phi(1)} \cdots x_{\phi(q)} y_\varepsilon$ in which θ, ϕ, and ε are strictly increasing. The derivatives of all such monomials generate d[A]; that is, elements of the form

$$\sum_{j=1}^{p} w_{\theta(1)} \cdots \hat{w}_{\theta(j)} \cdots w_{\theta(p)} x_{\phi(1)} \cdots x_{\phi(q)} y_\varepsilon$$

generate d[A]. Since A is an exterior algebra, all these elements have the form

$$e_\alpha \sum_{j=1}^{b} w_{\beta(1)} \cdots \hat{w}_{\beta(j)} \cdots w_{\beta(b)} x_{\beta(j)} x_{\gamma(1)} x_\gamma y_\varepsilon,$$

where $e_\alpha = w_{\alpha(1)} x_{\alpha(1)} \cdots w_{\alpha(a)} x_{\alpha(a)}$, $x_\gamma = x_{\gamma(2)} \cdots x_{\gamma(c)}$, and α, β, γ, and ε satisfy (1), (3), and (4). If β and γ do not satisfy (2), observe that

$$d\left(e_\alpha\left(\sum_{k=1}^{b} w_{\gamma(1)} w_{\beta(1)} \cdots \hat{w}_{\beta(k)} \cdots w_{\beta(b)} x_{\beta(k)}\right) x_\gamma y_\varepsilon\right)$$

$$= d(e_\alpha w_{\gamma(1)} d(w_{\beta(1)} \cdots w_{\beta(k)}) x_\gamma y_\varepsilon) = e_\alpha x_{\gamma(1)} d(w_{\beta(1)} \cdots w_{\beta(k)}) x_\gamma y_\varepsilon + 0$$

$$= e_\alpha x_{\gamma(1)}\left(\sum_{j=1}^{b} w_{\beta(1)} \cdots \hat{w}_{\beta(j)} \cdots w_{\beta(b)} x_{\beta(j)}\right) x_\gamma y_\varepsilon .$$

The remainder of the proof is obvious. ■

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