KÄHLER MANIFOLDS OF CONSTANT NULLITY

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1. INTRODUCTION

The purpose of this paper is to derive curvature conditions that guarantee the existence of a product structure for Kähler manifolds of constant nullity and for Kähler-immersed manifolds of constant relative nullity.

In the intrinsic-manifold case, we shall obtain the following results.

THEOREM A. *Let* M^n *be a complete, connected, and simply connected* C^∞ *Kähler manifold of constant positive nullity* μ, *and suppose that one of the following conditions is satisfied.*

(A1) $\mu = 2$.

(A2) *The restriction of the complex curvature tensor to the space of symmetric bivectors generated by vectors orthogonal to the space of nullity vectors at each point is a positive- or negative-definite Hermitian form on this space.*

Then M^n *is a metric product,* $M^n = C^\mu \times M^{n-\mu}$, *where* C^μ *is complete and flat, and* $M^{n-\mu}$ *is complete. Moreover,* C^μ *and* $M^{n-\mu}$ *are Kähler manifolds.*

COROLLARY. *The conclusion of Theorem* A *continues to hold if Condition* (A2) *is replaced by the condition that the curvatures of all holomorphic sections generated by vectors orthogonal to the space of nullity vectors at each point are* 1/2*-pinched.*

A Kähler immersion $\psi\colon M^d \to C^{d+k}$ is called n-*cylindrical* if $M^d = M^{d-n} \times C^n$ and $\psi = \overline{\psi} \times 1$, where 1 is the identity on C^n, ψ is a Kähler immersion of M^{d-n} into C^{d+k-n}, and C^m denotes complex m-space.

THEOREM B. *Let* M^d *be a complete, connected, and simply connected* C^∞ *Kähler manifold, Kähler-immersed in* C^{d+k} *with constant relative nullity* ν. *Then* ψ *is* ν*-cylindrical if one of the following conditions holds.*

(B1) $\nu = d - 2$.

(B2) *The curvatures of all holomorphic sections orthogonal to the nullity spaces are strictly negative.*

Nullity was defined by Chern and Kuiper [4]. Theorems A and B are Kähler analogues of similar theorems for Riemannian manifolds [5], [7]. I would like to thank Professor Yeaton H. Clifton for introducing me to the calculus of Kähler manifolds according to E. Cartan. Section (2) of this paper is a translation of this calculus into invariant language.

2. THE COMPLEX CURVATURE TENSOR

Let M be a Kähler manifold with almost-complex structure J [3]. We shall denote the tangent space to M at m by M_m, the Riemannian metric of M by $\langle\ ,\ \rangle$, the curvature transformation associated with the vectors x and y by R(x, y), the

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sectional curvature function by $K(\ ,\ )$, and the holomorphic curvature function by $H(\ )$, where $H(x) = K(x, Jx)$. Then, for any $x, y, z \in M_m$, we have the relations

$$R(x, y)\,Jz = JR(x, y)z\,, \tag{2.1}$$

$$R(Jx, Jy)z = R(x, y)z\,, \tag{2.2}$$

$$K(Jx, Jy) = K(x, y)\,, \tag{2.3}$$

$$\langle R(x, Jx)y, Jy \rangle = K(x, y)\,\|x \wedge y\|^2 + K(x, Jy)\,\|x \wedge Jy\|^2\,. \tag{2.4}$$

A Kähler manifold carries a Hermitian metric defined by the formula $\langle x, y\rangle_C = \langle x, y\rangle + i\langle x, Jy\rangle$, where i denotes $\sqrt{-1}$. In addition to being real-linear, $\langle\ ,\ \rangle_C$ satisfies the relations

$$\langle x, x\rangle_C = \langle x, x\rangle, \tag{2.5}$$

$$\langle x, y\rangle_C = \overline{\langle y, x\rangle_C}\,, \tag{2.6}$$

$$\langle Jx, y\rangle_C = i\langle x, y\rangle_C\,. \tag{2.7}$$

Definition 2.1. Let L be a real-linear operator on M_m. L is said to be *complex-linear* (respectively, *sesquilinear*), if $L(Jx) = JL(x)$ (respectively, $L(Jx) = -JL(x)$) for all $x \in M_m$.

Remarks. A similar definition holds for multilinear operators. It is easy to show that any real-linear operator on a space with an almost-complex structure can be written uniquely as the sum of a complex-linear operator and a sesquilinear operator.

Definition 2.2. For any $u, x \in M_m$, the *complex curvature transformation* $Q(u, x)$ associated with u and x is the real-linear operator on M_m defined by the relation

$$Q(u, x)y = R(u, x)y + R(Ju, x)Jy \qquad (y \in M_m)\,.$$

THEOREM 2.3. *For all* $u, x, y \in M_m$, *the complex curvature transformations have the properties*

$$Q(u, x)y = Q(u, y)x\,, \tag{2.8}$$

$$Q(u, x)Jy = JQ(u, x)y\,, \tag{2.9}$$

$$Q(u, Jx)y = JQ(u, x)y\,, \tag{2.10}$$

$$Q(Ju, x)y = -JQ(u, x)y\,, \tag{2.11}$$

$$2R(u, x)y = Q(u, x)y - Q(x, u)y\,. \tag{2.12}$$

Proof. These properties follow directly from the first Bianchi identity, the Kähler identities ((2.1) and (2.2)), and the fact that $R(u, x) = -R(x, u)$.

Definition 2.4. For $x, y, u, v \in M_m$, let $S(u, x, y, v) = \langle Q(u, x)y, v\rangle_C$.

THEOREM 2.5. *For* $x, y, u, v \in M_m$, S *has the properties*

$$S(u, x, y, v) = \overline{S(x, u, v, y)}, \tag{2.13}$$

$$S(u, Jx, y, v) = S(u, x, Jy, v) = iS(u, x, y, v), \tag{2.14}$$

$$S(Ju, x, y, v) = S(u, x, y, Jv) = -iS(u, x, y, v), \tag{2.15}$$

$$S(u, x, y, v) = S(u, y, x, v), \tag{2.16}$$

$$S(u, x, y, v) = S(v, x, y, u), \tag{2.17}$$

$$S(x, x, y, y) = \langle R(x, Jx)y, Jy \rangle, \tag{2.18}$$

$$S(x, x, x, x) = H(x)\, \|x\|^4 . \tag{2.19}$$

Proof. (2.13) follows from the symmetries (2.1) and (2.2) of the Riemannian curvature tensor [2]. (2.14) follows from (2.7), (2.9), and (2.10). (2.13) and (2.14) lead to (2.15). (2.16) follows from (2.8). (2.13) and (2.16) give (2.17). (2.18) and (2.19) follow from the symmetries of the Riemannian curvature tensor.

Remark. Just as the Riemannian curvature tensor may be regarded as a symmetric bilinear form on bivectors [2], S may be viewed as a Hermitian form on symmetric bivectors. If $a = x \oplus y$ and $b = u \oplus v$ are separable symmetric bivectors, we can set $H(a, b) = S(u, x, y, v)$. Then (2.16) and (2.17) show that $H(a, b)$ is well-defined; (2.14) shows that H is complex-linear in its first argument; and (2.13) shows that H is Hermitian with respect to permutations in its arguments. (2.19) shows that S is closely related to the holomorphic curvature. For these reasons, S is called the *complex-curvature tensor*. In fact, S is the invariant formulation for E. Cartan's complex-valued curvature form Ω.

3. THE CONULLITY OPERATORS

Hereafter, we shall assume that M is a Kähler manifold of constant positive nullity μ. Let $\mathscr{N}_m$ and $\mathscr{C}_m$ denote the nullity and conullity space at m, respectively [7]. If x is tangent to M, P(x) will denote the conullity component of x.

PROPOSITION 3.1. *Let* $x \in \mathscr{C}_m$, *and suppose that* U *and* V *are differentiable nullity vector fields on neighborhoods of* m *and that they agree at* m. *Then* $P(\nabla_x U) = P(\nabla_x V)$.

Proof. Let F be a frame field on a neighborhood of m whose first μ vector fields are nullity vector fields. Then, on the intersection of the domains of U, V, and F, we can write $U = u^\alpha F_\alpha$ and $V = v^\alpha F_\alpha$, where the repeated index indicates summation from 1 to μ, and where $u^\alpha(m) = v^\alpha(m)$ for each α. Thus,

$$\nabla_x U = x(u^\alpha)F_\alpha + u^\alpha \nabla_x F_\alpha ,$$

$$\nabla_x V = x(v^\alpha)F_\alpha + v^\alpha \nabla_x F_\alpha , \qquad \text{and}$$

$$P(\nabla_x U) = u^\alpha(m)\, P(\nabla_x F_\alpha) = v^\alpha(m)\, P(\nabla_x F_\alpha) = P(\nabla_x V).$$

Definition 3.2. Let $u \in \mathscr{N}_m$. The *conullity operator* T_u associated with u is the linear operator on $\mathscr{C}_m$ defined by the equation $T_u(x) = P(\nabla_x U)$, where $x \in \mathscr{C}_m$ and U is any nullity field in a neighborhood of m such that $U(m) = u$.

Remarks. $T_u(x)$ is well-defined, by Proposition 3.1, and T_u is linear because $\nabla_{ax} = a\nabla_x$. Moreover, T_u is complex-linear in u, because of the Kähler identity $\nabla_x J = J\nabla_x$.

THEOREM 3.3. *Let* T *be a conullity operator at* m. *Then, for all* x, y, z $\in \mathscr{C}_m$,

$$\mathfrak{S}\{Q(x, y)(T(z)) - Q(y, x)(T(z))\} = 0, \tag{3.1}$$

where $\mathfrak{S}$ *denotes the cyclic sum over* x, y, *and* z.

Proof. Suppose that T is the operator associated with $u \in \mathscr{N}_m$. By (2.12) and the second Bianchi identity,

$$\mathfrak{S}\{(\nabla_z Q)(x, y)(u) - (\nabla_z Q)(y, x)(u)\} = 0. \tag{3.2}$$

Let X, Y, and U be differentiable extensions of x, y, and u to a neighborhood of m, where U is a nullity field. By definition of $\nabla_z Q$, we have the relation

$$(\nabla_z Q)(X, Y)(U) = \nabla_z(Q(X, Y)(U)) - Q(\nabla_z X, Y)(U) - Q(X, \nabla_z Y)(U) - Q(X, Y)(\nabla_z U).$$

The right-hand side of this equation reduces to $-Q(X, Y)(P\nabla_z U)$, because U is a nullity field, and when we evaluate the resulting equation at m, we obtain the relation $(\nabla_z Q)(x, y)(u) = -Q(x, y)(T(z))$. The theorem is established if we substitute this value of $(\nabla_z Q)(x, y)(u)$ into (3.2).

Definition 3.4. Let L and N denote the linear and sesquilinear parts of T, respectively.

THEOREM 3.5. *For any* x, y, z $\in \mathscr{C}_m$, L, N, *and* Q *satisfy the relations*

$$Q(x, y)(L(z)) = Q(x, z)(L(y)), \tag{3.3}$$

$$Q(x, y)(N(z)) = Q(z, y)(N(x)). \tag{3.4}$$

Proof. When we write T = L + N, (3.1) becomes

$$\mathfrak{S}\{Q(x, y)(L(z)) - Q(y, x)(L(z)) + Q(x, y)(N(z)) - Q(y, x)(N(z))\} = 0. \tag{3.5}$$

When x, y, and z are replaced by Jx, Jy, and Jz, respectively, (3.5) becomes

$$\mathfrak{S}\{Q(x, y)(L(z)) - Q(y, x)(L(z)) - [Q(x, y)(N(z)) - Q(y, x)(N(z))]\} = 0. \tag{3.6}$$

Equations (3.5) and (3.6) lead to the relations

$$\mathfrak{S}\{Q(x, y)(L(z)) - Q(y, x)(L(z))\} = 0, \tag{3.7}$$

$$\mathfrak{S}\{Q(x, y)(N(z)) - Q(y, x)(N(z))\} = 0. \tag{3.8}$$

We obtain (3.3) when we replace x by Jx in (3.7) and add the resulting equation to (3.7). Similarly, (3.4) is obtained when we replace y by Jy in (3.8) and add the resulting equation to (3.8).

COROLLARY 3.6. *For any* v, x, y, z $\in \mathscr{C}_m$, L, N, *and* S *satisfy the relations*

$$S(x, L(y), z, v) = S(x, L(z), y, v), \tag{3.9}$$

$$S(x, y, N(z), v) = S(z, y, N(x), v), \tag{3.10}$$

$$S(x, y, N^2(z), v) = S(N^2(x), y, z, v). \tag{3.11}$$

Proof. (3.9) and (3.10) follow from (2.16) and the previous theorem. (3.11) follows from (3.10), (2.13), (2.16), and (2.17).

4. THE PRODUCT STRUCTURE

To prove Theorem A, it is sufficient to show that each conullity operator vanishes [7]. With the exception of the following theorem, information about a conullity operator T will come from the properties of its linear and sesquilinear parts L and N.

THEOREM 4.1. *Let* M *be a complete Kähler manifold of constant positive nullity, and let* T_u *be a conullity operator. Then the eigenvalues of* T_u *are zero.*

Proof. Let $T_u(x) = (a + bi)x = ax + bJx$. Then $J(T_u(x)) = -bx + aJx$, and by the remarks following Definition 3.2, $T_v(x) = (a^2 + b^2)x$, where $v = au - bJu$. However, the real eigenvalue $a^2 + b^2$ of T_v must vanish, by Theorem 3.1 of [7]. Thus, $a = b = 0$.

THEOREM 4.2. (a) *If* $H(x) \neq 0$ *for all nonzero* $x \in \mathscr{C}_m$, *then the eigenspaces of* L *are simple.*

(b) *If* $S(x, x, y, y) \neq 0$ *for all nonzero* $x, y \in \mathscr{C}_m$, *then* L *is a dilation.*

Proof. (a) Suppose that L has a multiple eigenspace with the eigenvalue α. Then there exist complex independent vectors x and y with $L(y) = \alpha y + x$ and $L(x) = \alpha x$. By (2.16) and (3.9),

$$\begin{aligned} H(x)\|x\|^4 &= S(x, x, x, x) = S(x, \alpha y - L(y), x, x) = \alpha S(x, y, x, x) - S(x, L(x), y, x) \\ &= \alpha\{S(x, y, x, x) - S(x, x, y, x)\} = 0. \end{aligned}$$

(b) By part (a), it is sufficient to show that the eigenvalues of L are equal. If $L(x) = \alpha x$ and $L(y) = \beta y$, then, by (2.14), (2.16), and (3.9),

$$\alpha S(x, x, y, y) = S(x, L(x), y, y) = S(x, x, L(y), y) = \beta S(x, x, y, y).$$

To get information about N, we first look at the complex-linear operator N^2.

THEOREM 4.3. (a) *If* $H(x) \neq 0$ *for all nonzero* $x \in \mathscr{C}_m$, *the eigenvalues of* N^2 *are real.*

(b) *If* $S(x, x, y, y) \neq 0$ *for all nonzero* $x, y \in \mathscr{C}_m$, *the eigenvalues of* N^2 *are nonnegative.*

Proof. (a) Let $N^2(x) = \alpha x$. Then, by (2.14), (2.15), and (3.11),

$$\alpha S(x, x, x, x) = S(x, x, \alpha x, x) = S(x, x, N^2(x), x) = S(N^2(x), x, x, x) = \overline{\alpha} S(x, x, x, x).$$

Thus, $(\alpha - \overline{\alpha})H(x) = 0$.

(b) Let $N^2(x) = rx$, with r real, and let $y = N(x)$. By part (a), (2.17), and (3.11),

$$S(x, x, y, y) = S(x, x, N(x), N(x)) = S(N(x), x, N(x), x) = S(x, x, N^2(x), x) = rH(x)\|x\|^4.$$

By a continuity argument, S(x, x, y, y) and H(x) have the same sign, and r is nonnegative.

THEOREM 4.4. *If* S(x, x, y, y) $\neq 0$ *for all nonzero* x, y $\in \mathscr{C}_m$, *then*

(a) N *has an eigenvector, and*

(b) N = 0 *if* N(x) = 0 *for one nonzero* x.

Proof. (a) Since N^2 is complex-linear, N^2 has an eigenvector x. By Theorem 4.3 (b), we can write $N^2(x) = r^2 x$, where r is real. If r = 0, either the vector x or the vector N(x) is an eigenvector of N. If $r \neq 0$, let y = rx + N(x). Then either y = 0, in which case x is an eigenvector of N with eigenvalue -r, or y is an eigenvector of N with eigenvalue r.

(b) Let $y \in \mathscr{C}_m$. Then by (3.10), S(x, x, N(y), N(y)) = S(N(x), x, y, N(y)) = 0. Thus, N(y) = 0.

Proof of Theorem A. Let T = L + N be a conullity operator at m. First suppose that condition (A1) holds. Then every nonzero conullity vector x at m is necessarily an eigenvector of T, because $\mathscr{C}_m$ is spanned by x and Jx. Thus, T = 0, by Theorem 4.1.

If condition (A2) holds, then L is a dilation, by Theorem 4.2 (b), and N has an eigenvector x, by Theorem 4.4 (a). Thus we can write $L(x) = \alpha x$ and $N(x) = \beta x$. Since $T(x) = (\alpha + \beta)x$ and $T(Jx) = (\alpha - \beta)Jx$, Theorem 4.1 implies that $\alpha + \beta = \alpha - \beta = 0$. Simple algebra gives $\alpha = \beta = 0$, so that L and N have zero for an eigenvalue. Thus L = 0 by Theorem 4.2 (b), and N = 0 by Theorem 4.4 (b).

Example. The purpose of this example is to show that the curvature conditions of Theorem A are different from the curvature conditions of Theorem (*) of [7]. Let x, Jx, y, and Jy be orthonormal vectors, and let $0 < \lambda < 1$. Set

$$R(x, Jx, x, Jx) = R(y, Jy, y, Jy) = 1, \qquad R(x, y, x, y) = 0,$$

$$R(x, Jy, x, Jy) = R(x, Jx, y, Jy) = \lambda.$$

Determine the remaining curvature components by the symmetries of the Riemannian curvature tensor and the Kähler identities. Then lengthy but straightforward calculations show that S(u, v, u, v) > 0 for all nonzero u and v. However, R is neither a positive- nor a negative-definite bilinear form on bivectors, because R(x, y, x, y) = 0.

Corollary to Theorem A. We shall assume that the holomorphic curvatures are positive; the negative case is similar. We shall show that *the pinching conditions in the statement of the corollary lead to the conclusion that* S(x, x, y, y) > 0 *for all* x, y $\in \mathscr{C}_m$. To simplify the notation, we write s(x, y) = S(x, x, y, y), and we let $\sigma(x)$ denote the holomorphic section determined by x.

We can assume that x and y are unit vectors. If $y \in \sigma(x)$, we can write $y = \alpha x$, where α is complex and $|\alpha| = 1$. By (2.14), (2.15), and (2.19),

$$s(x, y) = H(x) > \frac{1}{2} > 0.$$

If x, y, Jx, and Jy are orthonormal, the result follows from (2.4), (2.18), and (4.5) of [1]. The only remaining case occurs when $y \notin \sigma(x)$, but y has a component z in $\sigma(x)$. In this case, $z = \beta x$, where β is complex, and $s(z, y) = |\beta|^2 s(x, y)$. Thus, it suffices to show that s(z, y) > 0. Let $u = z/\|z\|$, and let v = -Ju, so that y and v are orthonormal. We can write $\langle y, Jv \rangle = \sin\theta$ and apply (4.2) and (4.3) of [1]. Then

$$s(v, y) = \frac{1}{4}\left(4 \cdot \frac{1}{2} - 2 + 2\sin^2\theta\right) > 0 \quad \text{and}$$

$$s(z, y) = s(\|z\| u, y) = \|z\|^2 s(v, y) > 0.$$

5. KÄHLER IMMERSIONS OF CONSTANT RELATIVE NULLITY

Let J and $\bar{J}$ denote the almost-complex structures belonging to the Kähler manifolds M and $\bar{M}$, respectively. We say that $\psi: M \to \bar{M}$ is a *Kähler immersion* of M into $\bar{M}$ if ψ is an isometric immersion, and if moreover $d\psi(Jx) = \bar{J}(d\psi(x))$ for all vectors x tangent to M. If M is Kähler-immersed in $\bar{M}$, we may view M as contained in $\bar{M}$, and we may regard J as the restriction of $\bar{J}$ to M. We shall denote both by J.

LEMMA 5.1. *Let* M *be Kähler-immersed in* $\bar{M}$, *and let* D_x *be a difference operator of the immersion. Then* D_x *is a complex-linear operator.*

Proof. The Kähler condition for M and $\bar{M}$ states that ∇_x and $\bar{\nabla}_x$ commute with J. Since $D_x = \bar{\nabla}_x - \nabla_x$, D_x also commutes with J.

Definition 5.2. A vector $y \in M_m$ is called a *relative nullity vector* if $D_x y = 0$ for all $x \in M_m$. The space of nullity vectors at m will be denoted by $\mathcal{N}_m$, and the subspace of M_m orthogonal to $\mathcal{N}_m$ will be denoted by $\mathscr{C}_m$.

Definition 5.3. Let $u \in \mathcal{N}_m$. The *relative conullity operator* T_u associated with u is the linear operator on $\mathscr{C}_m$ defined by the equation $T_u(x) = P(\nabla_x U)$, where P is the projection into $\mathscr{C}_m$, and where U is any relative nullity extension of u.

Remarks. Hereafter, we assume that $\bar{M}$ is complete and flat. If the relative nullity is constant, the distribution of relative nullity spaces is involutive, and the resulting integral manifolds are flat, totally geodesic, and complete. To establish Theorem B, it suffices to show that the relative conullity operators vanish.

LEMMA 5.4. *Let* T *be a relative conullity operator at* m. *Then, for all* $x, y \in \mathscr{C}_m$,

$$D_x(T(y)) = D_y(T(x)). \tag{5.1}$$

Proof. Let T be the conullity operator associated with u. Let X, Y, and U be extensions of x, y, and u to a neighborhood of m, where U is a nullity field. By the Codazzi-Mainardi equation [6],

$$\mathrm{Nor}\,(\bar{R}_{XY} U) = \mathrm{Nor}\,\{\bar{\nabla}_Y(D_X U) - \bar{\nabla}_X(D_Y U)\} + D_{[X,Y]}U + D_Y(\nabla_X U) - D_X(\nabla_Y U).$$

Since U is relatively null and $\bar{M}$ is flat, this equation reduces to $D_Y(P\nabla_X U) = D_X(P\nabla_Y U)$. At the point m, this equation reduces to (5.1).

LEMMA 5.5. *Let* T *be a relative conullity operator, and let* L *and* N *denote the linear and sesquilinear parts of* T. *Then, for all* $x, y \in \mathscr{C}_m$,

$$D_x(L(y)) = D_y(L(x)), \tag{5.2}$$

$$D_x(N(y)) = D_y(N(x)). \tag{5.3}$$

Proof. By (5.1), $D_x(L(y)) + D_x(N(y)) = D_y(L(x)) + D_y(N(x))$. This equation holds for all $x, y \in \mathscr{C}_m$. Hence, if we replace y by Jy and use the linearity properties of D_x, L, and N, we obtain the equation $D_x(L(y)) - D_x(N(y)) = D_y(L(x)) - D_y(N(x))$.

LEMMA 5.6. (a) $D_x(N(y)) = 0$ *for all* $x, y \in \mathscr{C}_m$.

(b) *If* $H(x) < 0$ *for all* $x \in \mathscr{C}_m$, *then* $N = 0$.

Proof. (a) The operator $G(x, y) = D_x(N(y))$ is sesquilinear in y, because N is sesquilinear and D_x is complex-linear. G is complex-linear in x because $D_x z$ is symmetric in x and z, and D_x is complex-linear. G is symmetric in x and y, by (5.3). Hence $G = 0$.

(b) The curvature equation of Gauss states that

$$K(x, y) = \overline{K}(x, y) + \frac{\langle D_x x, D_y y\rangle - \|D_x y\|^2}{\|x \wedge y\|^2}.$$

When $y = Jx$ and $\overline{M}$ is flat, this equation reduces to

$$H(x) = \frac{-2\,\|D_x x\|^2}{\|x \wedge Jx\|^2}.$$

If $x = N(y) \neq 0$, then $D_x x = 0$, by part (a), and therefore $H(x) = 0$.

LEMMA 5.7. *If* $H(x) < 0$ *for all* $x \in \mathscr{C}_m$, *the eigenspaces of* L *are simple.*

Proof. Suppose that L has a multiple eigenspace with eigenvalue α. Then there exist vectors x and y with $L(y) = \alpha y + x$ and $L(x) = \alpha x$. By (5.2), $D_x x = \alpha D_y x - \alpha D_x y = 0$.

Proof of Theorem B. If $H(x) < 0$ for all $x \in \mathscr{C}_m$, Lemmas 5.7 and 5.6 (b) show that $T = L + N$ is a dilation. However, the eigenvalues of T must be zero when M is complete [5]. Thus $T = 0$. If condition (B1) holds, then every vector is an eigenvector of T, so that each operator T again vanishes.

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