ON PERIODICALLY PERTURBED CONSERVATIVE SYSTEMS

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1. INTRODUCTION

The main result of this paper concerns the differential equation

(1.1) $$\frac{d^2 x}{dt^2} + \operatorname{grad} G(x) = p(t),$$

where $p \in C(R, R^n)$, p is 2π-periodic, and $G \in C^2(R^n, R)$. The equation (1.1) can be interpreted physically as the Newtonian equations of motion of a mechanical system subject to conservative internal forces and periodic external forces.

Specifically, we show that if there exist an integer N and numbers μ_N and μ_{N+1} such that

(1.2) $$N^2 < \mu_N \leq \mu_{N+1} < (N+1)^2$$

and

(1.3) $$\mu_N I_n \leq \left(\frac{\partial^2 G(a)}{\partial x_i \partial x_j} \right) \leq \mu_{N+1} I_n$$

for all $a \in R^n$, where I_n is the $n \times n$ identity matrix, then (1.1) has a 2π-periodic solution. This extends results of D. E. Leach [3] and W. S. Loud [4] in the one-dimensional case. Leach and Loud use polar coordinates in the plane, and their method is not applicable to higher dimensions, but they are able to establish uniqueness, which we do not consider here.

In the final section we are able to show both existence and uniqueness for a two-point boundary-value problem for (1.1). Both results depend on a preliminary lemma concerning Hammerstein operators; the lemma is a mild extension of a well known result of C. L. Dolph [1]. In the periodic case we also need an extension of Brouwer's fixed point theorem.

2. A PRELIMINARY LEMMA

Let H be a real Hilbert space, and let K: H → H be a completely continuous, linear symmetric, positive semidefinite operator. Let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

denote those values of λ for which the null space of $\lambda K - I$ (I denotes the identity operator on H) has positive dimension, and let the number of times each λ_n occurs

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in this sequence be equal to the dimension of the null space of $\lambda_n K - I$. Let $f: H \to H$ have a symmetric Gateaux derivative $f'(x)$ at each $x \in H$.

LEMMA. *If there exist numbers* μ_N *and* μ_{N+1} *such that*

$$\lambda_N < \mu_N \leq \mu_{N+1} < \lambda_{N+1} \tag{2.1}$$

for some integer N *and*

$$\mu_N I \leq f'(x) \leq \mu_{N+1} I \tag{2.2}$$

for all $x \in H$, *then the function* $F: H \to H$ *defined by the equation* $F(x) = x - Kf(x)$ *is bijective and has a continuous inverse.*

Proof. Let

$$\mu = \frac{1}{2}(\mu_{N+1} + \mu_N), \qquad \gamma = \frac{1}{2}(\mu_{N+1} - \mu_N), \tag{2.3}$$

and define $G(x) = f(x) - \mu x$. The key point in the proof is that the inequality

$$\|G(x) - G(y)\| \leq \gamma \|x - y\| \tag{2.4}$$

holds for all $x, y \in H$. After this is established, the remainder is essentially the same as Dolph's proof [1] for integral operators on L^2.

To prove the inequality, we note that at each $x \in H$, the function G has a Gateaux derivative $G'(x) = f'(x) - \mu I$, and by Lemma 3.3 of [7],

$$\|G(x) - G(y)\| \leq \|G'(z)\| \cdot \|x - y\|,$$

where z is a point on the line segment from y to x. Since $G'(z)$ is symmetric,

$$\|G'(z)\| = \sup\{|\langle G'(z)v, v\rangle|: v \in H, \|v\| = 1\}. \tag{2.5}$$

From (2.2) and (2.3) it follows that

$$-\gamma\|v\|^2 \leq \langle (f'(z) - \mu I)v, v\rangle \leq \gamma\|v\|^2$$

for all $v \in H$, and these inequalities, together with (2.5), yield (2.4).

Let $w \in H$ be arbitrary, and consider the equation

$$x = w + Kf(x), \tag{2.6}$$

which is equivalent to

$$x = [I - \mu K]^{-1} w + [I - \mu K]^{-1} KG(x) \equiv P(x). \tag{2.7}$$

If $\{v_m\}_1^\infty$ is an orthonormal sequence in H such that $v_m = \lambda_m K v_m$ $(m = 1, 2, \cdots)$, and if $y \in H$, then

$$K[I - \mu K]^{-1} y = K\left[y + \sum_1^\infty \frac{\mu\langle y, v_m\rangle v_m}{\lambda_m - \mu}\right] = \sum_1^\infty \frac{\langle y, v_m\rangle v_m}{\lambda_m} + \sum_1^\infty \frac{\mu\langle y, v_m\rangle v_m}{\lambda_m(\lambda_m - \mu)}$$

$$= \sum_{1}^{\infty} (\lambda_m - \mu)^{-1} \langle y, v_m \rangle v_m$$

(see for example [5, Sections 93 and 94]).

Consequently, by (2.1) and (2.3),

$$\|K[I - \mu K]^{-1}\| = \|[I - \mu K]^{-1}K\| = \sup_m |\lambda_m - \mu|^{-1}$$
$$= \max[(\mu - \lambda_N)^{-1}, (\lambda_{N+1} - \mu)^{-1}] = \alpha .$$

Since (2.1) and (2.3) imply that $\alpha\gamma < 1$ and

$$\|P(x) - P(y)\| \leq \alpha \|G(x) - G(y)\| \leq \alpha\gamma \|x - y\|$$

for $x, y \in H$, it follows that (2.7), and hence (2.6), has a unique solution.

Since $w \in H$ is arbitrary, the mapping $F: H \to H$, $F(x) = x - Kf(x)$ is one-to-one and onto. To show that F^{-1} is continuous, let $w_k = F(x_k)$ $(k = 1, 2)$. The results above imply that

$$x_1 - x_2 = [I - \mu K]^{-1}(w_1 - w_2) + [I - \mu K]^{-1}K(G(x_1) - G(x_2)),$$

so that

$$\|x_1 - x_2\| \leq \|[I - \mu K]^{-1}\| \cdot \|w_1 - w_2\| + \alpha\gamma \|x_1 - x_2\| ,$$

and hence $\|x_1 - x_2\| \leq L \|w_1 - w_2\|$, where $L = (1 - \alpha\gamma)^{-1} \|[I - \mu K]^{-1}\|$.

In Section 3 we shall need the following special case of the inequality above.

COROLLARY. *If, in addition to the hypothesis of the lemma,* $f(0) = 0$, *then* $x = w + Kf(x)$ *implies* $\|x\| \leq L \|w\|$, *where* L *depends only on* μ_N, μ_{N+1}, λ_N, λ_{N+1}, *and* K, *but not on* f.

3. A PERIODIC DIFFERENTIAL EQUATION

In this section, R^n will be regarded as the set of real $n \times 1$ matrices. If $a \in R^n$, then a^* will denote the $1 \times n$ matrix that is the transpose of a, and $|a|$ will denote the usual Euclidean norm of a.

Let H denote the real linear space of functions, defined on R with values in R^n, whose components are 2π-periodic and square integrable on bounded intervals. If for $x, y \in H$ we define

$$\langle x, y \rangle = \int_0^{2\pi} x^*(t)\, y(t)\, dt ,$$

then $(H, \langle\ \rangle)$ is a real Hilbert space. If $y \in H$, we denote the mean value $\frac{1}{2\pi} \int_0^{2\pi} y(t)\, dt$ of y by $M(y)$.

Elementary considerations show that for each $y \in H$ there exists a unique $Ky \in H$ such that Ky has an absolutely continuous derivative, such that $M(Ky) = 0$, and such that

$$\frac{d^2}{dt^2} Ky = -[y - M(y)] \text{ a.e.} \tag{3.1}$$

Indeed, if the Fourier series of y is $\sum_{n=0}^{\infty} (a_n \cos nt + b_n \sin nt)$, then

$$(Ky)(t) = \sum_{n=1}^{\infty} \frac{a_n \cos nt + b_n \sin nt}{n^2}.$$

The relations

$$a_n = \frac{1}{\pi} \int_0^{2\pi} y(s) \cos ns\, ds, \quad b_n = \frac{1}{\pi} \int_0^{2\pi} y(s) \sin ns\, ds \qquad (n = 1, 2, \cdots)$$

imply that

$$(Ky)(t) = \int_0^{2\pi} k(s, t) y(s) ds, \tag{3.2}$$

where

$$k(s, t) = \sum_{n=1}^{\infty} \frac{\cos n(t - s)}{n^2} = k(t, s). \tag{3.3}$$

Since the kernel k is continuous and symmetric, it follows as in the theory of integral equations that $K: H \to H$ is linear, completely continuous, and symmetric.

If y is a nonzero element of H such that $y = \lambda K(y)$, then $M(y) = 0$ and $d^2y/dt^2 + \lambda y = 0$; therefore the periodicity of y implies that $\lambda = m^2$ for some positive integer m.

We now turn to the main result.

THEOREM. *Let* $p \in C(R, R^n)$ *be* 2π*-periodic. If* $G \in C^2(R^n, R)$ *and there exist an integer* N *and numbers* μ_N *and* μ_{N+1} *such that*

$$N^2 < \mu_N \leq \mu_{N+1} < (N+1)^2, \tag{3.4}$$

and if

$$\mu_N I_n \leq \left(\frac{\partial^2 G(a)}{\partial x_i \partial x_j}\right) \leq \mu_{N+1} I_n \tag{3.5}$$

for all $a \in R^n$, *then there exists a* 2π*-periodic solution of* (1.1).

Proof. For convenience, let $Q(a)$ denote the Hessian matrix $\left(\frac{\partial^2 G(a)}{\partial x_i \partial x_j}\right)$. Let H be the space introduced in this section, and define $f: H \to H$ by the equation $f(x)(t) = \text{grad } G(x(t))$ for $x \in H$, $t \in R$. If x and y are in H and s is a point in $[0, 2\pi]$ such that $|x(s)|$ and $|y(s)|$ are bounded, then (3.5) implies that for $t > 0$

$$\left|\frac{1}{t}\left[\operatorname{grad} G(x(s)+ty(s)) - \operatorname{grad} G(x(s))\right] - Q(x(s))\,y(s)\right| \le 2\mu_{N+1}\,|y(s)|\,;$$

therefore by the chain rule and the dominated-convergence theorem,

$$\lim_{t\to 0}\left\|\frac{1}{t}\left[f(x+ty) - f(x)\right] - Q(x)y\right\|^2$$

$$= \lim_{t\to 0}\int_0^{2\pi}\left|\frac{1}{t}\left[\operatorname{grad} G(x(s)+ty(s)) - \operatorname{grad} G(x(s))\right] - Q(x(s))\,y(s)\right|^2 ds = 0\,.$$

It follows that at each $x \in H$ the mapping f has a Gateaux derivative $f'(x)$, where $f'(x)$ is the linear mapping $H \to H$ defined by $(f'(x)y)(t) = Q(x(t))\,y(t)$. If $y \in H$, then $t \in R$. The symmetry of Q implies that $f'(x)$ is symmetric, and by (3.5),

$$\mu_N\int_0^{2\pi}|y(s)|^2\,ds \le \int_0^{2\pi} y^*(s)\,Q(x(s))\,y(s)\,ds \le \mu_{N+1}\int_0^{2\pi}|y(s)|^2\,ds\,,$$

so that $\mu_N I \le f'(x) \le \mu_{N+1} I$ for all $x \in H$.

Let $K\colon H \to H$ be the operator introduced in this section. As we showed above, the only values of λ for which $x = \lambda Kx$ has nontrivial solutions are squares of positive integers; therefore by (3.4), f and K satisfy the hypothesis of the preliminary lemma. Thus for any $w \in H$ there exists a unique $x \in H$ such that $x = w + Kf(x)$. In particular, for each $a \in R^n$ there exists a unique $x_a \in H$ such that

$$x_a = a - Kp + Kf(x_a) = a + K[\operatorname{grad} G(x_a) - p]\,. \tag{3.6}$$

The definition of K implies that if

$$F_0(a) \equiv M[\operatorname{grad} G(x_a) - p]\,, \tag{3.7}$$

then

$$\frac{d^2x_a}{dt^2} = p(t) - \operatorname{grad} G(x_a) + F_0(a)\,.$$

To prove the theorem, it is therefore sufficient to establish the existence of $a_0 \in R^n$ such that $F_0(a_0) = 0$.

If L is the number given by the corollary to the preliminary lemma, then

$$\|x_{a_1} - x_{a_2}\| \le L\,\|a_1 - a_2\| = L\sqrt{2\pi}\,|a_1 - a_2|\,.$$

Hence, by (3.5) and the Schwarz inequality,

$$|F_0(a_1) - F_0(a_2)| \le \frac{1}{2\pi}\int_0^{2\pi}|\operatorname{grad} G(x_{a_1}(s)) - \operatorname{grad} G(x_{a_2}(s))|\,ds$$

$$\le \frac{\mu_{N+1}}{2\pi}\int_0^{2\pi}|x_{a_1}(s) - x_{a_2}(s)|\,ds \le \frac{\mu_{N+1}}{\sqrt{2\pi}}\,\|x_{a_1} - x_{a_2}\| \le \mu_{N+1}\,L\,|a_1 - a_2|\,,$$

and therefore F_0 is continuous.

From (3.2) and (3.3) it follows that $Kc = 0$ for every constant function c. Consequently (3.6) is equivalent to

$$x_a - a = v + K[f(x_a) - f(a)] \tag{3.8}$$

(here and in the remainder of the proof we write $K[-p] = v$). Let $a \in R^n$ be fixed, and define $\hat{f}: H \to H$ by $\hat{f}(x) = f(x+a) - f(a)$. Then $\hat{f}'(x) = f'(x+a)$, and therefore $\mu_N \leq \hat{f}'(x) \leq \mu_{N+1}$ for all $x \in H$. If $y = x_a - a$, then (3.8) is equivalent to

$$y = v + K\hat{f}(y).$$

But $\hat{f}$ and K satisfy the hypothesis of the preliminary lemma and $\hat{f}(0) = 0$; therefore the corollary is applicable to y. Thus if L is defined as above,

$$\|x_a - a\| \leq L\|v\|, \tag{3.9}$$

and since L is independent of a, (3.9) is true for all $a \in R^n$.

Let $F_1(a) = M[\text{grad } G(a) - p] = \text{grad } G(a) - M[p]$. From (3.5) and (3.7) it follows that

$$|F_0(a) - F_1(a)| \leq \frac{1}{2\pi}\int_0^{2\pi} |\text{grad } G(x_a(s)) - \text{grad } G(a)|\, ds$$

$$\leq \frac{\mu_{N+1}}{2\pi}\int_0^{2\pi} |x_a(s) - a|\, ds \leq \frac{\mu_{N+1}}{\sqrt{2\pi}}\|x_a - a\|,$$

and hence by (3.9)

$$|F_0(a) - F_1(a)| \leq L(\mu_{N+1}/\sqrt{2\pi})\|w\|. \tag{3.10}$$

From the identity

$$F_1(a) = \text{grad } G(a) - M[p] = \text{grad } G(0) - M[p] + \left[\int_0^1 Q(sa)\,ds\right] a$$

and (3.5) we infer that

$$a^* F_1(a) \geq \mu_N |a|^2 - |a| \cdot |\text{grad } G(0) - M[p]|;$$

hence $a^* F_1(a) \to +\infty$ as $|a| \to \infty$, and by virtue of (3.10), $a^* F_0(a) \to +\infty$ as $|a| \to \infty$.

Assume r to be so large that $a^* F_0(a) > 0$ if $|a| = r$. We may choose $\varepsilon > 0$ so small that for $|a| = r$

$$|a - \varepsilon F_0(a)|^2 = |a|^2 - 2\varepsilon a^* F_0(a) + \varepsilon^2 |F_0(a)|^2 < |a|^2 = r^2.$$

This inequality implies that if B_r is the closed ball of radius r about the origin in R^n, then the continuous mapping $\theta: R^n \to R^n$ defined by the equation $\theta(a) = a - \varepsilon F_0(a)$ maps the boundary of B_r into B_r.

By an extension of Brouwer's fixed point theorem due to B. Knaster, K. Kuratowski, and S. Mazurkiewicz [2] and to E. H. Rothe [6], there exists $a_0 \in B_r$ such that $\theta(a_0) = a_0$. Hence $F_0(a_0) = 0$, and by a previous remark the proof is complete.

4. A BOUNDARY-VALUE PROBLEM

In this section, we indicate a more direct application of the results of the second section. Let H be the real Hilbert space of functions defined on $[0, 2\pi]$ with values in R^n, with components in $L^2[0, \pi]$, and with inner product

$$\langle f, g \rangle = \int_0^{2\pi} f^*(t)\, g(t)\, dt .$$

Elementary arguments show that for each $f \in H$ there exists a unique $Kf \in H$ that is a solution of the boundary-value problem

$$x'' = -f \text{ (a. e.)}, \qquad x(0) = x(\pi) = 0 .$$

A direct computation shows that

$$K(s, t) = \int_a^b k(s, t) f(s)\, ds , \tag{4.1}$$

where

$$K(s, t) = \begin{cases} s(\pi - t)/\pi & (0 \le s \le t \le \pi), \\ t(\pi - s)/\pi & (0 \le t \le s \le \pi). \end{cases}$$

By standard arguments, (4.1) implies that $K: H \to H$ is completely continuous and symmetric. Moreover, if for some real λ, $u = \lambda K u$ and $u \neq 0$, then $\lambda = m^2$ for some positive integer m.

Suppose $G \in C^2(R^n, R)$ and $p \in C(R, R^n)$; then $y \in H$ is a solution of the two-point boundary-value problem

$$y'' + \operatorname{grad} G(y) = p(t), \qquad y(0) = a, \quad y(\pi) = b \tag{4.2}$$

if and only if y is a solution of the integral equation

$$y(t) = a + \frac{t}{\pi}(b - a) + \int_0^{\pi} k(s, t)[\operatorname{grad} G(y(s)) - p(s)]\, ds .$$

By reasoning as in the previous section, one shows that if for some integer N and numbers μ_N and μ_{N+1} the conditions (1.2) and (1.3) are satisfied, then the integral equation, and hence the boundary-value problem (4.2), has a unique solution.

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