AN EXTENSION OF WEYL'S THEOREM TO A CLASS OF NOT NECESSARILY NORMAL OPERATORS

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1. INTRODUCTION

A bounded linear operator T on a Hilbert space is called a *Fredholm operator* if its null space N(T) is finite-dimensional and its range R(T) is a closed subspace of finite codimension (see [4, p. 91]); the *index* of a Fredholm operator is defined as

$$i(T) = \dim N(T) - \dim R(T)^{\perp} \qquad (= \dim N(T) - \dim N(T^{*})).$$

Every invertible operator is trivially a Fredholm operator with index 0.

The *spectrum* $\sigma(T)$ of T is defined by the formula

$$\sigma(T) = \mathbf{C}\{\lambda\colon T - \lambda I \text{ is invertible}\}.$$

Analogously, the *Weyl spectrum* $\omega(T)$ of T is defined by the formula

$$(1) \qquad \omega(T) = \mathbf{C}\{\lambda\colon T - \lambda I \text{ is a Fredholm operator of index } 0\}.$$

Obviously, $\omega(T) \subset \sigma(T)$. The concept of Weyl spectrum is relevant only for infinite-dimensional spaces: $\omega(T) = \emptyset$ when the space is finite-dimensional, all operators being Fredholm operators of index 0. For technical reasons, it is convenient to allow finite-dimensional spaces. When the space is infinite-dimensional, $\omega(0) = \{0\}$.

The Weyl spectrum (also called the essential spectrum) occurs in the theory of perturbation by compact operators; it has the agreeable property of being invariant under such perturbation; that is, $\omega(T + K) = \omega(T)$ for all compact K (see [9]). We shall not use this property, but we remark that it implies that

$$\omega(T) = \bigcap \{\sigma(T + K)\colon K \text{ is compact}\}$$

(L. A. Coburn [3] used this formula to define Weyl spectrum), and that $\omega(T) = \{0\}$ when T is compact and the space is infinite-dimensional.

Modulo the above remarks, the Weyl spectrum tends to be large:

LEMMA 1. $\sigma(T) - \omega(T)$ *is either empty or consists of eigenvalues of finite multiplicity.*

Proof. If $\lambda \in \sigma(T) - \omega(T)$, then $S = T - \lambda I$ is singular and is a Fredholm operator of index 0. In particular, $\dim N(S) < \infty$, thus the problem is to show that $N(S) \neq \{0\}$. Assume to the contrary that S is injective; then $0 = i(S) = 0 - \dim R(S)^{\perp}$, and since R(S) is closed, this implies S is surjective. Thus S is bijective and therefore invertible [4, Problem 41], a contradiction.

Extending a classical result of H. Weyl for normal operators, Coburn [3] showed that if T is any hyponormal operator, then

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(2) $$\omega(T) = \sigma(T) - \pi_{00}(T),$$

where $\pi_{00}(T)$ denotes the set of isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity.

In general, if an operator T satisfies (2), one says that Weyl's theorem holds for T. This condition is clearly equivalent to

(2') $$\sigma(T) - \omega(T) = \pi_{00}(T).$$

It follows from Lemma 1 that if T has no eigenvalues of finite multiplicity, then $\omega(T) = \sigma(T)$, and in particular, Weyl's theorem holds for T. When the space is finite-dimensional, Weyl's theorem holds trivially for every T: $\pi_{00}(T) = \sigma(T)$ and $\omega(T) = \emptyset$.

V. Istrăţescu [5] showed that Weyl's theorem holds for a certain class of operators that are not necessarily hyponormal (the precise result is stated in the next section).

The aim of this note is to prove that Weyl's theorem holds for a class of operators that includes both the hyponormal operators and the class of operators considered by Istrăţescu.

2. STATEMENT OF RESULTS

Consider the following conditions that an operator T may satisfy:

(G_1) $\|(T - \lambda I)^{-1}\| \leq [\text{dist}(\lambda, \sigma(T))]^{-1}$ for all λ not in $\sigma(T)$. (Note that the right side is equal to the spectral radius of $(T - \lambda I)^{-1}$, so that we actually have equality; thus, T satisfies (G_1) if and only if $(T - \lambda I)^{-1}$ is normaloid for all λ not in $\sigma(T)$.)

(G_1') Every isolated point of $\sigma(T)$ is an eigenvalue of T.

(α) The restriction of T to any invariant subspace satisfies (G_1).

(α') The restriction of T to any reducing subspace satisfies (G_1).

(α'') The restriction of T to any invariant subspace is convexoid (that is, T is restriction-convexoid in the sense of [2]).

(α''') The restriction of T to any reducing subspace satisfies (G_1'); in other words, an isolated point of the spectrum of a direct summand of T is necessarily an eigenvalue for the summand.

(β) Each point of $\sigma(T)$ is a bare point of $\sigma(T)$ (that is, it lies on the circumference of some closed disc that contains $\sigma(T)$; see [7]).

(β') Each eigenvalue of T of finite multiplicity is a semibare point of $\sigma(T)$ (that is, it lies on the circumference of some closed disc that contains no other point of $\sigma(T)$; see [14]).

It is obvious that all of these conditions are translation-invariant: if T satisfies one of them, then so does $T - \mu I$ (T and $T - \mu I$ have the same invariant subspaces and the same reducing subspaces).

Since (G_1) $\Rightarrow$ (G_1') by a result of J. G. Stampfli [12, proof of Theorem C], we have the implication (α') $\Rightarrow$ (α'''). Since every operator satisfying (G_1) is convexoid (see [7, Theorem 2], [12, p. 474]) the implication (α) $\Rightarrow$ (α'') holds; also,

$(\alpha'') \Rightarrow (\alpha''')$ [2, Lemma 2]. Thus $(\alpha) \Rightarrow (\alpha') \Rightarrow (\alpha''')$ and $(\alpha) \Rightarrow (\alpha'') \Rightarrow (\alpha''')$. Obviously $(\beta) \Rightarrow (\beta')$ (consider tangent circles). Thus (α'''), (β') is the weakest pair of conditions.

Istrăţescu [5] showed that if an operator T satisfies (α) and (β), then Weyl's theorem holds for T. A hyponormal operator satisfies (α) (see Stampfli [12, Theorem 1]) but of course it need not satisfy (β); thus Istrăţescu's result complements but does not include Coburn's.

In the present note, we generalize Coburn's and Istrăţescu's results simultaneously by showing that Weyl's theorem holds for every operator T that satisfies (α''') and is reduced by each of its finite-dimensional eigenspaces. It follows that the conditions (α) and (β) in Istrăţescu's theorem may be replaced by the weaker conditions (α') and (β') (see Corollary 2 below).

3. WEYL'S THEOREM FOR A CLASS OF OPERATORS

In connection with reducing eigenspaces, we remark that N(T) reduces T if and only if $N(T) \subset N(T^*)$; the proof is elementary.

THEOREM. *If* T *is an operator that satisfies* (α''') *and is reduced by each of its finite-dimensional eigenspaces, then Weyl's theorem holds for* T.

Proof. Following Coburn [3], we first reformulate the problem: since $T - \lambda I$ also satisfies the hypotheses of the theorem, it suffices, in view of (2'), to show that

$$(*) \qquad 0 \in \pi_{00}(T) \Leftrightarrow T \text{ is a singular Fredholm operator of index } 0.$$

Next, we observe that each side of (*) implies that 0 is an eigenvalue of T of finite multiplicity. For the left side, this is trivial. On the other hand, if T is a singular Fredholm operator of index 0, then the proof of Lemma 1 shows that $N(T) \neq \{0\}$, so that 0 is an eigenvalue of finite multiplicity dim N(T). Thus, in view of the hypothesis of the theorem, each side of (*) implies that N(T) is a (nonzero, finite-dimensional) reducing subspace of T,

$$(3) \qquad N(T) \subset N(T^*) = R(T)^{\perp},$$

and, writing

$$(4) \qquad T = 0 \oplus T_1,$$

where 0 denotes the zero operator on N(T) and $T_1 = T \mid N(T)^{\perp}$ is injective, we have

$$(5) \qquad \sigma(T) = \{0\} \cup \sigma(T_1).$$

Proof of $\Rightarrow$. Assume $0 \in \pi_{00}(T)$. Of course, T is singular; it is to be shown that T is a Fredholm operator of index 0.

We assert that T_1 is invertible. Assuming to the contrary that $0 \in \sigma(T_1)$, we see from (5) that $\sigma(T) = \sigma(T_1)$. Then 0 is an isolated point of $\sigma(T_1)$ (because $0 \in \pi_{00}(T)$); therefore, by (α'''), it is an eigenvalue of T_1, a contradiction.

In particular, T_1 is surjective: $T(N(T)^{\perp}) = N(T)^{\perp}$. This obviously implies that $N(T)^{\perp} \subset R(T)$. On the other hand, (3) implies that $R(T) \subset R(T)^{\perp\perp} \subset N(T)^{\perp}$. Thus $R(T) = N(T)^{\perp}$, and in particular, R(T) is closed. Also, the subspace $R(T)^{\perp} = N(T)$ is finite-dimensional; therefore T is a Fredholm operator of index 0.

Proof of $\Leftarrow$. Assume T is a Fredholm operator of index 0. As we noted above, 0 is an eigenvalue of T of finite multiplicity; it remains only to show that 0 is an isolated point of $\sigma(T)$. In view of (5) and the fact that $\sigma(T_1)$ is closed, it will suffice to show that $0 \notin \sigma(T_1)$, that is, T_1 is invertible. Since T_1 is known to be injective, the proof will be complete if we show that T_1 is surjective, that is, $T(N(T)^{\perp}) = N(T)^{\perp}$.

Since T is a Fredholm operator, R(T) is closed. Moreover,

$$\dim N(T) = \dim R(T)^{\perp} < \infty,$$

because T has index 0; it then follows from (3) that $N(T) = R(T)^{\perp}$, and therefore $N(T)^{\perp} = R(T)$. Our task is thus reduced to showing that $T(R(T)) = R(T)$. Obviously, $T(R(T)) \subset R(T)$. Conversely, suppose $y \in R(T)$, say $y = Tx$. Write $x = Tu + z$, where $z \in R(T)^{\perp} = N(T)$. Then $y = Tx = T(Tu) + Tz = T(Tu) + 0 \in T(R(T))$.

It follows from the remarks in Section 2 that the theorem remains true with (α''') replaced by (α), (α'), or (α'').

COROLLARY 1 (Coburn [3, Theorem 3.1]). *If* T *is hyponormal, then Weyl's theorem holds for* T.

Proof. As we noted in Section 2, T is hyponormal $\Rightarrow (\alpha) \Rightarrow (\alpha''')$, and T is reduced by each of its eigenspaces [1, p. 168, Exercise 5]. It follows that Weyl's theorem holds for any seminormal operator; see Example 6 below.

The extension of Istrăţescu's result requires another lemma:

LEMMA 2. *If* T *satisfies* (G_1) *and* λ *is a semibare point of* $\sigma(T)$, *then* $N(T - \lambda I) = N(T^* - \bar{\lambda} I)$ *and therefore* $N(T - \lambda I)$ *reduces* T.

Proof. Since $T - \lambda I$ also satisfies (G_1), one can suppose $\lambda = 0$. Choose a complex number $\lambda_0 \neq 0$ such that the disc $\{\lambda: |\lambda - \lambda_0| \leq |\lambda_0|\}$ meets $\sigma(T)$ only at 0. Obviously, $\operatorname{dist}(\lambda_0, \sigma(T)) = |\lambda_0|$; therefore $\|(T - \lambda_0 I)^{-1}\| \leq 1/|\lambda_0|$, by (G_1). Thus, setting $W = -\lambda_0(T - \lambda_0 I)^{-1}$, we see that $\|W\| \leq 1$; that is, W is a contraction. (In fact, $\|W\| = 1$, but we do not need this.)

Suppose $Tx = 0$. Then $(T - \lambda_0 I)^{-1} x = -\lambda_0^{-1} x$, by elementary algebra; thus $Wx = x$. By an elementary property of contractions (see F. Riesz and B. Sz.-Nagy [8]), W and W^* have the same fixed points; thus $W^* x = x$, and therefore $T^* x = 0$. Thus $Tx = 0$ implies $T^* x = 0$, and the argument is reversible.

The following corollary extends the result of Istrăţescu [5] cited above.

COROLLARY 2. *If* T *satisfies* (α') *and* (β'), *then Weyl's theorem holds for* T.

Proof. As we noted in Section 2, $(\alpha') \Rightarrow (\alpha''')$. If λ is an eigenvalue of T of finite multiplicity, then λ is a semibare point of $\sigma(T)$, by the hypothesis (β'); therefore $N(T - \lambda I)$ reduces T, by Lemma 2. Incidentally, (G_1), (α'''), (β') are sufficient for the proof; the full force of (α') is not used.

4. EXAMPLES

1. In the theorem, one cannot dispense with (α''') altogether. For example, let $T = T_1 \oplus T_2$, where T_1 is the one-dimensional zero operator and T_2 is an injective generalized nilpotent compact operator (for example, let $T_2 e_n = (1/n) e_{n+1}$; see [1, p. 178, Example 1]). Since T is compact, $\omega(T) = \{0\}$, as we remarked in the introduction; $\sigma(T) = \{0\}$ and $\pi_{00}(T) = \{0\}$, and therefore $\sigma(T) - \omega(T) = \emptyset \neq \pi_{00}(T)$.

Thus Weyl's theorem fails for T, even though the only eigenspace of T is reducing. Of course, T does not satisfy (α'''), since 0 is an isolated point of $\sigma(T_2)$ but is not an eigenvalue of T_2.

2. The spectrum in Example 1 was necessarily stingy, because Weyl's theorem holds trivially for every compact operator T with infinite spectrum: $\omega(T) = \{0\}$ and $\pi_{00}(T) = \sigma(T) - \{0\}$.

3. If R(T) is closed, then so is R(T*) (see [13, p. 227, Problem 7]); it follows that T is a Fredholm operator if and only if T^* is a Fredholm operator, and therefore $\omega(T^*) = (\omega(T))^-$. Also, $\sigma(T^*) = (\sigma(T))^-$. However, in general

$$\pi_{00}(T^*) \neq (\pi_{00}(T))^- ;$$

thus it is to be expected that Weyl's theorem may hold for T but not for T^*. Indeed, if T_2 is the operator described in Example 1, then $\sigma(T_2) = \omega(T_2) = \{0\}$ and $\pi_{00}(T_2) = \emptyset$ (thus Weyl's theorem holds for T_2), but $\pi_{00}(T_2^*) = \{0\}$ (thus Weyl's theorem fails for T_2^*).

4. If Weyl's theorem holds for T and $\sigma(T)$ has no isolated points, then Weyl's theorem holds for T^*; this follows at once from the formulas in Example 3 and the fact that $\pi_{00}(T) = \pi_{00}(T^*) = \emptyset$. An example is the unilateral shift $Te_n = e_{n+1}$ [4, Problem 67].

5. If T is reduced by its finite-dimensional eigenspaces, then it can be written in the form $T = T_1 \oplus T_2$, where T_1 is normal and $\omega(T_2) = \sigma(T_2)$ (see [3, Corollary 3.3] for the case where T is hyponormal). The proof runs as follows. The finite-dimensional eigenspaces of T are mutually orthogonal (see the remark at the beginning of Section 3); let M be their closed linear span (that is, their orthogonal direct sum). Then M reduces T, $T_1 = T \mid M$ is normal, and T_1 has no new eigenvalues (in other words, the eigenvalues of T_1 are precisely the eigenvalues of T of finite multiplicity; see [1, p. 164, Example 5]). Let $T_2 = T \mid M^\perp$. Since T_1 has no new eigenvalues, it follows that if λ is an eigenvalue of T_2, then $N(T - \lambda I) \subset M^\perp$ and therefore $N(T - \lambda I) = N(T_2 - \lambda I)$. Thus the eigenvalues of T_2 are precisely the eigenvalues of T of infinite multiplicity (for both T and T_2). In particular, T_2 has no eigenvalues of finite multiplicity; therefore $\omega(T_2) = \sigma(T_2)$, by the remark following (2'). Moreover, $T^* = T_1^* \oplus T_2^*$, where T_1^* is normal and $\omega(T_2^*) = \sigma(T_2^*)$ (see the formulas in Example 3); for example, such a decomposition of T^* is available when T is hyponormal (see also Example 6 below). Incidentally, if $T = T_1 \oplus T_2$ with T_1 normal, then $\omega(T) = \omega(T_1) \cup \omega(T_2)$; this follows at once from (1) and the fact that if $T_1 - \lambda I$ is a Fredholm operator, then by normality it must have index 0.

6. If T satisfies (G_1) (equivalently, T^* satisfies (G_1)) and λ is an isolated point of $\sigma(T)$ (equivalently, $\bar{\lambda}$ is an isolated point of $\sigma(T^*)$), then

$$N(T - \lambda I) = N(T^* - \bar{\lambda} I),$$

by Lemma 2, since an isolated point is obviously semibare. (The fact that λ, $\bar{\lambda}$ are eigenvalues of T, T^* [12, proof of Theorem C] is immaterial here.) It follows at once that if T satisfies (G_1), then $\pi_{00}(T^*) = (\pi_{00}(T))^-$, and therefore Weyl's theorem holds for T if and only if it holds for T^* (see the formulas in Example 3). In particular, if T is hyponormal, then Weyl's theorem holds for T^* (see Corollary 1); thus Weyl's theorem holds for every seminormal operator.

7. The final result can be put informally as follows: If a generalized nilpotent operator N is perturbed by a compact operator K, and the resulting operator $N + K$ is "pleasant", then N must also be compact.

Specifically, suppose $T = N + K$, where $\sigma(N) = \{0\}$ and K is compact, and suppose that (i) Weyl's theorem holds for T, and (ii) T satisfies either (α') or (α''). Then T is a compact normal operator, and therefore $N = T - K$ is compact.

The proof is as follows. If the underlying Hilbert space is finite-dimensional, then either of (α'), (α'') implies that T is normal [12, Theorem C], [2, Theorem 2]. Assume that the space is infinite-dimensional. Then $\emptyset \neq \omega(N) \subset \sigma(N) = \{0\}$; thus $\omega(N) = \{0\}$, and therefore $\omega(T) = \omega(N + K) = \omega(N) = \{0\}$. Citing (i), we have the relation

$$(*) \qquad \pi_{00}(T) = \sigma(T) - \{0\};$$

in particular, every nonzero point of $\sigma(T)$ is an isolated point of $\sigma(T)$; therefore $\sigma(T)$ has at most one limit point (namely, 0). It follows that T is normal and the underlying Hilbert space is the orthogonal direct sum of the eigenspaces of T; this is proved in [2, Lemmas 3, 4], under the assumption that T satisfies (α''), and the proof is easily adapted to operators satisfying (α'). Since the eigenspaces belonging to nonzero eigenvalues are finite-dimensional, by (*), and since the nonzero eigenvalues either are finite in number or form a null sequence, T is clearly compact (see [1, p. 187, Exercise 7]).

Conditions (i) and (ii) hold if T is seminormal (see Example 6) or if T satisfies (α') and (β') (see Corollary 2). The result proved above is due to J. Schwartz in the case where T is assumed to be self-adjoint [10, Lemma 2.2] and to K. Kitano in the case where T is assumed to be normal [6, Lemma 2.3].

The result can also be formulated in terms that are suggestive of the theory of spectral operators: If a compact operator is the sum of a "pleasant" operator and a generalized nilpotent, $K = T + N$, then T and N are also compact and T is normal.

Addendum. That Weyl's theorem holds for a seminormal operator (see Example 6 above) is proved in a forthcoming book by Professor Schechter. In his proof also, the key observation is that $N(T - \lambda I) = N(T^* - \bar{\lambda} I)$ for an isolated point λ of $\sigma(T)$; the argument for this involves showing that the idempotent operator

$$P = (1/2\pi i) \int_C (zI - T)^{-1} dz,$$

where C is a sufficiently small circle enclosing λ, is the projection with range $N(T - \lambda I)$ (see Stampfli [11, Theorem 2]).

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