ON CONSISTENCY WITH RESPECT TO FUNCTIONALS OF ℓ - ℓ TRANSFORMATIONS

G. E. Peterson

1. INTRODUCTION

Let ℓ denote the space of absolutely convergent series of complex numbers. Many theorems concerning ℓ - ℓ methods of summation are stated relative to the *natural linear functional* $\sigma \in \ell'$, that is, the complex-valued functional defined by $\sigma(x) = \sum_n x_n$ for every $x \in \ell$. For example, H. I. Brown and V. F. Cowling have proved a theorem [1, Theorem 3] relating the concepts of perfectness and consistency (with respect to σ). One may ask what happens to these theorems if the natural linear functional is replaced by an arbitrary linear functional. The present research grew out of an attempt to answer this question as it pertains to the theorem of Brown and Cowling.

Let s denote a space of sequences, and let A and B denote matrices that represent mappings from s into ℓ; in other words, let A and B represent s - ℓ transformations. If $F \in \ell'$, we say that A and B are *F-consistent* on s provided $F(Ax) = F(Bx)$ for each $x \in s$. If A is an ℓ - ℓ transformation, we say that *F-consistency is extendable from* ℓ *to* ℓ_A (see [1] and [3] for notation not defined here) provided A and B are F-consistent on ℓ_A whenever B is an ℓ - ℓ method that is F-consistent with A on ℓ and satisfies the condition $\ell_B \supseteq \ell_A$. Brown and Cowling proved [1, Theorem 3] that σ-consistency is extendable from ℓ to ℓ_A if and only if A is perfect. We show (see Theorems 3, 4, and 5) that this result extends to a functional $F \in \ell'$ if and only if F has the representation $F(x) = \sum_n h_n x_n$, where $h \notin c_0$. We are then able to define a class of transformations that is included in the class of ℓ - ℓ transformations as a proper subset but includes the perfect transformations as a proper subset (Theorem 7). In the process of proving these theorems, we establish some other results. For example, Theorem C gives necessary and sufficient conditions for A to map the space of absolutely convergent sequences into ℓ, and Theorem 1 is a generalization of the lemma of [1].

2. MATRIX TRANSFORMATIONS OF SEQUENCE SPACES

Let E^∞ (respectively, ac, c_0, c, m) denote the space consisting of all complex sequences $x = \{x_n\}$ such that $x_n = 0$ for all but finitely many n (respectively, such that $\sum_n |x_n - x_{n+1}| < \infty$, $\lim_n x_n = 0$, $\lim_n x_n$ exists, $\sup_n |x_n| < \infty$). If s is a sequence space and $A = (a_{nk})$ is an infinite matrix, then s_A denotes the space of all sequences x such that $Ax \equiv \left\{ \sum_k a_{nk} x_k \right\} \in s$. If s_1 and s_2 are sequence spaces, we say that A defines an s_1 - s_2 transformation if and only if $Ax \in s_2$ for every $x \in s_1$. The unit sequences e^k are defined by the equations $(e^k)_n = \delta_{nk}$. The dual of the FK-space s is denoted by s'.

THEOREM A [2, p. 29]. *The matrix* A *defines an* ℓ - c *transformation if and only if*

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$$\sup_{n,k} |a_{nk}| < \infty \quad \textit{and} \quad \lim_n a_{nk} \textit{ exists for each } k.$$

If these conditions are satisfied, then $\lim_n \sum_k a_{nk}x_k = \sum_k \lim_n a_{nk}x_k$ *for every* $x \in \ell$.

THEOREM B [2, p. 21]. *The matrix* A *defines a* c_0 - c *transformation if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty \quad \textit{and} \quad \lim_n a_{nk} \textit{ exists for each } k.$$

If these conditions are satisfied, then $\lim_n \sum_k a_{nk}x_k = \sum_k \lim_n a_{nk}x_k$ *for every* $x \in c_0$.

THEOREM C. *The matrix* B *defines an* ac - ℓ *transformation if and only if*

$$\sup_j \sum_{n=1}^{\infty} \left| \sum_{k=j}^{\infty} b_{nk} \right| < \infty.$$

Proof. Let $x \in$ ac and $y \in \ell$ be related by the equation $y_k = x_k - x_{k-1}$ $(x_0 = 0)$, and write $\sum_{k=1}^m b_{nk}x_k$ in the form $\sum_{j=1}^m y_j \sum_{k=j}^m b_{nk}$. An application of Theorem A with $a_{mj} = \sum_{k=j}^m b_{nk}$ shows that $\sum_k b_{nk}$ exists for each n and that $(Bx)_n = \sum_{j=1}^\infty y_j \sum_{k=j}^\infty b_{nk}$. Thus, B defines an ac - ℓ transformation if and only if the matrix D given by $d_{nj} = \sum_{k=j}^\infty b_{nk}$ defines an ℓ - ℓ transformation. The result now follows from a theorem due to Knopp, Lorentz, and Mears [1, p. 357].

3. THEOREMS ABOUT FUNCTIONALS

THEOREM 1. *Let* $F \in \ell'$ *have the representation* $F(x) = \sum_n h_n x_n$ $(h \in m$ *but* $h \notin c_0)$. *Let* A *be an* ℓ - ℓ *transformation, and let* $f \in \ell'_A$. *Then there exists a transformation* B *such that* $\ell_B \supseteq \ell_A$ *and* $F(Bx) = f(x)$ *for all* $x \in \ell_A$.

Our proof is similar to the proof of the lemma in [1], which is the special case $F = \sigma$ of this theorem.

Proof. Since $h \notin c_0$, there exist an increasing sequence $j = \{j(n)\}$ of integers and a number δ such that $|h_{j(n)}| \geq \delta > 0$ for every n. If $b_n = 1/h_{j(n)}$, then $b \in m$. Let $f \in \ell'_A$; then

$$f(x) = \sum_n t_n \sum_k a_{nk}x_k + \sum_k a_k x_k$$

for some $t, a \in m$ [1, p. 358, equation (4)]. Define

$$b_{nk} = \begin{cases} b_m(2^{-m}a_k + t_m a_{mk}) & (n = j(m)), \\ 0 & (n \notin \{j(m)\}). \end{cases}$$

For $x \in \ell_A$ we have the relations

$$\sum_n |(Bx)_n| = \sum_n \left|\sum_k b_{j(n),k} x_k\right| = \sum_n \left|\sum_k (b_n 2^{-n} a_k x_k + b_n t_n a_{nk} x_k)\right|$$

$$\leq \sum_n |b_n| 2^{-n} \left|\sum_k a_k x_k\right| + \sum_n |b_n t_n| \left|\sum_k a_{nk} x_k\right| < \infty .$$

Thus $\ell_B \supseteq \ell_A$. Similarly, for every $x \in \ell_A$,

$$F(Bx) = \sum_n h_n \sum_k b_{nk} x_k = \sum_n h_{j(n)} \sum_k b_n(2^{-n} a_k + t_n a_{nk}) x_k$$

$$= \sum_n \sum_k 2^{-n} a_k x_k + \sum_n \sum_k t_n a_{nk} x_k = \sum_k a_k x_k + \sum_n \sum_k t_n a_{nk} x_k = f(x).$$

The theorem is proved.

We shall now show that the restriction $h \notin c_0$ cannot be omitted in Theorem 1.

THEOREM 2. *Let* $F \in \ell'$ *have the representation* $F(x) = \sum_n h_n x_n$ $(h \in c_0)$. *Then there exist an* ℓ - ℓ *transformation* A *and an* $f \in \ell'_A$ *such that no transformation* B *satisfies the two conditions* $\ell_B \supseteq \ell_A$ *and* $F(Bx) = f(x)$ *for all* $x \in \ell_A$. *In fact, no* B *satisfies the weaker conditions* $\ell_B \supseteq \ell_A$ *and* $F(Bx) = f(x)$ *for all* $x \in \ell$.

Proof. Consider the matrix

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 & . & . & . \\ 2 & -1 & 0 & 0 & . & . & . \\ 0 & 2 & -1 & 0 & . & . & . \\ 0 & 0 & 2 & -1 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

By the theorem of Knopp, Lorentz, and Mears, A is an ℓ - ℓ method, and we can easily verify that the sequence $\{2^k\}$ belongs to ℓ_A. For every $x \in \ell_A$, define f by the formula $f(x) = \sum_n \sum_k a_{nk} x_k$. By using the closure theorem of Banach and Steinhaus [3, p. 225], we can easily show that $f \in \ell'_A$. Now suppose there is a method B such that $\ell_B \supseteq \ell_A$ and $F(Bx) = f(x)$ for all $x \in \ell$. Then $\sum_k b_{nk} 2^k$ exists for every n, since $\ell_B \supseteq \ell_A$. This implies that $\lim_k b_{nk} 2^k = 0$ and hence

(1) $$\lim_k b_{nk} = 0 \quad \text{for every } n.$$

Upon applying the relation $F(Bx) = f(x)$ to the sequences $x = e^k$, we obtain the relation

(2) $$\sum_n h_n b_{nk} = \sum_n a_{nk} = 1 \quad \text{for every } k.$$

Furthermore, since B is an ℓ - ℓ method,

(3) $$\sup_k \sum_n |b_{nk}| < \infty .$$

By (1) and (3), B^t (the transpose of B) is a c_0 - c_0 method. But this contradicts (2), which says that B^t maps $h \in c_0$ into $\{1\} \notin c_0$.

4. THE MAIN THEOREMS

THEOREM 3. *If* A *is perfect, then* F-*consistency is extendable from* ℓ *to* ℓ_A *for every* $F \in \ell'$.

THEOREM 4. *Let* A *be an* ℓ - ℓ *transformation, and let* $F \in \ell'$ *have the representation* $F(x) = \sum_n h_n x_n$ ($h \in m$ *but* $h \notin c_0$). *If* F-*consistency is extendable from* ℓ *to* ℓ_A, *then* A *is perfect.*

These theorems generalize the sufficient conditions and the necessary conditions in Theorem 3 of [1]. The proofs are trivial modifications of the proof of [1, Theorem 3], and we omit them. In the proof of Theorem 4, our Theorem 1 takes the place of the lemma of [1].

We shall now show that the restriction $h \in c_0$ cannot be omitted in Theorem 4.

THEOREM 5. *Let* $F \in \ell'$ *have the representation* $F(x) = \sum_n h_n x_n$ ($h \in c_0$). *Then there exists an* ℓ - ℓ *transformation* A *that is not perfect but for which* F-*consistency is extendable from* ℓ *to* ℓ_A.

The proof of Theorem 5 is based on Theorem 6.

THEOREM 6. *If the matrix* B *defines an* ac - ℓ *transformation and* $F \in \ell'$ *has the representation* $F(x) = \sum_n h_n x_n$ ($h \in c_0$), *then*

$$F(Bx) = \sum_k x_k \sum_n h_n b_{nk}$$

for every $x \in$ ac.

Proof. Suppose $x \in$ ac, and let $y_k = x_k - x_{k-1}$. First we shall show that the matrix $C = (c_{jn})$, given by

$$c_{jn} = \sum_{k=1}^{j} b_{nk} x_k,$$

defines a c_0 - c transformation. It is clear that $\lim_j c_{jn}$ exists for every n. Furthermore, by Theorem C,

$$\sum_n |c_{jn}| = \sum_{n=1}^{\infty} \left| \sum_{k=1}^{j} y_k \sum_{i=k}^{j} b_{ni} \right| \leq \sum_{k=1}^{j} |y_k| \sum_{n=1}^{\infty} \left| \sum_{i=k}^{\infty} b_{ni} - \sum_{i=j+1}^{\infty} b_{ni} \right| \leq M < \infty,$$

where the constant M does not depend on j. Thus C defines a c_0 - c transformation. By Theorem B,

$$\sum_k x_k \sum_n h_n b_{nk} = \lim_j \sum_n h_n c_{jn} = \sum_n h_n \lim_j c_{jn} = \sum_n h_n \sum_k b_{nk} x_k = F(Bx).$$

Proof of Theorem 5. The matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & . & . & . \\ -1 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & -1 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & -1 & 1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}$$

defines an ℓ - ℓ method, which is not perfect, because A is reversible but is not of type M^* [1, Theorem 2]. (The sequence $\{1\}$ is a left annihilator of A.) Suppose B is a matrix such that $\ell_B \supseteq \ell_A$ and $F(Bx) = F(Ax)$ on ℓ. By applying the latter relation to the unit sequence $x = e^k$, we find that

$$\sum_n h_n b_{nk} = \sum_n h_n a_{nk} \quad \text{for every } k.$$

Since $\ell_A = ac$, both A and B define ac - ℓ transformations. By Theorem 6, $F(Bx) = \sum_k x_k \sum_n h_n b_{nk} = \sum_k x_k \sum_n h_n a_{nk} = F(Ax)$ for every $x \in \ell_A$. Thus F-consistency is extendable from ℓ to ℓ_A. This completes the proof of Theorem 5.

Not every ℓ - ℓ method has the property in Theorem 5. That is, there exist an ℓ - ℓ method A and an $F \in \ell'$ with the representation $F(x) = \sum_n h_n x_n$ ($h \in c_0$) such that F-consistency is not extendable from ℓ to ℓ_A. For example, let A be the matrix used in the proof of Theorem 2, let

$$B = \begin{bmatrix} 8^{-1} & 8^{-2} & 8^{-3} & 8^{-4} & . & . & . \\ -2^{-2} & 0 & 0 & 0 & . & . & . \\ 0 & -2^{-4} & 0 & 0 & . & . & . \\ 0 & 0 & -2^{-6} & 0 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix},$$

and let $F(x) = \sum_n 2^{-n} x_n$. Both A and B define ℓ - ℓ transformations. The matrix

$$C = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & . & . & . \\ 0 & 2 & -1 & 0 & 0 & . & . & . \\ 0 & 0 & 2 & -1 & 0 & . & . & . \\ 0 & 0 & 0 & 2 & -1 & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}$$

has the same summability field as A (that is, $\ell_C = \ell_A$), and C has the right inverse

$$D = \begin{bmatrix} 1/2 & 1/4 & 1/8 & 1/16 & . & . & . \\ 0 & 1/2 & 1/4 & 1/8 & . & . & . \\ 0 & 0 & 1/2 & 1/4 & . & . & . \\ 0 & 0 & 0 & 1/2 & . & . & . \\ . & . & . & . & . & . & . \end{bmatrix}.$$

Since D defines an ℓ - ℓ transformation, it follows as in [3, p. 4, Problems 20 and 21] that $\ell_C = \ell \oplus \{2^k\}$. Thus $\ell_A = \ell \oplus \{2^k\}$. Furthermore, $\{2^k\} \in \ell_B$, and hence $\ell_B \supseteq \ell_A$. Now

$$\sum_n 2^{-n} a_{nk} = \sum_n 2^{-n} b_{nk} = 0 \qquad \text{for every k.}$$

Therefore $F(Ax) = F(Bx)$ for every $x \in \ell$. However, $F(A\{2^k\}) = -1$ and $F(B\{2^k\}) = 0$. Thus F-consistency is not extendable from ℓ to ℓ_A.

We summarize our results in the following theorem.

THEOREM 7. *Let* $\mathscr{F}_1$ *represent the class of all elements* F *in* ℓ', *and let* $\mathscr{F}_2$ *and* $\mathscr{F}_3$ *denote the subclasses of* $\mathscr{F}_1$ *whose elements have the form* $F(x) = \sum_n h_n x_n$ *with* $h \in c_0$ *and* $h \in E^\infty$, *respectively. For* $i = 1, 2, 3$, *let* K_i *consist of all matrices* A *such that* F-*consistency is extendable from* ℓ *to* ℓ_A *for each* $F \in \mathscr{F}_i$. *Then* K_1 *consists of all perfect* ℓ - ℓ *transformations*, K_3 *consists of all* ℓ - ℓ *transformations, and* $K_1 \subset K_2 \subset K_3$ *(strict inclusions).*

Proof. We have proved everything except the statement that K_3 consists of all ℓ - ℓ transformations. Let A be an arbitrary ℓ - ℓ transformation, and let $F \in \ell'$ have the representation $F(x) = \sum_n h_n x_n$ $(h \in E^\infty)$. Let B be an ℓ - ℓ method such that $\ell_B \supseteq \ell_A$ and $F(Bx) = F(Ax)$ for every $x \in \ell$. Then $\sum_n h_n b_{nk} = \sum_n h_n a_{nk}$ for all k. For every $x \in \ell_A$, we have the identities

$$F(Bx) = \sum_n h_n \sum_k b_{nk} x_k = \sum_k x_k \sum_n h_n b_{nk}$$

$$= \sum_k x_k \sum_n h_n a_{nk} = \sum_n h_n \sum_k a_{nk} x_k = F(Ax).$$

The interchanges in the order of summation are permissible because $\sum_n$ is a finite sum and $\sum_k$ exists for each n.

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University of Missouri
St. Louis, Missouri 63121