A STABILITY THEOREM FOR FRAMES IN HILBERT SPACE

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The point of departure for this note is the well-known theorem [1, p. 4] stating that if $\{x_i\}_1^\infty$ and $\{y_i\}_1^\infty$ are frames (that is, orthonormal sequences) in a Hilbert space H and if $\sum \|x_i - y_i\|^2 < \infty$, then the completeness of $\{x_i\}$ implies that of $\{y_i\}$ (for a generalization, see [2]). We may regard this theorem as a global extension of the trivial local stability theorem in which the symbol ∞ in the inequality is replaced with 1. In this note, we prove that any class of frames, stable under a small perturbation of a certain type, is automatically stable under a more radical perturbation.

1. SOME DEFINITIONS AND THE THEOREM

H will be a complex Hilbert space of arbitrary dimension, and A will be a fixed set such that $\mathrm{card}(A) \leq \dim(H)$. An A-frame is an H-valued function on A whose image is an orthonormal set of H.

Let R^+ be the nonnegative real numbers. Let ϕ be a function defined on $\prod_A R^+$ into $R^+ \cup \{\infty\}$ such that

(1) if $\xi_a \leq K\eta_a$ for all $a \in A$ and if $\phi\{\eta_a\} < \infty$, then $\phi\{\xi_a\} < \infty$,

(2) if for each $a \in A$, f_a is a continuous function from a topological space X into R^+, if $f_a(x) \leq \xi_a$ for all $a \in A$ and $x \in X$, and if $\phi\{\xi_a\} < \infty$, then $\phi\{f_a\}$ is continuous.

THEOREM. *Let* C *be a class of* A*-frames. Suppose there exists a* $\delta > 0$ *such that if* $\{x_a\} \in C$ *and* $\{y_a\}$ *is an* A*-frame with* $\phi\{\|x_a - y_a\|\} < \delta$, *then* $\{y_a\} \in C$. *Suppose further that* $\{x_a\} \in C$ *and that* $\{z_a\}$ *is an* A*-frame with* $\phi\{\|z_a - x_a\|\} < \infty$. *Then* $\{z_a\} \in C$.

Assumption (2), which is a generalized M-test, assures us that the perturbations defined by ϕ are small. We can obtain the prototypical theorem mentioned in the introductory paragraph by taking A to be the positive integers, $\phi\{\xi_j\} = \sum \xi_j^2$, $\delta = 1$, and C the class of complete frames. The theorem is an almost immediate consequence of the following lemma.

LEMMA. *Let* $\{x_a\}$ *and* $\{y_a\}$ *be* A*-frames. Then for each* t *with* $0 \leq t \leq 1$, *there exists an* A*-frame* $\{x_a(t)\}$ *satisfying the conditions*

(i) $x_a(0) = x_a$, $x_a(1) = y_a$,

(ii) $x_a(t)$ *is a strongly continuous function of* t,

(iii) $\|x_a(t) - x_a\| \leq 2\|y_a - x_a\|$ *for all* $a \in A$.

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2. PROOFS

First we prove the theorem, assuming the lemma. Since

$$\phi\{\|x_a - z_a\|\} < \infty \qquad \text{and} \qquad \|x_a(t) - x_a(u)\| \leq 4\|x_a - z_a\|,$$

(1) and (2) imply that $\phi\{\|x_a(t) - x_a(u)\|\} = \psi(t, u)$ is continuous on the unit square. Pick n so large that

$$\psi(k/n, k - 1/n) = |\psi(k/n, k - 1/n) - \psi(k/n, k/n)| < \delta,$$

so that $\{x_a(k/n)\} \in C$ if $\{x_a(k - 1/n)\} \in C$. Then $\{x_a(1)\} = \{z_a\} \in C$.

We now prove the lemma. Assume first that

$$\dim(H \ominus \operatorname{span}\{x_a\}) \leq \dim(H \ominus \operatorname{span}\{y_a\}).$$

Then there is an isometry V on H with $Vx_a = y_a$ for all $a \in A$.

Next note that for any Hilbert-space isometry W and any complex number λ with $|\lambda| < 1$, $W_\lambda = (W + \lambda I)(I + \bar{\lambda} W)^{-1}$ is an isometry because

$$\begin{aligned} W_\lambda^* W_\lambda &= (I + \lambda W^*)^{-1}(W^* + \bar{\lambda} I)(I + \bar{\lambda} W)^{-1}(W + \lambda I) \\ &= (I + \lambda W^*)^{-1} W^* (I + \bar{\lambda} W)(I + \bar{\lambda} W)^{-1}(I + \lambda W^*) W = I. \end{aligned}$$

Let $J = H \ominus V(H)$, and write $H = \sum_0^\infty V^k J \oplus K$, where the $V^k J$ and K are pairwise orthogonal. The restrictions of V to K and $H \ominus K$ will be denoted by S and R, respectively. For $0 < t \leq 1$, define $x_a(t) = (R_{1-t} \oplus S^t)x_a$, where R_{1-t} is the isometry defined as above and

$$S^t = \int_{-\pi}^{\pi} e^{i\lambda t}\, dE(\lambda),$$

$E(\lambda)$ being the resolution of the identity for S. The family S^t is the usual curve in U(K) (the unitary operators on K) joining the unitary operator S with I. Clearly, $\{x_a(t)\}$ is an A-frame, and $x_a(1) = (R \oplus S)x_a = Vx_a = y_a$. We define $x_a(0)$ to be x_a, and x_a^1 and x_a^2 to be the projections of x_a on $H \ominus K$ and K, respectively.

We have the estimate

$$\begin{aligned} \|x_a(t) - x_a\|^2 &= \|(R_{1-t} - I)x_a^1\|^2 + \|(S^t - I)x_a^2\|^2 \\ &= t^2\|(I + (1 - t)R)^{-1}(R - I)x_a^1\|^2 + \|(S^t - I)x_a^2\|^2 \\ &\leq t^2\|(I + (1 - t)R)^{-1}\|^2\,\|(R - I)x_a^1\|^2 + \|S - I)x_a^2\|^2 \\ &\leq \|(R - I)x_a^1\|^2 + \|(S - I)x_a^2\|^2 = \|y_a - x_a\|^2. \end{aligned}$$

Since R_{1-t} is norm-continuous for $0 < t \leq 1$, we have only to show that $R_u x \to x$ as $u \to 1$. If $x = V^n y$, where $y \in J$, and if $\|y\| = 1$, then

$$R_u x - x = (1 - u)\left\{\left(\sum_1^\infty (-1)^{j-1}(u^{j-1} + u^j)V^{j+n}y\right) - V^n y\right\},$$

and therefore

$$\|R_u x - x\|^2 = (1 - u)^2 \left\{ 1 + \sum (1 + u)^2 u^{2(j-1)} \right\} = o(1).$$

Hence R_u tends strongly to I on linear combinations of the $V^n x$, and hence on all elements of $H \ominus K$.

Finally, if $\dim(H \ominus \operatorname{span}\{x_a\}) > \dim(H \ominus \operatorname{span}\{y_a\})$, we form the $\{y_a(t)\}$ guaranteed by the above argument. Then $x_a(t) = y_a(1 - t)$ has properties (i) and (ii), and

$$\|x_a(t) - x_a\| \leq \|y_a(1 - t) - y_a\| + \|y_a - x_a\| \leq 2\|y_a - x_a\|.$$

This completes the proof.

3. SOME REMARKS

1. If A consists of the positive integers and $\phi\{\xi_j\} = \left(\sum \lambda_j \xi_j^2\right)^{1/2}$, where $\lambda_j > 0$, then the A-frames may be metrized by

$$d(\{x_i\}, \{y_i\}) = \phi\{\|x_i - y_i\|\}/[1 + \phi\{\|x_i - y_i\|\}],$$

with the usual convention that $\infty/(1 + \infty) = 1$. Then the lemma implies that the components are the balls of radius less than unity.

2. If in the proof of the lemma we define $x_a(t)$ as $(V + (1 - t)I)(I + (1 - t)V)^{-1} x_a$, then $\lim x_a(t)$ is not necessarily x_a, but $(I - 2P)x_a$, where P is the projection on the eigenspace of V corresponding to the eigenvalue -1.

REFERENCES

1. P. R. Halmos, *A Hilbert space problem book.* Van Nostrand, Princeton, N. J., 1967.

2. Y. Sibuya, *On biorthogonal systems.* Michigan Math. J. 13 (1966), 165-168.

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