ON MAPS WITH NORMAL STRUCTURE

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INTRODUCTION

In this note, we show that smooth imbeddings that are homotopic through smooth imbeddings have fiber-homotopy-equivalent normal sphere bundles, provided the codimension is at least 3.

The method is to consider maps having (generalized) tubular neighborhoods. Each such neighborhood gives rise to a corresponding normal fibering, which in the smooth case is fiber-homotopy equivalent to the normal sphere bundle. The invariance under homotopy is deduced from a uniqueness theorem for such neighborhoods. The situation for codimension 2 is discussed in the last section.

1. NORMAL STRUCTURES

Definition 1. Suppose K is a finite complex, M^n is a manifold, and $f: K \to M$ is a map. A T-*neighborhood* for f is a compact manifold $N^n \subset M^n$ such that $f(K) \subset \operatorname{int} N$ and $f: K \to N$ is a homotopy equivalence. Two T-neighborhoods N and N' for f are said to be *equivalent* if there is a homotopy equivalence of pairs $h: (N, \partial N) \to (N', \partial N')$ such that f and hf are homotopic as maps from K to N'. An equivalence class $\mathcal{N}$ of T-neighborhoods for f is called a *normal structure* if each open neighborhood of f(K) contains a member of $\mathcal{N}$. The *formal codimension* of $\mathcal{N}$ is the least integer k such that $\pi_k(N, \partial N) \neq 0$ $(N \in \mathcal{N})$.

THEOREM 1. *A map* $f: K \to M$ *admits at most one normal structure of formal codimension greater than or equal to* 3.

Proof. Let $\mathcal{N}$ and $\mathcal{N}'$ be normal structures for f, of formal codimension at least 3. Let $N \in \mathcal{N}$, and choose $N' \in \mathcal{N}'$ so that $N' \subset \operatorname{int} N$ and $N - \operatorname{int} N' = W$ is a manifold. We now show that W is an h-cobordism. Since the formal codimension is at least 3, we have isomorphisms $\pi_1 \partial N' \to \pi_1 N'$ and $\pi_1 \partial N \to \pi_1 N$. The theorem of Van Kampen applied to $N = N' \cup W$ shows that $\pi_1 \partial N' \to \pi_1 W$ is an isomorphism, and it follows easily that $\pi_1 \partial N \to \pi_1 W$ is an isomorphism. Passing to universal covering spaces, we see that we can obtain $(\tilde{W}, \partial\tilde{N}')$ from $(\tilde{N}, \tilde{N}')$ by excising the part over $\operatorname{int} \tilde{N}'$. Since $N' \to N$ is a homotopy equivalence, $H_*(\tilde{N}, \tilde{N}') = 0$, and therefore $\partial\tilde{N}' \to \tilde{W}$ and $\partial N' \to W$ are homotopy equivalences. Poincaré duality gives $H_*(\tilde{W}, \partial\tilde{N}) = 0$, and it follows that $\partial N \to W$ is a homotopy equivalence. Now let $r: W \to \partial N'$ be a deformation retraction. Taking r and $1_{N'}$ together, we obtain $h: N \to N'$, which is an equivalence of T-neighborhoods.

Remark. W may fail to be an h-cobordism in codimension 2; take $f: S^1 \to S^1 \times D^2 = N$ to be a trefoil knot homotopic to the zero section, and N' to be a smooth tubular neighborhood of f. Notice that N and N' are equivalent as T-neighborhoods for f.

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2. NORMAL FIBERINGS

Definition 2. Let $f\colon K \to M$ be a map with T-neighborhood N. Let ξ be the Hurewicz fibering over N associated with the inclusion $\partial N \to N$. Define the *normal fibering* $\nu(f, N)$ to be the induced fibering $f^* \xi$. If $\mathcal{N}$ is a normal structure for f, denote by $\nu(f, \mathcal{N})$ the fiber-homotopy equivalence class of $\nu(f, N)$ $(N \in \mathcal{N})$. When the formal codimension is at least 3, abbreviate this to $\nu(f)$.

We now give a definition of isotopy that seems appropriate in our category of maps with normal structure.

Definition 3. Let f and g be maps from K to M^n with normal structures $\mathcal{N}$ and $\mathcal{N}'$. We say that f and g are *locally homotopic* provided there exist a map $G\colon K \times I \to M$ and a compact submanifold $Q^n \subset M$ such that

$$G_0 = f, \quad G_1 = g, \quad Q \supset G(K \times I), \quad Q \in \mathcal{N}, \quad Q \in \mathcal{N}'.$$

The maps f and g are *weakly locally homotopic* if there exists a sequence $f_0, f_1, \cdots, f_p$ of maps with normal structure such that $f_0 = f$, $f_p = g$, and f_i is locally homotopic to f_{i+1}, for $0 \leq i \leq p - 1$.

THEOREM 2. *If* $(f, \mathcal{N})$ *and* $(g, \mathcal{N}')$ *are weakly locally homotopic, then* $\nu(f, \mathcal{N}) = \nu(g, \mathcal{N}')$, *that is, representatives are fiber-homotopy equivalent.*

Proof. We may assume that $(f, \mathcal{N})$ and $(g, \mathcal{N}')$ are locally homotopic. Let G and Q satisfy Definition 3. If we replace $\partial Q \to Q$ by a fibering ϕ, then the restrictions of $G^* \phi$ to $K \times 0$ and $K \times 1$ are $\nu(f, Q)$ and $\nu(g, Q)$, and these belong to $\nu(f, \mathcal{N})$ and $\nu(g, \mathcal{N}')$.

3. CONSEQUENCES OF UNIQUENESS

The following theorem establishes the relationship between two ideas of isotopy.

THEOREM 3. *Let* $F\colon K \times I \to M$ *be a continuous family of maps, with each* F_t *having a normal structure of formal codimension at least* 3. *Then* F_0 *and* F_1 *are weakly locally homotopic.*

Proof. Recall that the normal structure $\mathcal{N}(t)$ for each F_t is unique (Theorem 1). For each t, select $N(t) \in \mathcal{N}(t)$, and select an open interval U(t) around t such that $F(K \times U) \subset N(t)$. Pick a subcover $U(t_1), \cdots, U(t_p)$ of [0, 1] such that $t_i \leq t_{i+1}$ and $U(t_i) \cap U(t_{i+1}) \neq \emptyset$; pick $r_i \in U(t_i) \cap U(t_{i+1})$. Then $F(K \times [r_i, r_{i+1}]) \subset N(t_i)$. Now $N(t_i)$ is a T-neighborhood for both F_{r_i} and $F_{r_{i+1}}$, has formal codimension at least 3, and is therefore in both $\mathcal{N}(t_i)$ and $\mathcal{N}(t_{i+1})$, by Theorem 1. It follows that F_{r_i} and $F_{r_{i+1}}$ are locally homotopic; moreover, F_0 is locally homotopic to F_{r_1}, and F_1 is locally homotopic to F_{r_p}, and so F_0 and F_1 are weakly locally homotopic. ■

The following theorems are immediate consequences of Theorems 2 and 3.

THEOREM 4. *Let* $f_t\colon K \to M$ *be a continuous family of maps, each* f_t *having a normal structure of formal codimension* $k \geq 3$. *Then* $\nu(f_0) = \nu(f_1)$.

THEOREM 5. *Let* f *and* g *be smooth imbeddings* $V^{n-k} \to M^n$, *with* V *closed and* $k \geq 3$, *that are homotopic through smooth imbeddings. Then the normal sphere bundles are fiber-homotopy equivalent.*

4. CODIMENSION 2

The difficulty in codimension 2 appears to be the possible failure of Theorem 1 for $k = 2$. However, a weaker version of Theorem 1 is sometimes sufficient. We give an example.

THEOREM 6. *Let* f *and* g *be smooth imbeddings* $V^{n-2} \to M^n$, *with orientable normal bundles that are homotopic through smooth imbeddings. Then the normal sphere bundles* E_0 *and* E_1 *for* f *and* g *are fiber-homotopy equivalent.*

LEMMA. *Let* $\partial N' \to N'$ *be orientable (as a fibering). Then there is a retraction* $r\colon W \to \partial N'$ *inducing homology isomorphisms.*

Proof. Let $\rho\colon W \to N'$ be the restriction of a deformation retraction $N \to N'$. Consider the diagram

$$\begin{array}{ccc} W & \xrightarrow{\rho} & N' \\ \uparrow & & \uparrow \\ \partial N' & \xrightarrow{1} & \partial N'. \end{array}$$

The obstructions to deforming ρ into $\partial N'$ relative to $\partial N'$ lie in $H^*(W, \partial N'; \pi_* F)$, where F is the homotopy fiber of $\partial N' \to N'$. Now the orientability of $\partial N' \to N'$ implies that the coefficients are untwisted, and by excision of int N' we obtain the relation $H^*(W, \partial N'; \pi_* F) = 0$. Now deform ρ into a retraction $r\colon W \to \partial N'$. Since $i_*\colon H_* \partial N' \to H_* W$ is an isomorphism, r_* is also an isomorphism.

Proof of Theorem 6. We may reduce the problem to that of Theorem 1, by using an argument similar to the proof of Theorem 3. The lemma then gives a fiberwise map $\phi\colon E_0 \to E_1$ inducing homology isomorphisms. Let F_0 and F_1 be the fibers (of the homotopy type of S^1). Let $\alpha\colon F_0 \to F_1$ be induced by ϕ. Consider the diagram

$$\begin{array}{ccc} H_2(S \wedge F_0) & \xrightarrow{(S\alpha)_*} & H_2(S \wedge F_1) \\ \downarrow & & \downarrow \\ H_2(T_0) & \xrightarrow[\bar{\phi}_*]{} & H_2(T_1) \end{array} ,$$

where T_0 and T_1 are the Thom complexes. Because the bundles are orientable, the vertical maps are isomorphisms. By a familiar exact-sequence argument, $\bar{\phi}_*$ is an isomorphism. It follows that $\deg \alpha = \pm 1$, and therefore ϕ is a fiber-homotopy equivalence.

Remark. If V is replaced by a complex K, then the above proof does not work, for F_0 and F_1 may be infinite-dimensional [1].

In the nonorientable case, we have at present only the following weak version of Theorem 6.

THEOREM 7. *Let* E_0 *and* E_1 *be the normal sphere bundles of smooth imbeddings*

$$f\colon V^{n-2} \to M^n \quad \textit{and} \quad g\colon V^{n-2} \to M^n$$

that are homotopic through smooth imbeddings. Then $H_*(E_0; Z_2) \approx H_*(E_1; Z_2)$.

Proof. Apply the method of Theorem 3 to reduce the problem to the situation in Theorem 1. Now Poincaré duality for W shows that $H_*(W, \partial N; Z_2) = 0$. We then have isomorphisms $H_*(\partial N'; Z_2) \to H_*(W; Z_2) \leftarrow H_*(\partial N; Z_2)$.

REFERENCE

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