LIFTING COMMUTING OPERATORS

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1. INTRODUCTION

All Hilbert spaces considered in this paper are assumed to be complex, and all operators under consideration are assumed to be bounded and linear. The algebra of all (bounded, linear) operators on a Hilbert space $\mathscr{H}$ will be denoted by $\mathscr{L}(\mathscr{H})$, and if $S \in \mathscr{L}(\mathscr{H})$ and $\mathscr{M} \subset \mathscr{H}$ is an invariant subspace for S, then the operator in $\mathscr{L}(\mathscr{M})$ obtained by restricting S to $\mathscr{M}$ will be denoted by $S \mid \mathscr{M}$.

It is known [5] that if T is a contraction (that is, $\|T\| \leq 1$) acting on a Hilbert space $\mathscr{H}$, then there exists a unique minimal co-isometry V acting on a Hilbert space $\mathscr{K} \supset \mathscr{H}$ such that $T = V \mid \mathscr{H}$. Sarason showed in [4] that in case V^* is the unilateral shift of multiplicity one, then for every X in $\mathscr{L}(\mathscr{H})$ commuting with T, there exists a Y in $\mathscr{L}(\mathscr{K})$ such that $YV = VY$, $Y\mathscr{H} \subset \mathscr{H}$, $Y \mid \mathscr{H} = X$, and $\|Y\| = \|X\|$. Sarason's proof makes use of duality techniques and other properties of H^p-spaces. More recently, Sz.-Nagy and Foiaş generalized Sarason's result to the case that T is an arbitrary contraction [7, Theorem 2]. Their proof is based on the structure theory for contractions as set forth in [5].

In this paper we give an alternate and somewhat simpler proof of the above-mentioned lifting theorem of Sz.-Nagy and Foiaş. Our proof is matricial in character, and it employs an interesting generalization of a result of Douglas [1]. We then use [7, Theorem 2] to characterize the commutant of a contraction in terms of the commutant of its minimal strong unitary dilation.

Actually, in [7] Sz.-Nagy and Foiaş treat the more general case of two contractions T_1 and T_2 and an intertwining operator X satisfying the equation $XT_1 = T_2 X$. However, we show in Section 5 that this more general result can easily be derived from the special case $T_1 = T_2$ by the use of a matricial device.

2. ON OPERATOR EQUATIONS

The basic tool for our attack on the lifting theorem is a generalization of the following lemma, which is contained in [1, Theorem 1].

LEMMA 2.1. *Suppose that* $\mathscr{G}$, $\mathscr{H}$, *and* $\mathscr{K}$ *are Hilbert spaces, that* A *is an operator mapping* $\mathscr{G}$ *into* $\mathscr{K}$, *and that* B *is an operator mapping* $\mathscr{H}$ *into* $\mathscr{K}$. *Then there exists a contraction* Z *mapping* $\mathscr{G}$ *into* $\mathscr{H}$ *and satisfying* $A = BZ$ *if and only if* $AA^* \leq BB^*$.

The following generalization is of interest in its own right.

THEOREM 1. *Let* $\mathscr{H}_0$, $\mathscr{H}_1$, $\mathscr{H}_2$, *and* $\mathscr{K}$ *be Hilbert spaces, and for* $0 \leq i \leq 2$ *let* A_i *be an operator mapping* $\mathscr{H}_i$ *into* $\mathscr{K}$. *Then there exist operators* Z_1 *and* Z_2 *that map* $\mathscr{H}_0$ *into* $\mathscr{H}_1$ *and* $\mathscr{H}_2$, *respectively, and that satisfy the two conditions*

Received June 16, 1968.
This Research was partially supported by the National Science Foundation. R. G. Douglas and Carl Pearcy are Alfred P. Sloan Research Fellows.

$$A_1 Z_1 + A_2 Z_2 = A_0, \tag{1}$$

$$Z_1^* Z_1 + Z_2^* Z_2 \leq 1_{\mathscr{H}_0} \tag{2}$$

if and only if

$$A_1 A_1^* + A_2 A_2^* \geq A_0 A_0^*. \tag{3}$$

Proof. Consider the operator from $\mathscr{H}_1 \oplus \mathscr{H}_2$ to $\mathscr{K} \oplus \mathscr{K}$ defined by the 2×2 matrix $\begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}$ and the operator from $\mathscr{H}_0 \oplus \mathscr{H}_0$ to $\mathscr{K} \oplus \mathscr{K}$ defined by $\begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$. If there exist operators $Z_1: \mathscr{H}_0 \to \mathscr{H}_1$ and $Z_2: \mathscr{H}_0 \to \mathscr{H}_2$ satisfying (1) and (2), then clearly

$$\begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \end{pmatrix}^*\begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \end{pmatrix} \leq \begin{pmatrix} 1_{\mathscr{H}_0} & 0 \\ 0 & 1_{\mathscr{H}_0} \end{pmatrix},$$

where the matrix $\begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \end{pmatrix}$ denotes the obvious operator from $\mathscr{H}_0 \oplus \mathscr{H}_0$ to $\mathscr{H}_1 \oplus \mathscr{H}_2$. The inequality (2) implies that $\begin{pmatrix} Z_1 & 0 \\ Z_2 & 0 \end{pmatrix}$ is a contraction, and hence by Lemma 2.1 we have the inequality

$$\begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}^* \geq \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}^*, \tag{4}$$

which immediately yields (3).

On the other hand, (3) implies (4), and thus by Lemma 2.1 there exists a contraction $\begin{pmatrix} Z_1 & C_1 \\ Z_2 & C_2 \end{pmatrix}$ from $\mathscr{H}_0 \oplus \mathscr{H}_0$ into $\mathscr{H}_1 \oplus \mathscr{H}_2$ satisfying the equation

$$\begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Z_1 & C_1 \\ Z_2 & C_2 \end{pmatrix} = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix};$$

this implies that $A_1 Z_1 + A_2 Z_2 = A_0$. Moreover, since $\begin{pmatrix} Z_1 & C_1 \\ Z_2 & C_2 \end{pmatrix}$ is a contraction,

$$\begin{pmatrix} Z_1 & C_1 \\ Z_2 & C_2 \end{pmatrix}^*\begin{pmatrix} Z_1 & C_1 \\ Z_2 & C_2 \end{pmatrix} \leq \begin{pmatrix} 1_{\mathscr{H}_0} & 0 \\ 0 & 1_{\mathscr{H}_0} \end{pmatrix},$$

and it follows immediately that $Z_1^* Z_1 + Z_2^* Z_2 \leq 1_{\mathscr{H}_0}$.

Similarly, one can prove the following theorem by using $n \times n$ matrices.

THEOREM 2. *Let $\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_n$, and $\mathcal{K}$ be Hilbert spaces, and for $0 \leq i \leq n$, let A_i be an operator mapping $\mathcal{H}_i$ into $\mathcal{K}$. A necessary and sufficient condition that there exist operators $Z_1, Z_2, \cdots, Z_n$ ($Z_i: \mathcal{H}_0 \to \mathcal{H}_i$) satisfying*

$$\sum_{i=1}^{n} A_i Z_i = A_0 \quad \textit{and} \quad \sum_{i=1}^{n} Z_i^* Z_i \leq 1_{\mathcal{H}_0}$$

is that $\sum_{i=1}^n A_i A_i^* \geq A_0 A_0^*$.

The next result, which we shall use in the proof of the main theorem, is due to Sz.-Nagy and Foiaş [6]. Here we present a somewhat simpler proof based on Lemma 2.1.

PROPOSITION 2.2. *For $i = 1, 2$, let T_i be a contraction on a Hilbert space $\mathcal{H}_i$, and let X be an operator mapping $\mathcal{H}_2$ into $\mathcal{H}_1$. A necessary and sufficient condition that the operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$ defined by the matrix $\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$ be a contraction is that there exist a contraction C mapping $\mathcal{H}_2$ into $\mathcal{H}_1$ such that*

$$X = \sqrt{1_{\mathcal{H}_1} - T_1 T_1^*}\, C \sqrt{1_{\mathcal{H}_2} - T_2^* T_2}\,.$$

Proof. The operator defined on $\mathcal{H}_1 \oplus \mathcal{H}_2$ by the matrix $\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$ is a contraction if and only if

$$\begin{pmatrix} T_1 T_1^* & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & X \\ 0 & T_2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ X^* & T_2^* \end{pmatrix} = \begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}\begin{pmatrix} T_1^* & 0 \\ X^* & T_2^* \end{pmatrix} \leq \begin{pmatrix} 1_{\mathcal{H}_1} & 0 \\ 0 & 1_{\mathcal{H}_2} \end{pmatrix},$$

or equivalently, if and only if

$$\begin{pmatrix} 0 & X \\ 0 & T_2 \end{pmatrix}\begin{pmatrix} 0 & X \\ 0 & T_2 \end{pmatrix}^* \leq \begin{pmatrix} 1_{\mathcal{H}_1} - T_1 T_1^* & 0 \\ 0 & 1_{\mathcal{H}_2} \end{pmatrix}. \tag{5}$$

By Lemma 2.1, the last inequality is equivalent to the existence of a contraction

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

acting on $\mathcal{H}_1 \oplus \mathcal{H}_2$ and satisfying

$$\begin{pmatrix} 0 & X \\ 0 & T_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1_{\mathcal{H}_1} - T_1 T_1^*} & 0 \\ 0 & 1_{\mathcal{H}_2} \end{pmatrix}\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}. \tag{6}$$

If (6) holds, then clearly $K_{22} = T_2$ and $K_{21} = 0$, and it is therefore not difficult to see that (5) is equivalent to the existence of a contraction $M = \begin{pmatrix} 0 & K_{12} \\ 0 & T_2 \end{pmatrix}$ in $\mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $\sqrt{1_{\mathcal{H}_1} - T_1 T_1^*}\, K_{12} = X$.

If M is such a contraction, then $M^* M \le 1_{\mathscr{H}_1 \oplus \mathscr{H}_2}$, from which it follows that $K_{12}^* K_{12} \le 1_{\mathscr{H}_2} - T_2^* T_2$. Thus by Lemma 2.1 there exists a contraction C mapping $\mathscr{H}_1$ into $\mathscr{H}_2$ and satisfying $K_{12} = C\sqrt{1_{\mathscr{H}_2} - T_2^* T_2}$; hence

$$X = \sqrt{1_{\mathscr{H}_1} - T_1 T_1^*}\, C \sqrt{1_{\mathscr{H}_2} - T_2^* T_2},$$

as desired. On the other hand, if $X = \sqrt{1_{\mathscr{H}_1} - T_1 T_1^*}\, C \sqrt{1_{\mathscr{H}_2} - T_2^* T_2}$, where C is a contraction that maps $\mathscr{H}_1$ into $\mathscr{H}_2$, then, setting $K_{12} = C\sqrt{1_{\mathscr{H}_2} - T_2^* T_2}$, we have by Lemma 2.1 the inequality $K_{12}^* K_{12} \le 1_{\mathscr{H}_2} - T_2^* T_2$. This implies that $M^* M \le 1_{\mathscr{H}_1 \oplus \mathscr{H}_2}$, and thus that M is a contraction. The proof is complete.

3. THE COMMUTANT OF A PARTIAL ISOMETRY

It is easy to see that if Q is a partial isometry acting on a Hilbert space $\mathscr{K}$, then $\mathscr{K}$ can be decomposed as a direct sum $\mathscr{K} = \mathscr{H} \oplus \mathscr{G}$ in such a way that the 2×2 matrix for Q corresponding to this decomposition of $\mathscr{K}$ has the form $Q = \begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}$, where T is a contraction on $\mathscr{H}$, S maps $\mathscr{G}$ into $\mathscr{H}$, and $SS^* + TT^* = 1_{\mathscr{H}}$. (Take $\mathscr{H}$ to be the range of Q.) The following theorem relates the commutant of such a partial isometry Q to the commutant of the contraction T, and it plays the key role in our proof of the lifting theorem.

THEOREM 3. *Suppose that $\mathscr{G}$ and $\mathscr{H}$ are Hilbert spaces, that T is a contraction on $\mathscr{H}$, and that S is an operator mapping $\mathscr{G}$ into $\mathscr{H}$ such that $SS^* + TT^* = 1_{\mathscr{H}}$. Let Q be the partial isometry $Q = \begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}$ on $\mathscr{H} \oplus \mathscr{G}$, and let X be an operator on $\mathscr{H}$ that commutes with T. Then there exists an operator $Y = \begin{pmatrix} X & A \\ 0 & B \end{pmatrix}$ on $\mathscr{H} \oplus \mathscr{G}$ such that Y commutes with Q and $\|Y\| = \|X\|$.*

Proof. We may assume without loss of generality that $\|X\| = 1$. It follows from an elementary matrix calculation and Proposition 2.2 that to produce a matrix Y with the desired properties, we need only prove the existence of a contraction B in $\mathscr{L}(\mathscr{G})$ and an operator A mapping $\mathscr{G}$ into $\mathscr{H}$ such that

$$TA + SB = XS, \tag{7}$$

and such that

$$A = \sqrt{1_{\mathscr{H}} - XX^*}\, C \sqrt{1_{\mathscr{G}} - B^* B} \tag{8}$$

for some contraction C mapping $\mathscr{G}$ into $\mathscr{H}$. We first seek an A of the form $A = \sqrt{1_{\mathscr{H}} - XX^*}\, K$, where K is an operator from $\mathscr{G}$ into $\mathscr{H}$; thus (7) becomes

$$T\sqrt{1_{\mathscr{H}} - XX^*}\, K + SB = XS. \tag{9}$$

Since $1_{\mathscr{H}} \ge XX^*$ and $SS^* + TT^* = 1_{\mathscr{H}}$, we deduce that

$$\begin{aligned} T(1_{\mathscr{H}} - XX^*)T^* + SS^* &\ge TT^* - TXX^*T^* + SS^* - (1_{\mathscr{H}} - XX^*) \\ &= XX^* - XTT^*X^* = XSS^*X^*, \end{aligned}$$

and Theorem 1 implies the existence of operators K: $\mathscr{G} \to \mathscr{H}$ and B: $\mathscr{G} \to \mathscr{G}$ satisfying (9) and $K^*K + B^*B \leq 1_{\mathscr{G}}$. This latter inequality implies that B is a contraction and that $K^*K \leq 1_{\mathscr{G}} - B^*B$. An application of Lemma 2.1 yields a contraction C^* mapping $\mathscr{G}$ into $\mathscr{H}$ such that $K^* = \sqrt{1_{\mathscr{G}} - B^*B}\,C^*$. Thus $K = C\sqrt{1_{\mathscr{G}} - B^*B}$, so that (8) is satisfied and the proof is complete.

4. THE MAIN THEOREM

We begin this section by reminding the reader of certain relations between a contraction and its associated unique minimal co-isometric extension. It is proved in [5] that if T is a contraction on a Hilbert space $\mathscr{H}$, then there exists a co-isometry V acting on a Hilbert space $\mathscr{K} \supset \mathscr{H}$ such that $V\mathscr{H} \subset \mathscr{H}$ and $V \mid \mathscr{H} = T$. If the smallest reducing subspace for V that contains $\mathscr{H}$ is $\mathscr{K}$ itself, then V is said to be a *minimal co-isometric extension* of T, and such a V is unique up to isomorphism [5]. The minimal co-isometric extension of a contraction T on $\mathscr{H}$ may be realized matricially as follows. Let $\mathscr{D}_*$ be the Hilbert space that is the closure of the range of $1_{\mathscr{H}} - TT^*$, and denote by D_* the operator $\sqrt{1_{\mathscr{H}} - TT^*}$, regarded as an operator from $\mathscr{D}_*$ into $\mathscr{H}$. It follows (see [5, pp. 16-17]) that the unique minimal co-isometric extension V of T is given matricially as the operator

$$V = \begin{pmatrix} T & D_* & 0 & 0 & \cdots & & \\ & 0 & 1_{\mathscr{D}_*} & 0 & \cdots & & \\ & & 0 & 1_{\mathscr{D}_*} & & & \\ & & & 0 & \ddots & & \\ & \text{\Large 0} & & & \ddots & 1_{\mathscr{D}_*} & \\ & & & & & 0 & \ddots \\ & & & & & & \ddots \end{pmatrix} \tag{10}$$

acting on the Hilbert space $\mathscr{K} = \mathscr{H} \oplus \mathscr{D}_* \oplus \mathscr{D}_* \oplus \cdots$. Note that this matrix is essentially the lower right-hand corner of the Schäffer matrix for the strong unitary dilation of T [5].

With these preliminaries out of the way, we now offer our proof of the very pretty lifting theorem of Sz.-Nagy and Foias.

THEOREM 4. *Let* T *be a contraction on the Hilbert space* $\mathscr{H}$, *and let* V *be the unique minimal co-isometric extension of* T *acting on the Hilbert space* $\mathscr{K}$ *containing* $\mathscr{H}$. *For every operator* X *in* $\mathscr{L}(\mathscr{H})$ *that commutes with* T, *there exists an operator* Y *in* $\mathscr{L}(\mathscr{K})$ *such that*

$$YV = VY, \quad Y\mathscr{H} \subset \mathscr{H}, \quad Y \mid \mathscr{H} = X, \quad \|Y\| = \|X\|.$$

Proof. As noted above, V can be realized as the matrix (10) acting on the Hilbert space $\mathscr{K} = \mathscr{H} \oplus \mathscr{D}_* \oplus \mathscr{D}_* \oplus \cdots$. We begin the argument by defining an increasing sequence of subspaces of $\mathscr{K}$ as follows. Let

$$\mathscr{H}_0 = \mathscr{H} \oplus (0) \oplus (0) \oplus \cdots,$$

and for $n \geq 1$, let

$$\mathscr{H}_n = \mathscr{H} \oplus \underbrace{\mathscr{D}_* \oplus \cdots \oplus \mathscr{D}_*}_{n-1 \text{ copies}} \oplus (0) \oplus \cdots .$$

Observe that each $\mathscr{H}_n$ is an invariant subspace for V, and define $V_n = V \mid \mathscr{H}_n$. We can then regard the operator V_n as the $n \times n$ matrix

$$\begin{pmatrix} T & D_* & & & & \\ & 0 & 1_{\mathscr{D}_*} & & & \\ & & 0 & 1_{\mathscr{D}_*} & & \\ & & & 0 & \ddots & \\ & & & & \ddots & \ddots \\ & & & & & \ddots & 1_{\mathscr{D}_*} \\ & & & & & & 0 \end{pmatrix}$$

Observe that for $n \geq 0$, $\mathscr{H}_n$ is an invariant subspace for the operator $V_{n+1} \in \mathscr{L}(\mathscr{H}_{n+1})$. We may write $\mathscr{H}_{n+1} = \mathscr{H}_n \oplus \mathscr{D}_*$, and relative to this decomposition, the operator V_{n+1} can be written as a 2×2 matrix $V_{n+1} = \begin{pmatrix} V_n & S_n \\ 0 & 0 \end{pmatrix}$, where S_n is an operator mapping $\mathscr{D}_*$ into $\mathscr{H}_n$. An easy matricial calculation shows that for each $n \geq 0$, $V_n V_n^* + S_n S_n^* = 1_{\mathscr{H}_n}$. This implies that V_n is a partial isometry for $n \geq 1$ and that Theorem 3 is applicable. By induction, we define a sequence $\{Y_n\}_{n=0}^{\infty}$ of operators as follows. Let $Y_0 = X$, and apply Theorem 3 with V_1 in place of Q to obtain an operator $Y_1 \in \mathscr{L}(\mathscr{H}_1)$ such that $Y_1 V_1 = V_1 Y_1$, $Y_1 \mathscr{H}_0 \subset \mathscr{H}_0$, $Y_1 \mid \mathscr{H}_0 = Y_0$, and $\|Y_1\| = \|X\|$. Suppose now that for $1 \leq k \leq n$, we have defined an operator Y_k in $\mathscr{L}(\mathscr{H}_k)$ such that

$$(p_k) \quad Y_k V_k = V_k Y_k, \quad Y_k \mathscr{H}_{k-1} \subset \mathscr{H}_{k-1}, \quad Y_k \mid \mathscr{H}_{k-1} = Y_{k-1}, \quad \|Y_k\| = \|X\|.$$

Applying Theorem 3 with V_{n+1} in place of Q and Y_n in place of X, we obtain an operator Y_{n+1} in $\mathscr{L}(\mathscr{H}_{n+1})$ that satisfies (p_{n+1}). Thus, by induction, we obtain a sequence $\{Y_n\}_{n=1}^{\infty}$ such that for each n, the operator Y_n belongs to $\mathscr{L}(\mathscr{H}_n)$ and satisfies (p_n). We may consider each Y_n as an operator on $\mathscr{K}$ by defining $Y_n x = 0$ for x in $\mathscr{K} \ominus \mathscr{H}_n$, and it follows easily that for vectors x in the linear manifold $\bigcup_{n=1}^{\infty} \mathscr{H}_n$, which is dense in $\mathscr{K}$, the sequence $\{Y_n x\}$ is strongly Cauchy. Since $\|Y_n\| = \|X\|$ for all n, the sequence $\{Y_n\}$ must be strongly convergent—say to the operator Y in $\mathscr{L}(\mathscr{K})$. It is obvious from the properties of the sequence $\{Y_n\}$ that $Y\mathscr{H}_n \subset \mathscr{H}_n$ and $Y \mid \mathscr{H}_n = Y_n$ for each n. In particular, $Y \mid \mathscr{H}_0 = X$, as was to be proved. Furthermore, it is clear that $\|Y\| = \|X\|$, and since the sequence $\{V_n\}$ (regarded as a sequence of operators on $\mathscr{K}$ in the obvious way) converges strongly to V, we see that YV = VY. This completes the proof.

The following corollary of Theorem 4 removes the restriction of minimality on V.

COROLLARY 4.1. *Let* T *be a contraction on a Hilbert space* $\mathscr{H}$, *and let* V *be a co-isometric extension of* T *acting on the Hilbert space* $\mathscr{K}$ *containing* $\mathscr{H}$. *For every operator* X *in* $\mathscr{L}(\mathscr{H})$ *that commutes with* T, *there exists an operator* Y *in* $\mathscr{L}(\mathscr{K})$ *such that*

$$YV = VY, \quad Y\mathscr{H} \subset \mathscr{H}, \quad Y \mid \mathscr{H} = X, \quad \|Y\| = \|X\|.$$

Proof. Let $\mathscr{K}_1 \subset \mathscr{K}$ be the smallest reducing subspace for V that contains $\mathscr{H}$, and let $V_1 = V \mid \mathscr{K}_1$. Then V_1 is a minimal co-isometric extension of T, and by virtue of the uniqueness of such an extension, Theorem 4 is applicable. Thus there exists on operator Y_1 in $\mathscr{L}(\mathscr{K}_1)$ such that $Y_1 V_1 = V_1 Y_1$, $Y_1 \mathscr{H} \subset \mathscr{H}$, $Y_1 \mid \mathscr{H} = X$, and $\|Y_1\| = \|X\|$. Let Y be the operator in $\mathscr{L}(\mathscr{K})$ defined by setting $Yx = Y_1 x$ for $x \in \mathscr{K}_1$ and $Yx = 0$ for $x \in \mathscr{K} \ominus \mathscr{K}_1$. It follows easily that Y has the desired properties, and the proof is complete.

5. INTERTWINING OPERATORS

Theorem 4 takes a somewhat more general appearance if we consider two contractions T_1 and T_2 acting on Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and an operator X mapping $\mathscr{H}_1$ into $\mathscr{H}_2$ and satisfying $XT_1 = T_2X$. Such an operator X is called an *intertwining operator* between T_1 and T_2, and the problem is to extend X to an intertwining operator between the minimal co-isometric extensions of T_1 and T_2.

THEOREM 5 (Sz.-Nagy and Foiaş). *Suppose that for* i = 1, 2, T_i *is a contraction acting on a Hilbert space* $\mathscr{H}_i$, *and* V_i *is the unique minimal co-isometric extension of* T_i *acting on the Hilbert space* $\mathscr{K}_i \supset \mathscr{H}_i$. *Let* X *be an operator that maps* $\mathscr{H}_1$ *into* $\mathscr{H}_2$ *and satisfies the equation* $XT_1 = T_2X$. *Then there exists an operator* Y *mapping* $\mathscr{K}_1$ *into* $\mathscr{K}_2$ *such that*

$$YV_1 = V_2 Y, \quad Y\mathscr{H}_1 \subset \mathscr{H}_2, \quad Y \mid \mathscr{H}_1 = X, \quad \|Y\| = \|X\|.$$

Proof. A simple matricial calculation using the fact that $XT_1 = T_2X$ shows that the operators

$$\hat{T} = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \quad \text{and} \quad \hat{X} = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$$

on $\mathscr{H}_1 \oplus \mathscr{H}_2$ commute. It is easy to see that the minimal co-isometric extension of $\hat{T}$ is the operator $\hat{V} = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$ acting on the space $\mathscr{K}_1 \oplus \mathscr{K}_2$. An application of Theorem 4 yields the existence of an operator $\hat{Y} = \begin{pmatrix} Z_1 & Z_2 \\ Y & Z_3 \end{pmatrix}$ on $\mathscr{K}_1 \oplus \mathscr{K}_2$ such that $\hat{Y}\hat{V} = \hat{V}\hat{Y}$, $\hat{Y} \mid \mathscr{H} = \hat{X}$, and $\|\hat{Y}\| = \|\hat{X}\|$. It follows immediately that the operator Y has the desired properties, and the proof is complete.

6. THE MINIMAL UNITARY DILATION

If T is a contraction on a Hilbert space $\mathscr{H}$, and U is a unitary operator on a Hilbert space $\mathscr{K} \supset \mathscr{H}$ such that $T^n = P_{\mathscr{H}} U^n \mid \mathscr{H}$ $(1 \leq n < \infty)$, where $P_{\mathscr{H}} \in \mathscr{L}(\mathscr{K})$ is the projection onto $\mathscr{H}$, then U is said to be a *strong unitary dilation* of T. Every contraction T has a strong unitary dilation U, and if U is assumed to be minimal in the sense that the smallest reducing subspace for U containing $\mathscr{H}$ is $\mathscr{K}$ itself, then U is unique up to isomorphism [5]. The relation between the unique minimal co-isometric extension of T and the unique minimal unitary dilation of T is given by the following known lemma, which we prove without appealing to the structure theory for contractions.

LEMMA 6.1. *Suppose that* T *is a contraction on a Hilbert space* $\mathscr{H}$, *and let* V *be the unique minimal co-isometric extension of* T *acting on a Hilbert space* $\mathscr{K}_+ \supset \mathscr{H}$. *Let* W^* *be the unique minimal co-isometric extension of* V^* *acting on the Hilbert space* $\mathscr{K}$. *Then* W^* *is unitary, and* W *is the minimal strong unitary dilation of* T.

Proof. To show that W^* (and therefore W) is unitary, it suffices to show that W^* has no null space. Regard W^* as the matrix (10) with V^* replacing T. Since $1_{\mathscr{K}_+} - V^*V$ is the projection on the orthocomplement of the range of V^*, it follows that the only vectors x and y in $\mathscr{K}_+$ for which $V^*x + \sqrt{1_{\mathscr{K}_+} - V^*V}\,y = 0$ are $x = y = 0$. That W^* has no null space follows immediately. We next show that W is a strong unitary dilation of T. From the hypotheses we see that $T^n = V^n \mid \mathscr{H}$ and that $V^{*n} = (W^n)^* \mid \mathscr{K}_+$. It follows immediately that $V^n = P_{\mathscr{K}_+} W^n \mid \mathscr{K}_+$, where $P_{\mathscr{K}_+} \in \mathscr{L}(\mathscr{K})$ is the projection onto $\mathscr{K}_+$; therefore

$$T^n = P_{\mathscr{H}}(P_{\mathscr{K}_+} W^n \mid \mathscr{K}_+) \mid \mathscr{H} = P_{\mathscr{H}} W^n \mid \mathscr{H},$$

where $P_{\mathscr{H}} \in \mathscr{L}(\mathscr{K})$ is the projection onto $\mathscr{H}$. This proves that W is a strong unitary dilation of T. To prove minimality, we must show that the smallest reducing subspace $\mathscr{S}$ for W that contains $\mathscr{H}$ is $\mathscr{K}$ itself. To this end, note that $\mathscr{S}$ must contain the linear manifold $\mathscr{M}$ consisting of all vectors of the form

$$\sum_{n=0}^{N} W^{*n} x_n \qquad (N \geq 0,\ x_0, \cdots, x_N \in \mathscr{H}).$$

Since $V^{*n} = W^{*n} \mid \mathscr{K}_+$, it follows that the closure of $\mathscr{M}$ is the smallest reducing subspace for V containing $\mathscr{H}$, and thus, by the minimality of V, the closure of $\mathscr{M}$ is $\mathscr{K}_+$. Thus $\mathscr{S}$ contains $\mathscr{K}_+$, and since $\mathscr{S}$ is a reducing subspace for W^*, the minimality of W^* implies that $\mathscr{S} = \mathscr{K}$, as desired. The proof is complete.

The following lemma is contained in [3, Lemma 0].

LEMMA 6.2. *If* T *is a contraction on the Hilbert space* $\mathscr{H}$ *and* W *is the minimal strong unitary dilation of* T *on the Hilbert space* $\mathscr{K}$, *then there exist invariant subspaces* $\mathscr{M} \subset \mathscr{K}$ *and* $\mathscr{N} \subset \mathscr{K}$ *for* W *such that* $\mathscr{M} \supset \mathscr{N}$ *and* $\mathscr{H} = \mathscr{M} \ominus \mathscr{N}$. *Furthermore* $\mathscr{M}$ *can be taken to be the smallest invariant subspace for* W *that contains* $\mathscr{H}$.

We have discussed the minimal strong unitary dilation of a contraction because we wish to relate the commutant of a contraction to the commutant of its minimal strong unitary dilation. (By the *commutant* of an operator $A \in \mathscr{L}(\mathscr{H})$ we mean the algebra $\{X \in \mathscr{L}(\mathscr{H}): XA = AX\}$.) The connection is given by the following theorem, which we believe to be new.

THEOREM 6. *Suppose that* T *is a contraction on a Hilbert space* $\mathscr{H}$ *and that* W *is the unique minimal strong unitary dilation of* T *on the Hilbert space* $\mathscr{K} \supset \mathscr{H}$. *Write* $\mathscr{H} = \mathscr{M} \ominus \mathscr{N}$, *where* $\mathscr{M}$ *is the smallest invariant subspace for* W *containing* $\mathscr{H}$, *and* $W\mathscr{N} \subset \mathscr{N}$. *Then the commutant of* T *consists of all operators of the form* $P_{\mathscr{H}}Y \mid \mathscr{H}$, *where* $Y \in \mathscr{L}(\mathscr{K})$ *is in the commutant of* W *and satisfies* $Y\mathscr{M} \subset \mathscr{M}$ *and* $Y\mathscr{N} \subset \mathscr{N}$. *Furthermore, if* X *is in the commutant of* T, *then* X *can be written as* $X = P_{\mathscr{H}}Y \mid \mathscr{H}$, *where* Y *has the properties described above and also satisfies the condition* $\|Y\| = \|X\|$.

Proof. We first show that if $Y \in \mathscr{L}(\mathscr{K})$ is in the commutant of W and has $\mathscr{M}$ and $\mathscr{N}$ for invariant subspaces, then $P_{\mathscr{H}}Y \mid \mathscr{H}$ is in the commutant of $T = P_{\mathscr{H}}W \mid \mathscr{H}$. This follows from an imposing but easily verified equation, valid for any two operators A and B in $\mathscr{L}(\mathscr{K})$ having $\mathscr{M}$ and $\mathscr{N}$ for invariant subspaces:

$$\begin{aligned} P_{\mathscr{H}}AB \mid \mathscr{H} &= P_{\mathscr{H}}(AB \mid \mathscr{M}) \mid \mathscr{H} = [P_{\mathscr{H}}(AB \mid \mathscr{M})^* \mid \mathscr{H}]^* \\ &= [(AB \mid \mathscr{M})^* \mid \mathscr{H}]^* = [\{(A \mid \mathscr{M})(B \mid \mathscr{M})\}^* \mid \mathscr{H}]^* \\ &= [(B \mid \mathscr{M})^* (A \mid \mathscr{M})^* \mid \mathscr{H}]^* = [(A \mid \mathscr{M})^* \mid \mathscr{H}]^* [(B \mid \mathscr{M})^* \mid \mathscr{H}]^* \\ &= [P_{\mathscr{H}}(A \mid \mathscr{M}) \mid \mathscr{H}][P_{\mathscr{H}}(B \mid \mathscr{M}) \mid \mathscr{H}] = (P_{\mathscr{H}}A \mid \mathscr{H})(P_{\mathscr{H}}B \mid \mathscr{H}) . \end{aligned}$$

To complete the proof, it suffices to show that if $X \in \mathscr{L}(\mathscr{H})$ is in the commutant of T, then there exists $Y \in \mathscr{L}(\mathscr{K})$ in the commutant of W such that

(a) $\|Y\| = \|X\|$,

(b) $X = P_{\mathscr{H}}Y \mid \mathscr{H}$, and

(c) $\mathscr{M}$ and $\mathscr{N}$ are invariant subspaces for Y.

By virtue of Lemma 6.1 and the uniqueness theorem for the minimal strong unitary dilation of T, we may assume that there exist a Hilbert space $\mathscr{K}_+$ $(\mathscr{H} \subset \mathscr{K}_+ \subset \mathscr{K})$ and a co-isometry $V \in \mathscr{L}(\mathscr{K}_+)$ such that V is the minimal co-isometric extension of T and W^* is the minimal co-isometric extension of V^*. Applying Theorem 4 to X, one obtains an operator $Y_+ \in \mathscr{L}(\mathscr{K}_+)$ such that $Y_+V = VY_+$, $Y_+\mathscr{H} \subset \mathscr{H}$, $Y_+ \mid \mathscr{H} = X$, and $\|Y_+\| = \|X\|$. Clearly, $Y_+^* V^* = V^* Y_+^*$, and another application of Theorem 4 yields the existence of an operator $Y^* \in \mathscr{L}(\mathscr{K})$ such that $Y^* W^* = W^* Y^*$, $Y^* \mathscr{K}_+ \subset \mathscr{K}_+$, $Y^* \mid \mathscr{K}_+ = Y_+^*$, and $\|Y^*\| = \|Y_+^*\|$. We assert that the operator Y satisfies (a), (b), and (c). That (a) is satisfied is obvious; to see that (b) is satisfied, note that $Y_+ = P_{\mathscr{K}_+} Y \mid \mathscr{K}_+$, and thus

$$X = Y_+ \mid \mathscr{H} = P_{\mathscr{H}}(P_{\mathscr{K}_+}Y \mid \mathscr{K}_+) \mid \mathscr{H} = P_{\mathscr{H}}Y \mid \mathscr{H}.$$

We begin the argument to show that (c) is satisfied by defining $\mathscr{K}_- = \mathscr{K} \ominus \mathscr{K}_+$, so that $\mathscr{K} = \mathscr{K}_+ \oplus \mathscr{K}_-$. Since $V\mathscr{H} \subset \mathscr{H}$, we see that $V^*(\mathscr{K}_+ \ominus \mathscr{H}) \subset \mathscr{K}_+ \ominus \mathscr{H}$, and since W^* extends V^*, we have the relation $W^*(\mathscr{K}_+ \ominus \mathscr{H}) \subset \mathscr{K}_+ \ominus \mathscr{H}$. Thus $\mathscr{H} \oplus \mathscr{K}_-$ is an invariant subspace for W, and we prove that $\mathscr{H} \oplus \mathscr{K}_- = \mathscr{M}$ by showing that $\mathscr{H} \oplus \mathscr{K}_-$ is the smallest invariant subspace for W that contains $\mathscr{H}$. Clearly,

$$\bigvee_{n=1}^{\infty} W^n \mathscr{H} \subset \mathscr{K}_- \oplus \mathscr{H}.$$

Suppose that this inclusion is proper. Then some nonzero vector x in $\mathscr{K}_-$ is orthogonal to $\bigvee_{n=0}^{\infty} W^n \mathscr{H}$. Since

$$\bigvee_{n=1}^{\infty} W^{*n} \mathscr{H} = \bigvee_{n=1}^{\infty} V^{*n} \mathscr{H} \subset \mathscr{K}_+,$$

x is also orthogonal to $\bigvee_{n=1}^{\infty} W^{*n} \mathscr{H}$, and thus to $\bigvee_{n=-\infty}^{\infty} W^n \mathscr{H}$. But $\bigvee_{n=-\infty}^{\infty} W^n \mathscr{H}$ is a reducing subspace for W, and it contains $\mathscr{H}$; hence, by minimality, $\bigvee_{n=-\infty}^{\infty} W_n \mathscr{H} = \mathscr{K}$. Thus x is orthogonal to $\mathscr{K}$, a contradiction. This proves that $\bigvee_{n=0}^{\infty} W^n \mathscr{H} = \mathscr{H} \oplus \mathscr{K}_-$, and thus that $\mathscr{H} \oplus \mathscr{K}_- = \mathscr{M}$. It follows that $\mathscr{K}_- = \mathscr{N}$, and we can complete the argument by showing that $\mathscr{K}_-$ and $\mathscr{H} \oplus \mathscr{K}_-$ are invariant subspaces for Y. Since Y^* is an extension of Y_+^*, $Y^* \mathscr{K}_+ \subset \mathscr{K}_+$, and thus $Y \mathscr{K}_- \subset \mathscr{K}_-$. Finally, the inclusion $Y_+ \mathscr{H} \subset \mathscr{H}$ implies that $Y_+^*(\mathscr{K}_+ \ominus \mathscr{H}) \subset \mathscr{K}_+ \ominus \mathscr{H}$, and since Y^* extends Y_+^*, we conclude that $Y^*(\mathscr{K}_+ \ominus \mathscr{H}) \subset \mathscr{K}_+ \ominus \mathscr{H}$, which implies that $Y(\mathscr{H} \oplus \mathscr{K}_-) \subset \mathscr{H} \oplus \mathscr{K}_-$.

7. CONCLUDING REMARKS

(I) Let T be a contraction on a Hilbert space, and denote by $\mathscr{A}_T$ the weakly closed algebra (*not* *-algebra) generated by T and 1. Let $\mathscr{A}_T'$ denote the commutant of T, and let $\mathscr{A}_T''$ denote the commutant of $\mathscr{A}_T'$. The relations

$$\mathscr{A}_T \subset \mathscr{A}_T'' \subset \mathscr{A}_T'$$

are elementary, and it is not difficult to provide examples of operators T such that the above three algebras are distinct. The theorems discussed in this paper relate $\mathscr{A}_T'$ to $\mathscr{A}_V'$ and $\mathscr{A}_W'$, where V and W are the minimal co-isometric extension of T and the minimal strong unitary dilation of T, respectively. Is there a relation between $\mathscr{A}_T''$ and the algebras $\mathscr{A}_V''$ and $\mathscr{A}_W''$?

(II) It would be possible to organize this paper slightly differently, so that Theorem 4 would become a corollary of Corollary 4.1 instead of vice versa. To accomplish this, the operator V of Corollary 4.1 would be taken to be the matrix (10), with $1_{\mathscr{H}}$ replacing $1_{\mathscr{D}_*}$ everywhere. The operator V would then act on the Hilbert space $\mathscr{H} \oplus \mathscr{H} \oplus \cdots$. The alternate organization would have the advantage of requiring a less general form of Theorem 2, and the arguments would involve fewer Hilbert spaces.

(III) Consider the class (L) of all operators V on a Hilbert space $\mathscr{K}$ with the property that whenever $\mathscr{H} \subset \mathscr{K}$ is an invariant subspace for V and $X \in \mathscr{L}(\mathscr{H})$ commutes with $V | \mathscr{H}$, then X can be lifted to an operator on $\mathscr{K}$ that commutes with V. It follows from remark (II) above that (L) contains all of the co-isometries in $\mathscr{L}(\mathscr{K})$. On the other hand, Douglas showed in [2] that not all normal operators belong to (L). It would be interesting to characterize the class (L).

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