SEMIFREE DIFFERENTIABLE ACTIONS OF S^1 ON HOMOTOPY (4k + 3)-SPHERES

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1. INTRODUCTION

We shall call an action of S^1 *semifree* if it is free outside the set F of fixed points. It is well-known that if S^1 acts semifreely on a homotopy (4k + 3)-sphere Σ^{4k+3} with fixed point set F of codimension 4, then the orbit space has a natural differentiable structure and is a homotopy (4k + 2)-sphere. In this paper, we study the semifree differentiable actions of S^1 on homotopy (4k + 3)-spheres ($k \geq 2$), the fixed point sets consisting of the homotopy (4k - 1)-spheres. The only complete result in this direction is the following theorem of Montgomery and Yang [7, Theorem 3].

THEOREM. *On any homotopy 7-sphere, there are infinitely many differentiably distinct, semifree, differentiable actions of the circle group* S^1, *each having* S^3 *as the fixed point set.*

The following five theorems are immediate consequences of the main theorem.

THEOREM 1. *If there exists a semifree differentiable action of* S^1 *on a homotopy* (4k + 3)*-sphere* Σ^{4k+3}, *and if its fixed point set is a homotopy* (4k - 1)*-sphere* Σ^{4k-1} *and its orbit space is a homotopy* (4k + 2)*-sphere* Σ^{4k+2}, *then there exist infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *on* Σ^{4k+3} *with fixed point set* Σ^{4k-1} *and orbit space* Σ^{4k+2}.

THEOREM 2. *Every homotopy sphere* Σ^{4k+3} *in* bP_{4k+4} *admits infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *with fixed point sets of codimension* 4. *For example, let* Σ_M^{4k+3} *and* Σ_M^{4k-1} *be the Milnor spheres of dimensions* 4k + 3 *and* 4k - 1, *respectively. Then, for each integer* $n \geq 1$, *the homotopy sphere* $n\Sigma_M^{4k+3}$ *admits infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *with fixed point set* $n\Sigma_M^{4k-1}$. (For the notation bP_n, see [5, p. 510].)

THEOREM 3. *There exist infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *on* S^{4k+3} *with fixed point set* S^{4k-1} *and orbit space* S^{4k+2}.

THEOREM 4. (i) *For each homotopy sphere* Σ^7 *in* $4\theta_7$, *there exist infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *on* S^{11} *with fixed point set* Σ^7.

(ii) *On each homotopy sphere* Σ^{11} *in* $2\theta_{11}$, *there are infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *with orbit space* S^{10}.

(iii) *On each homotopy sphere in* $4\theta_{11}$, *there are infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *with fixed point set* S^7 *and orbit space* S^{10}.

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THEOREM 5. (i) *For each homotopy sphere* Σ^{11} *in* $32\theta_{11}$, *there exist infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *on* S^{15} *with fixed point set* Σ^{11}.

(ii) *Each homotopy sphere in* bP_{16} *admits infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *with orbit space* S^{14}.

(iii) *On each homotopy sphere in* $32bP_{16} \approx Z_{254}$, *there are infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *with fixed point set* S^{11} *and orbit space* S^{14}.

(iv) *On each homotopy* 15-*sphere, there are infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *whose fixed point sets are of codimension* 4.

The proofs used in this paper depend heavily on the results of Montgomery and Yang [8] and of Levine [6], but they are more algebraic in character. Throughout the paper, we assume $k \geq 2$.

2. PRELIMINARIES AND NOTATION

Definition 2.1. By a *standard action* of S^1 on S^{4k+3} we mean the following. Let

$$S^{4k+3} = \{(z_1, \cdots, z_{2k+2}) \in C^{2k+2} \mid \Sigma_{i=1}^{2k+2} |z_i|^2 = 1\},$$

and let S^1 act on S^{4k+3} on the last two coordinates via the linear action $S^1 \subset U(2)$. Under this action, the fixed point set is $S^{4k-1} \times \{0\}$, which we identify with S^{4k-1}. The orbit space is easily seen to be diffeomorphic to S^{4k+2}. The imbedding of S^{4k-1} onto the submanifold $S^{4k-1} \times \{0\}$ of S^{4k+3} is called a *standard imbedding*.

PROPOSITION 2.2 (Montgomery and Yang [8, Proposition 4]). *Let* Σ^{n-1} *and* Σ^{n-4} *be homotopy spheres of dimension* $n - 1$ *and* $n - 4$ $(n \geq 7)$, *and let* f *be an imbedding of* Σ^{n-4} *into* Σ^{n-1}. *Then there exists a semifree, differentiable action of the circle group* S^1 *on a homotopy* n-*sphere* Σ^n *whose fixed point set is diffeomorphic to* Σ^{n-4} *and whose orbit space is diffeomorphic to* Σ^{n-1}.

Definition 2.3. We shall denote the set of isotopy classes of knotted $(4k - 1)$-spheres in S^{4k+2} by $\theta^{4k+2,4k-1}$; it is an infinite abelian group of rank 1 [6]. We denote the equivalence class of $(S^{4k+2}, \Sigma^{4k-1})$ by $[S^{4k+2}, \Sigma^{4k-1}]$. Let (S^1, Σ^{4k+3}, F) be a semifree, differentiable action of S^1 on a homotopy $(4k + 3)$-sphere Σ^{4k+3} that has fixed point set $F \in \theta_{4k-1}$. The equivariant diffeomorphism class of (S^1, Σ^{4k+3}, F) is denoted by $\{S^1, \Sigma^{4k+3}, F\}$. Let $\Sigma_*^{4k+3,4k-1}$ be the totality of diffeomorphism classes $\{S^1, \Sigma^{4k+3}, F\}$ modulo the subset

$$\{\{S^1, S^{4k+3}, S^{4k-1}\} \mid \text{the imbedding of } S^{4k-1} \text{ into } S^{4k+3} \text{ is isotopic to the standard imbedding}\}.$$

Denote the equivalence class of $\{S^1, \Sigma^{4k+3}, F\}$ by $[S^1, \Sigma^{4k+3}, F]$. It is easy to verify that $\Sigma_*^{4k+3,4k-1}$ is an abelian group under the connected sum operation

$$[S^1, \Sigma_1^{4k+3}, F_1] + [S^1, \Sigma_2^{4k+3}, F_2] = [S^1, \Sigma_1^{4k+3} \# \Sigma_2^{4k+3}, F_1 \# F_2].$$

Furthermore, if there are two semifree, differentiable actions $(S^1, \Sigma_1^{4k+3}, F_1)$ and $(S^1, \Sigma_2^{4k+3}, F_2)$, then

$$(\Sigma_1^{4k+3} \# \Sigma_2^{4k+3})/S^1 = (\Sigma_1^{4k+3}/S^1) \# (\Sigma_2^{4k+3}/S^1).$$

Thus the subset $\Sigma_{**}^{4k+3,4k-1}$ of $\Sigma_*^{4k+3,4k-1}$ defined by

$$\Sigma_{**}^{4k+3,4k-1} = \{[S^1, \Sigma^{4k+3}, F] \in \Sigma_*^{4k+3,4k-1} \mid \Sigma^{4k+3}/S^1 = S^{4k+2}\}$$

is a well defined subgroup of $\Sigma_*^{4k+3,4k-1}$.

LEMMA 2.4. *The groups* $\Sigma_{**}^{4k+3,4k-1}$ *and* $\Sigma_*^{4k+3,4k-1}$ *are infinite.*

Proof. Define a homomorphism ϕ: $\Sigma_{**}^{4k+3,4k-1} \to \theta^{4k+2,4k-1}$ by

$$\phi[S^1, \Sigma^{4k+3}, F] = [\Sigma^{4k+3}/S^1, F].$$

By Proposition 2.2, ϕ is onto. Since $\theta^{4k+2,4k-1}$ is infinite, $\Sigma_{**}^{4k+3,4k-1}$ is infinite.

Definition 2.5. Define the homomorphisms

$$\alpha\colon \Sigma_*^{4k+3,4k-1} \to \theta_{4k+3}, \quad \beta\colon \Sigma_*^{4k+3,4k-1} \to \theta_{4k-1}, \quad \gamma\colon \Sigma_*^{4k+3,4k-1} \to \theta_{4k+2},$$

$$\alpha^*\colon \Sigma_{**}^{4k+3,4k-1} \to \theta_{4k+3}, \quad \text{and} \quad \beta^*\colon \Sigma_{**}^{4k+3,4k-1} \to \theta_{4k-1}$$

as follows:

$$\alpha[S^1, \Sigma^{4k+3}, \Sigma^{4k-1}] = \Sigma^{4k+3}, \quad \beta[S^1, \Sigma^{4k+3}, \Sigma^{4k-1}] = \Sigma^{4k-1},$$

$$\gamma[S^1, \Sigma^{4k+3}, \Sigma^{4k-1}] = \Sigma^{4k+3}/S^1,$$

$$\alpha^* = \alpha \mid \Sigma_{**}^{4k+3,4k-1}, \quad \text{and} \quad \beta^* = \beta \mid \Sigma_{**}^{4k+3,4k-1}.$$

LEMMA 2.6. (i) $\operatorname{Im}\alpha \supset bP_{4k+4}$ *and* $\operatorname{Im}\beta \supset bP_{4k}$.

(ii) $\operatorname{Im}\alpha^* \supset (\operatorname{ord}\theta_{4k+2})bP_{4k+4}$ *and* $\operatorname{Im}\beta^* \supset (\operatorname{ord}\theta_{4k+2})bP_{4k}$. *In particular,*

$$\operatorname{Im}\alpha^{*11,7} \supset 2\theta_{11}, \quad \operatorname{Im}\alpha^{*15,11} \supset 2bP_{16},$$

$$\operatorname{Im}\beta^{*11,7} \supset 2\theta_7, \quad \operatorname{Im}\beta^{*15,11} \supset 2\theta_{11},$$

where $\alpha^{*4k+3,4k-1}$ *means* $\alpha^*\colon \Sigma_{**}^{4k+3,4k-1} \to \theta_{4k+3}$, *and so forth.*

(iii) γ *is surjective.*

Proof. Let us recall the explicit description of homotopy spheres in bP_{4k+4} given by Brieskorn and Hirzebruch [3], [4]:

$$\Sigma_{3,6n-1}^{4k+3} = \{(z_1, \cdots, z_{2k+3}) \in C^{2k+3} \mid z_1^3 + z_2^{6n-1} + z_3^2 + \cdots + z_{2k+3}^2 = 0,$$
$$|z_1|^2 + \cdots + |z_{2k+3}|^2 = 1\} \approx n\Sigma_M^{4k+3}.$$

Consider the action of S^1 on the last two variables of $\Sigma_{3,6n-1}^{4k+3}$ via the representation $S^1 \subset U(2)$, which gives $\Sigma_{3,6n-1}^{4k-1}$ as a fixed point set and a homotopy $(4k+2)$-sphere as orbit space. Applying Proposition 2.2, we obtain (i). The action induces an action of S^1 on the homotopy sphere $(\operatorname{ord}\theta_{4n+2})\Sigma_{3,6n-1}^{4k+3}$ with fixed point set $(\operatorname{ord}\theta_{4n+2})\Sigma_{3,6n-1}^{4k-1}$ and orbit space S^{4k+2}. This implies (ii), by Proposition 2.2. The particular cases follow from the fact (see [5]) that

$\theta_7 = Z_{28}, \quad \theta_{10} = Z_6, \quad \theta_{11} = Z_{992}, \quad \theta_{14} = Z_2, \quad bP_{16} = Z_{8128}.$

We turn to the proof of (iii). For each homotopy sphere $\Sigma^{4k+2} \in \theta_{4k+2}$, there exists a sequence of imbeddings

$$S^{4k-1} \to D^{4k} \to D^{4k+2} \to \Sigma^{4k+2}.$$

Again we can apply Proposition 2.2 to the pair $(\Sigma^{4k+2}, S^{4k-1})$ to get an element $[S^1, \Sigma^{4k+3}, S^{4k-1}] \in \Sigma_*^{4k+3,4k-1}$ such that

$$\gamma[S^1, \Sigma^{4k+3}, S^{4k-1}] = \Sigma^{4k+2}.$$

Problem: Determine the groups Im α, Im α^*, Im β, and Im β^*.

3. THE MAIN THEOREM

LEMMA 3.1. *The quotient groups* $\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta$, $\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\gamma$, *and* $\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\beta \cap \mathrm{Ker}\,\gamma$ *are finite; hence, the subgroups* $\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta$, $\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\gamma$, *and* $\mathrm{Ker}\,\beta \cap \mathrm{Ker}\,\gamma$ *are infinite.*

Proof. The groups $\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\alpha$, $\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\beta$, and $\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\gamma$ are finite because

$$\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\alpha \approx \mathrm{Im}\,\alpha, \qquad \Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\beta \approx \mathrm{Im}\,\beta,$$
$$\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\gamma \approx \mathrm{Im}\,\gamma,$$

and because Im α, Im β, and Im γ are finite [5]. Consider the exact sequence

$$0 \to \mathrm{Ker}\,\alpha/\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \to \Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \to \Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\alpha \to 0.$$

By the second isomorphism theorem

$$\mathrm{Ker}\,\alpha/\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \approx (\mathrm{Ker}\,\alpha + \mathrm{Ker}\,\beta)/\mathrm{Ker}\,\beta,$$

and the right-hand member is a subgroup of the finite group $\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\beta$. Similar proofs apply to the other two cases. This completes the proof of the lemma.

LEMMA 3.2. *The group* $\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \cap \mathrm{Ker}\,\gamma$ *is finite. Hence the group* $\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \cap \mathrm{Ker}\,\gamma$ *is infinite.*

Proof. The sequence

$$0 \to \frac{\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta}{\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \cap \mathrm{Ker}\,\gamma} \to \frac{\Sigma_*^{4k+3,4k-1}}{\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \cap \mathrm{Ker}\,\gamma} \to \frac{\Sigma_*^{4k+3,4k-1}}{\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta} \to 0$$

is exact, and the group $(\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta)/(\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \cap \mathrm{Ker}\,\gamma)$ is isomorphic to a subgroup of the finite group $\Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\gamma$. Lemma 3.1 now gives the result.

MAIN THEOREM. *Let the homotopy spheres* Σ^{4k+3}, Σ^{4k-1}, *and* Σ^{4k+2} *belong to* Im α, Im β, *and* θ_{4k+2}, *respectively. Then*

(i) Σ^{4k+3} *admits infinitely many differentiably distinct, semifree, differentiable actions of* S^1 *whose fixed point sets have codimension* 4.

(ii) *If* $\alpha^{-1}(\Sigma^{4k+3}) \cap \beta^{-1}(\Sigma^{4k-1}) \neq \emptyset$, *then there are infinitely many differentiably nonequivalent, semifree actions of* S^1 *on* Σ^{4k+3} *with fixed point set* Σ^{4k-1}.

(iii) *If* $\alpha^{-1}(\Sigma^{4k+3}) \cap \gamma^{-1}(\Sigma^{4k+2}) \neq \emptyset$, *then there exist infinitely many distinct, semifree, differentiable actions of* S^1 *on* Σ^{4k+3} *with orbit space* Σ^{4k+2}.

(iv) *If* $\beta^{-1}(\Sigma^{4k-1}) \cap \gamma^{-1}(\Sigma^{4k+2}) \neq \emptyset$, *then there exists at least one homotopy* $(4k+3)$*-sphere that admits infinitely many differentiably distinct, semifree actions of* S^1 *whose fixed point set is* Σ^{4k-1} *and whose orbit space is* Σ^{4k+2}.

(v) *If* $\alpha^{-1}(\Sigma^{4k+3}) \cap \beta^{-1}(\Sigma^{4k-1}) \cap \gamma^{-1}(\Sigma^{4k+2}) \neq \emptyset$, *then there are infinitely many differentiably distinct, differentiable actions of* S^1 *on* Σ^{4k+3} *with fixed point set* Σ^{4k-1} *and orbit space* Σ^{4k+2}.

Proof. (i) is clear. We give only the proof of (v), because the proofs of (ii) to (v) are exactly the same. Let

$$\psi\colon \Sigma_*^{4k+3,4k-1} \to \Sigma_*^{4k+3,4k-1}/\mathrm{Ker}\,\alpha \cap \mathrm{Ker}\,\beta \cap \mathrm{Ker}\,\gamma$$

be the natural map. Then, for each element

$$[S^1, \Sigma^{4k+3}, \Sigma^{4k-1}] \in \alpha^{-1}(\Sigma^{4k+3}) \cap \beta^{-1}(\Sigma^{4k-1}) \cap \gamma^{-1}(\Sigma^{4k+2}),$$

$\psi^{-1}(\psi[S^1, \Sigma^{4k+3}, \Sigma^{4k-1}])$ contains infinitely many elements, by Lemma 3.2, and clearly

$$\psi^{-1}(\psi[S^1, \Sigma^{4k+3}, \Sigma^{4k-1}]) \subset \alpha^{-1}(\Sigma^{4k+3}) \cap \beta^{-1}(\Sigma^{4k-1}) \cap \gamma^{-1}(\Sigma^{4k+2}).$$

By definition, the set $\alpha^{-1}(\Sigma^{4k+3}) \cap \beta^{-1}(\Sigma^{4k-1}) \cap \gamma^{-1}(\Sigma^{4k+2})$ contains all the elements $[S^1, \Sigma^{4k+3}, \Sigma^{4k-1}]$ with fixed point set Σ^{4k-1} and orbit space Σ^{4k+2}. This completes the proof of the main theorem.

Theorem 1 is a restatement of part (v) of the Main Theorem. Theorem 2 follows from Lemma 2.6 and parts (i) and (ii) of the Main Theorem, and Theorem 3 can be deduced either from Lemma 3.2 or from part (v) of the Main Theorem.

Proof of Theorem 4. It is known that S^1 can act semifreely on S^{11} in such a way that the set of fixed points is any prescribed homotopy sphere in $4\theta_7$ and its orbit space is S^{10} [8]. This implies (i). Part (ii) follows from Lemma 2.6, because $\mathrm{Im}\,\alpha^{*11,7} \supset 2\theta_{11}$. Since $84\theta_{11} = 4\theta_{11}$, we can deduce (iii) by using the Brieskorn-Hirzebruch description of homotopy spheres in θ_{11}.

Proof of Theorem 5. By using again the Brieskorn-Hirzebruch equations, we see that S^1 can act semifreely on S^{15} with fixed point set in $32\theta_{11}$ and with orbit space S^{14}. This proves (i). To prove (ii) and (iv), it suffices to show that the homomorphism

$$\alpha\colon \Sigma_*^{15,11} \to \theta_{15}$$

is surjective. This will imply that S^1 can act on $2\theta_{15} = bP_{16}$ with orbit space in $2\theta_{14}$. By [1] and [2], there exists a homotopy sphere $\Sigma^{15} \notin bP_{16}$ such that $\Sigma^{15} \in \mathrm{Im}\,\alpha$. Thus $\mathrm{Im}\,\alpha \supset \{\Sigma^{15}, bP_{16}\}$. Hence, $\mathrm{Im}\,\alpha = \theta_{15}$, because $\theta_{15}/bP_{16} = Z_2$ [5]. Finally, S^1 can act semifreely on $992bP_{16} = 32bP_{16}$ with fixed point set in $992\theta_{11}$ and orbit space in $992\theta_{14}$. This completes the proof of (iii).

COROLLARY 3.3. $\mathrm{Im}\,\alpha^{15,11} = \theta_{15}$ *and* $\mathrm{Im}\,\alpha^{*15,11} \supset bP_{16}$.

Problems. Determine all possible homotopy spheres Σ^{4k+3}, Σ^{4k-1}, and Σ^{4k+2} that satisfy one of the conditions

$$\alpha^{-1}(\Sigma^{4k+3}) \cap \beta^{-1}(\Sigma^{4k-1}) \neq \emptyset,$$

$$\alpha^{-1}(\Sigma^{4k+3}) \cap \beta^{-1}(\Sigma^{4k-1}) \cap \gamma^{-1}(\Sigma^{4k+2}) \neq \emptyset,$$

$$\cdots.$$

Notice that if $\Sigma^{4k-1} \in bP_{4k}$, then $\Sigma^{4k-1} \times D^3 \approx S^{4k-1} \times D^3$ [9]. Thus Σ^{4k-1} can be imbedded in S^{4k+2}. Therefore $\operatorname{Im} \beta^* \supset bP_{4k}$ (see [8]).

Part (iv) of the Main Theorem implies the following.

THEOREM 3.4. *Corresponding to each homotopy sphere Σ^{4k-1} in bP_{4k}, there exists at least one homotopy $(4k+3)$-sphere that admits infinitely many differentiably distinct, semifree actions of S^1 whose fixed point set is Σ^{4k-1} and whose orbit space is S^{4k+2}.*

REFERENCES

1. G. E. Bredon, *A Π_*-module structure for Θ_* and applications to transformation groups.* Ann. of Math. (2) 86 (1967), 434-448.
2. ———, *Exotic actions on spheres.* Proceedings of the Conference on Transformation Groups, Springer-Verlag, 1968 (to appear).
3. E. Brieskorn, *Beispiele zur Differentialtopologie von Singularitäten.* Invent. Math. 2 (1966/67), 1-14.
4. F. Hirzebruch, *Singularities and exotic spheres.* Séminaire Bourbaki, 1966/67, No. 314.
5. M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres, I.* Ann. of Math. (2) 77 (1963), 504-537.
6. J. Levine, *A classification of differentiable knots.* Ann. of Math. (2) 82 (1965), 15-50.
7. D. Montgomery and C. T. Yang, *Differentiable actions on homotopy seven spheres I.* Trans. Amer. Math. Soc. 122 (1966), 480-498.
8. ———, *Differentiable transformation groups on homotopy spheres.* Michigan Math. J. 14 (1967), 33-46.
9. S. Smale, *On the structure of manifolds.* Amer. J. Math. 84 (1962), 387-399.

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