ON A CONJECTURE RELATED TO THE SUSPENSION OF HOMOTOPY 3-SPHERES AND FAKE CUBES

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1. INTRODUCTION

One of the major outstanding problems in 3-dimensional topology is the Poincaré conjecture: If H^3 is a homotopy 3-sphere, then H^3 is topologically equivalent to the 3-sphere S^3. In [8], it was shown that if H^3 is a homotopy 3-sphere bounding a contractible combinatorial 4-manifold, then the double suspension $\Sigma^2 H^3$ is topologically equivalent to S^5. Here we state a conjecture equivalent to the conjecture that if F^3 is a fake cube, then $\operatorname{int} F^3 \times (0, 1) = E^4$. Of course, one would like to obtain the result that $F^3 \times [0, 1] = I^4$. If we let $2F^3$ denote the double of F^3, it would then follow from [3] (since $2F^3 = \partial(F^3 \times [0, 1])$ and $\partial F^3 = S^2$) that $F^3 = I^3$, and this would give the Poincaré conjecture.

The following problem was suggested to the author by E. H. Connell as a possible means of showing that the suspension of a homotopy 3-sphere is S^4.

CONJECTURE. *If* H^3 *is a homotopy* 3-*sphere and* B_1 *and* B_2 *are two disjoint, piecewise linear* 3-*cells in* H^3 *under some combinatorial triangulation of* H^3, *then some homeomorphism* h *taking* $H^3 \times E^1$ *onto itself has the property that*

$$H^3 \times E^1 = h(\operatorname{int} B_1 \times E^1) \cup (\operatorname{int} B_2 \times E^1).$$

We prove here that this conjecture is equivalent to the statement that the suspension of a homotopy 3-sphere is S^4. Also, we show that such a solution leads to a number of additional results, and we obtain a partial solution to the conjecture. More specifically, the following results are obtained. (In what follows, H^3 will always denote a homotopy 3-sphere, and F^3 a fake cube. If M is a manifold with nonempty boundary, 2M will denote the double of M. If X is a topological space, ΣX will denote the suspension of X.) Assuming the conjecture, we show that

$$\operatorname{int}(F^3 \times I) = E^4 = (H^3 - \{pt.\}) \times E^1, \quad 2(F^3 \times I) = S^4, \quad F^3 \times I^2 = I^5, \quad \Sigma(2F^3) = S^4,$$

and that the 3-dimensional Poincaré conjecture is equivalent to the conjecture that every triangulation of S^4 is combinatorial. Conversely, if we assume that $\operatorname{int}(F^3 \times I) = E^4$, $(H^3 - \{pt.\}) \times E^1 = E^4$, or $\Sigma(2F^3) = S^4$, then the conjecture is true. Making use of a theorem of [10], we strengthen two of the above results by showing that the conjecture implies that $\Sigma F^3 = I^4$ and $\Sigma H^3 = S^4$. Finally, by using the engulfing theorem of [14] and the product structure on $H^3 \times E^1$, we prove a weak form of our conjecture (see Theorem 5). Since this conjecture is equivalent to the statement that $\Sigma H^3 = S^4$, we shall call this conjecture the ΣH-Conjecture.

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2. DEFINITIONS AND PRELIMINARIES

We shall use the terminology of [7] and [15]. For example, if the complex T collapses simplically to the subcomplex L, this will be denoted by $T \searrow L$. We shall use T' and L' to denote the *first barycentric subdivision* of the complexes T and L, respectively. Also, we shall use the concept of regular neighborhoods. The regular neighborhoods used here can always be considered to be the canonical regular neighborhoods. That is, if K is a combinatorial n-manifold, L is a finite subcomplex of K, and U is an open set in K containing L, then by a *regular neighborhood of* L *in* U we shall mean the simplicial neighborhood of L in some n^{th} barycentric subdivision of K mod L $(n \geq 2)$, say $N(L, (K \bmod L)^n)$, such that

$$N(L, (K \bmod L)^n) \subset U.$$

Here we obtain $(K \bmod L)'$ (or $(K \bmod L)^1$) by starring the simplexes of K - L (using barycenters) in order of decreasing dimension, and $(K \bmod L)^n$ is inductively defined by the formula $(K \bmod L)^n = ((K \bmod L)^{n-1} \bmod L)'$.

I^n, E^n, and S^n will denote spaces homeomorphic to the unit n-cube, Euclidean n-space, and the n-sphere, respectively. The symbol = indicates topological equivalence. A *homotopy* 3-*sphere* H^3 is a closed, connected, simply connected 3-manifold. A *homotopy* 4-*sphere* H^4 is a closed, connected 4-manifold such that $\pi_i(H^4) = 0$ for i = 1, 2, 3. A *fake cube* F^3 is a compact, contractible 3-manifold with nonempty boundary such that $\partial F^3 = S^2$. By [1], whenever we are considering an H^3 or F^3, we may assume that these 3-manifolds also have combinatorial triangulations.

If X is a topological space, then ΣX and CX will denote the *suspension* of X and the *cone* over X, respectively. That is,

$$CX = (X \times I)/\{X \times 1\} \quad \text{and} \quad \Sigma X = (X \times [-1, 1])/(\{X \times -1\}, \{X \times 1\}).$$

If M^m is an m-manifold and N^n is an n-manifold without boundary, then $M^m \subset N^n$ $(m \leq n)$ is *locally flat* if for each point $p \in \text{int } M^m$ there exists a neighborhood U of p in N^n such that

$$(U, U \cap M^m) = (E^n, E^m) \quad \text{as pairs,}$$

and if for each point $q \in \partial M^m$ there exists a neighborhood V of q in N^n such that

$$(V, V \cap M^m) = (E^n, E^m_+) \quad \text{as pairs.}$$

Finally, if X is a subset of E^n (or S^n), then X is said to be *cellular* if there exists a sequence $\{B_i^n\}_{i=1}^{\infty}$ of n-cells such that

$$B_{i+1}^n \subset \text{int } B_i^n \quad \text{and} \quad X = \bigcap_{i=1}^{\infty} B_i^n.$$

If $X \subset E^n$ (or $X \subset S^n$) is cellular, then clearly $E^n/\{X\} = E^n$ (or $S^n/\{X\} = S^n$).

3. MAIN RESULTS

THEOREM 1. *Suppose that the* ΣH-*Conjecture is true, that* H^3 *is a homotopy* 3-*sphere, that* $F^3 = H^3 - \text{int}\,\Delta^3$, *where* Δ^3 *is a* 3-*simplex of* H^3, *and that* p *is a point of* int Δ^3. *Then*

$$(H^3 - \{p\}) \times E^1 = E^4 = \text{int}\, F^3 \times E^1 .$$

Proof. Since $H^3 - \{p\}$ is homeomorphic to int F^3, it suffices to show that int $F^3 \times E^1 = E^4$. We first apply the ΣH-Conjecture. Let $B_2 = \Delta^3$, and let B_1 be a combinatorial 3-cell contained in int F^3. By the ΣH-Conjecture, there exists a homeomorphism h carrying $H^3 \times E^1$ onto itself such that

$$h(\text{int}\, B_1 \times E^1) \cup (\text{int}\, \Delta^3 \times E^1) = H^3 \times E^1 .$$

Hence $h(\text{int}\, B_1 \times E^1)$ contains $F^3 \times E^1$.

Let ΣH^3 denote the 2-point compactification of $H^3 \times E^1$; that is, let $\Sigma H^3 = H^3 \times E^1 \cup \{\omega\} \cup \{-\omega\}$ (we can think of ΣH^3 as the suspension of H^3 with suspension points ω and $-\omega$). Let

$$C = h(B_1 \times E^1) \cup \{\omega\} \cup \{-\omega\} \subset \Sigma H^3 \quad \text{and} \quad \Sigma F^3 = F^3 \times E^1 \cup \{\omega\} \cup \{-\omega\} \subset \Sigma H^3$$

(we can also think of ΣF^3 as the suspension of F^3 with suspension points ω and $-\omega$). Since C is homeomorphic to $B_1 \times E^1 \cup \{\omega\} \cup \{-\omega\}$, it is a 4-cell containing ΣF^3. Let S^4 be the 4-sphere 2C, where 2C denotes the double of C. Now $\Sigma F^3 \subset S^4$ and $\Sigma \partial F^3$ is an embedded 3-sphere that is locally flat except perhaps at the two points ω and $-\omega$. Let A be the flat arc in $\Sigma \partial F^3$ obtained as the suspension of some point $q \in \partial F^3$. Since A is locally flat in S^4 except perhaps at the two suspension points ω and $-\omega$, it is flat [6]. Therefore, by shrinking A to a point we obtain a 3-sphere $\Sigma \partial F^3/A$ embedded in the 4-sphere S^4/A in such a way that it is locally flat except perhaps for the point $\{A\}$. It follows by [4] and [5] that $\Sigma \partial F^3/A$ is flat in S^4/A; hence, $\text{int}(\Sigma F^3/A) = E^4$. Since $\text{int}(\Sigma F^3/A)$ is homeomorphic to $\text{int}(\Sigma F^3)$ and the latter expression is homeomorphic to int $F^3 \times E^1$, we see that int $F^3 \times E^1$ is homeomorphic to E^4.

Remark 1. The F^3 above is clearly a fake cube. It follows easily that if F^3 is any fake cube or p is any point of H^3, then the ΣH-Conjecture implies that

$$\text{int}(F^3 \times I) = \text{int}\, F^3 \times E^1 = E^4 = (H^3 - \{p\}) \times E^1 .$$

COROLLARY 1. *If the* ΣH-*Conjecture holds and* F^3 *is a fake cube, then* $2(F^3 \times I) = S^4$ *and* $F^3 \times I^2 = I^5$.

Proof. $2(F^3 \times I) = S^4$, since $\text{int}(F^3 \times I) = E^4$. That is, since $\partial(F^3 \times I)$ has an open collar U in $F^3 \times I$, $(F^3 \times I) \cup U = E^4$, where the U is the collar in the other copy of $F^3 \times I$ in $2(F^3 \times I)$. Hence, $2(F^3 \times I)$ is the union of two open subsets, each of which is homeomorphic to E^4, and it follows directly from [3] that $2(F^3 \times I) = S^4$. $F^3 \times I^2 = I^5$, since $\partial(F^3 \times I^2) = S^4$ and $\text{int}(F^3 \times I^2) = E^5$. That is,

$$\partial(F^3 \times I^2) = \partial([F^3 \times I] \times I) = 2(F^3 \times I) = S^4 .$$

Since $\text{int}(F^3 \times I^2) = E^5$, it follows by the above argument that $2(F^3 \times I^2) = S^5$. Since $\partial(F^3 \times I^2) = S^4$, another application of [3] gives the result $F^3 \times I^2 = I^5$.

THEOREM 2. *Suppose that the* ΣH-*Conjecture holds, that* F^3 *is a fake cube, that* $2F^3$ *is the double of* F^3, *and that* $\Sigma(2F^3)$ *is the suspension of* $2F^3$. *Then* $\Sigma(2F^3) = S^4$.

Proof. By Corollary 1, $2(F^3 \times I) = S^4$. Since there is a 2-complex $\tilde{K}^2$ in int F^3 such that F^3 collapses to $\tilde{K}^2$, there is a 2-complex K^2 in int$(F^3 \times I)$ such that $F^3 \times I \searrow K^2$ $\left(\text{namely, } \tilde{K}^2 \times \frac{1}{2}\right)$. Since int$(F^3 \times I) = E^4$, K^2 is cellular in int$(F^3 \times I)$. That is, corresponding to each open set $U \subset \text{int}(F^3 \times I)$ containing K^2, there is a regular neighborhood N of K^2 lying in U. Since N and $F^3 \times I$ are regular neighborhoods of K^2, N is homeomorphic to $F^3 \times I$, by [7] or [15]. Thus $K^2 \subset \text{int}\, N = E^4$, and hence some 4-ball in int $N \subset N \subset U$ contains K^2 in its interior.

Now, since K^2 is cellular in $F^3 \times I$, $(F^3 \times I)/K^2$ is homeomorphic to $F^3 \times I$ (where $(F^3 \times I)/K^2$ denotes the space obtained by shrinking K^2 to a point). Since $F^3 \times I$ is a regular neighborhood of K^2, $(F^3 \times I) - K^2$ is combinatorially equivalent to $\partial(F^3 \times I) \times [0, 1)$ [7]. Since $\partial(F^3 \times I) = 2F^3$, it follows that

$$(F^3 \times I)/K^2 = 2F^3 \times [0, 1) \cup \{K^2\}$$

is the cone over $2F^3$ from the point $\{K^2\}$. Thus, shrinking the copy of K^2 in each half of $2(F^3 \times I) = S^4$, we see that

$$S^4 = (F^3 \times I) \cup (F^3 \times I) = (F^3 \times I)/K^2 \cup (F^3 \times I)/K^2$$

$$= \{K^2\} \cup 2F^3 \times (-1, 1) \cup \{K^2\} = \Sigma(2F^3).$$

COROLLARY 2. *The 3-dimensional Poincaré conjecture is equivalent to the conjecture that every triangulation of* S^4 *is combinatorial.*

Proof. The 3-dimensional Poincaré conjecture implies that every homotopy 3-sphere H^3 is combinatorially equivalent to S^3. Now suppose K is a complex that triangulates S^4 and v is a vertex of K. Then the link of v in K, lk(v, K), is a simply connected combinatorial 3-manifold [2]; hence lk(v, K) is a homotopy 3-sphere. Therefore, assuming the Poincaré conjecture, we can conclude that lk(v, K) is combinatorially equivalent to S^3, and hence that K is a combinatorial triangulation of S^4.

Now suppose every triangulation of S^4 is combinatorial, and let H^3 be a homotopy 3-sphere. Let $F^3 = H^3 - \text{int}\, \Delta^3$. By Theorem 2, $\Sigma(2F^3) = S^4$ and the triangulation of F^3 gives us a triangulation of $S^4 = \Sigma(2F^3)$. If v is the vertex corresponding to one of the suspension points, then $\text{lk}(v, \Sigma(2F^3)) = 2F^3$. Since every triangulation of S^4 is combinatorial, this implies that $2F^3 = S^3$. Since $\partial F^3 = S^2$, it follows by [3] that $F^3 = I^3$. Hence, $H^3 = F^3 \cup \Delta^3 = I^3 \cup \Delta^3 = S^3$.

COROLLARY 3. *If for some* $n \geq 4$ *every triangulation of* S^n *is combinatorial, then the 3-dimensional Poincaré conjecture is true.*

Proof. If every triangulation of S^n is combinatorial, then every triangulation of a k-manifold $(k \leq n)$ without boundary is combinatorial (since the suspension of a "bad" sphere is again a "bad" sphere and the suspension of a link of a vertex in a k-manifold is a triangulated k-sphere). Since $n \geq 4$, this implies that every triangulation of S^4 is combinatorial. Hence, by Corollary 2, the 3-dimensional Poincaré conjecture is true.

The following corollary was suggested to the author by M. L. Curtis.

COROLLARY 4. *If* H^4 *is a combinatorial homotopy* 4-*sphere having a* 2-*spine* K^2 *such that* K^2 *can be embedded piecewise linearly in some combinatorial* 3-*manifold, then* $H^4 = S^4$.

Proof. The assumption that K^2 is a 2-spine of H^4 means that there is some combinatorial 4-cell $B^4 \subset H^4$ such that if $F^4 = H^4 - \text{int } B^4$, then there exists a $K^2 \subset \text{int } F^4$ such that $F^4 \searrow K^2$. Now suppose K^2 can be embedded piecewise linearly in some combinatorial 3-manifold M^3 (with or without boundary). If N^3 is a regular neighborhood of K^2 in M^3, then, since K^2 is contractible, N^3 is a fake cube. By Corollary 1, $N^3 \times I^2 = I^5$. We note that $N^3 \times I^2$ is a combinatorial manifold with boundary, and we are only assuming that it is topologically equivalent to I^5. Since any contractible, combinatorial 5-manifold with boundary can be piecewise linearly embedded in E^5, we may suppose that both $N^3 \times I^2$ and $F^4 \times I$ are piecewise linearly embedded in E^5. That is, if W^5 is any contractible combinatorial 5-manifold with boundary, then $2W^5$ is a combinatorial 5-manifold topologically equivalent to S^5. By [13], $2W^5 - \{pt.\}$ is combinatorially equivalent to E^5. Hence, W can be piecewise linearly embedded in E^5. Since $N^3 \times I^2$ and $F^4 \times I$ are regular neighborhoods of the given embeddings of the corresponding copies of K^2, it follows by [11] and [15] or by [7] that $F^4 \times I$ is combinatorially equivalent to $N^3 \times I^2 = I^5$. The result from [7] or [11] that we use is that any two piecewise linear embeddings of a contractible 2-complex in E^5 are equivalent under a piecewise linear homeomorphism and that hence their regular neighborhoods are piecewise linearly homeomorphic. Since $S^4 = \partial(F^4 \times I) = 2F^4$, it follows by the generalized Schoenflies theorem that $F^4 = I^4$ and hence $H^4 = S^4$.

Recall that if X is a topological space, then CX denotes the cone over X. That is, $CX = (X \times I)/\{X \times 1\}$.

COROLLARY 5. *If* F^3 *is a fake cube, then* $C[\partial(F^3 \times I)] \cup F^3 \times I = S^4$ *and* $C(F^3 \times I) = I^5$.

Proof.

$$C[\partial(F^3 \times I \times 0)] \cup (F^3 \times I \times 0) = \partial[C(F^3 \times I)]$$

(where $\partial[C(F^3 \times I)]$ denotes the mod 2 boundary of $C(F^3 \times I)$) and

$$C[\partial(F^3 \times I \times 0)] \cap (F^3 \times I \times 0) = \partial(F^3 \times I \times 0) = 2F^3.$$

From the proof of Theorem 2, it follows that $F^3 \times I = C(2F^3) = C[\partial(F^3 \times I \times 0)]$. Hence $\partial[C(F^3 \times I)] = 2(F^3 \times I) = S^4$. Also,

$$\text{int}(C(F^3 \times I)) = (\text{int } F^3) \times (0, 1) \times (0, 1) = \text{int}(F^3 \times I^2) = E^5$$

(by Corollary 1 or by [13]).

We would now like to consider $2[C(F^3 \times I)]$. We know that

$$S^4 = \partial[C(F^3 \times I)] \subset 2[C(F^3 \times I)]$$

and that each of the two complementary domains of $\partial[C(F^3 \times I)]$ in $2[C(F^3 \times I)]$ is homeomorphic to E^5. We would like to know that $2[C(F^3 \times I)] = S^5$. For then,

$$S^4 = \partial[C(F^3 \times I)] \subset 2[C(F^3 \times I)] = S^5,$$

and S^4 is locally flat in S^5 modulo the common vertex ($= \{F^3 \times I \times 1\}$) of the two cones making up S^5. Hence, by [4] and [5], $\partial[C(F^3 \times I)]$ is flat in $2[C(F^3 \times I)]$, and therefore $C(F^3 \times I) = I^5$. However, it is not immediately clear that $2[C(F^3 \times I)] = S^5$. To obtain this conclusion, we must consider $2[C(F^3 \times I)]$ in a different manner.

Consider $2(F^3 \times I^2)$, which by Corollary 1 is homeomorphic to S^5. As in the proof of Theorem 2, there exists a 2-complex $\tilde{K}^2 \subset \text{int } F^3$ such that $F^3 \searrow \tilde{K}^2$. Let

$$K^2 = \tilde{K}^2 \times \frac{1}{2} \times 1 \subset 2(F^3 \times I^2).$$

Let N be a regular neighborhood of $F^3 \times I \times 1$ in $2(F^3 \times I^2)$. Then N and $F^3 \times I$ are also regular neighborhoods of K^2. Hence $N = (F^3 \times I^2) = I^5$, and it follows that $F^3 \times I \times 1$ is cellular in $2(F^3 \times I^2)$. Thus

$$S^5 = 2(F^3 \times I^2) = 2(F^3 \times I^2)/\{F^3 \times I \times 1\} = 2[C(F^3 \times I)],$$

and the proof of Corollary 5 is now complete.

THEOREM 3. *Let* H^3 *be a homotopy* 3-*sphere, and suppose* F^3 *is a fake cube in* H^3 *obtained by removing the interior of a piecewise linear* 3-*cell. If* $\text{int}(F^3 \times I) = E^4$, $(H^3 - \{\text{pt.}\}) \times E^1 = E^4$, *or* $\Sigma(2F^3) = S^4$, *then the* ΣH-*Conjecture is true.*

Proof. We have already noted that if $\text{int}(F^3 \times I) = E^4$, then

$$(H^3 - \{\text{pt.}\}) \times E^1 = E^4$$

(Theorem 1) and $\Sigma(2F^3) = S^4$ (Theorem 2). Clearly, $(H^3 - \{\text{pt.}\}) \times E^1 = E^4$ implies that $\text{int}(F^3 \times I) = E^4$. Also, if $\Sigma(2F^3) = S^4$, it follows as in the proof of Theorem 1 (since $\Sigma\partial F^3$ is a 3-sphere in the 4-sphere $\Sigma(2F^3)$ that is locally flat except perhaps for the two suspension points) that each complementary domain of $\Sigma\partial F^3$ in $\Sigma(2F^3)$ is homeomorphic to E^4. Therefore, $\Sigma(2F^3) = S^4$ implies that $\text{int}(F^3 \times I) = E^4$. Hence it suffices to show that if $\text{int}(F^3 \times I) = E^4$, then the ΣH-Conjecture is true.

Let H^3 be a homotopy 3-sphere, and let B_1 and B_2 be two disjoint, piecewise linear 3-cells in H^3. Let $\hat{B}_2$ be a piecewise linear 3-cell in $\text{int } B_2$, and let F^3 be the fake cube $H^3 - \text{int } \hat{B}_2$. We now consider the inclusion

$$B_1 \times \left(-\frac{1}{2}, \frac{1}{2}\right) \subset \text{int } F^3 \times \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Since the 4-cell $B_1 \times \left[-\frac{1}{4}, \frac{1}{4}\right]$ and the compact set $(H^3 - \text{int } B_2) \times 0$ each lie in $\text{int } F^3 \times \left(-\frac{1}{2}, \frac{1}{2}\right) = E^4$, there exist a 4-cell C and a homeomorphism f_1 taking $\text{int } F^3 \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ onto itself such that

(1) $\left(B_1 \times \left[-\frac{1}{4}, \frac{1}{4}\right]\right) \cup ((H^3 - \text{int } B_2) \times 0) \subset \text{int } C \subset C \subset \text{int } F^3 \times \left(-\frac{1}{2}, \frac{1}{2}\right)$,

(2) $f_1\left(\text{int } B_1 \times \left(-\frac{1}{4}, \frac{1}{4}\right)\right) \supset (H^3 - \text{int } B_2) \times 0$, and

(3) f_1 *is the identity map outside* C.

Since f_1 is the identity outside C, f_1 extends by the identity to all of $H^3 \times [-1, 1]$. We denote the extended homeomorphism by g_1.

Thus g_1 is a homeomorphism taking $H^3 \times [-1, 1]$ onto itself and satisfying the two conditions

(1) $g_1(\text{int } B_1 \times [-1, 1]) \supset (H^3 - \text{int } B_2) \times 0$,

(2) $g_1 = \text{identity on } \hat{B}_2 \times [-1, 1] \cup H^3 \times \left[-1, -\frac{1}{2}\right] \cup H^3 \times \left[\frac{1}{2}, 1\right]$.

Unfortunately, $g_1(\text{int } B_1 \times [-1, 1])$ may fail to contain $\text{int } B_1 \times [-1, 1]$. We now consider the region $H^3 \times (-1, 1)$. Since

$$g_1(\text{int } B_1 \times (-1, 1)) \supset (H^3 - \text{int } B_2) \times 0,$$

there exists a piecewise linear 3-cell $\hat{B}_1 \subset \text{int } B_1$ such that

$$g_1(\text{int } \hat{B}_1 \times (-1, 1)) \supset (H^3 - \text{int } B_2) \times 0.$$

Let p be a point of $\text{int } \hat{B}_1$, and consider the 1-dimensional subpolyhedron of $H^3 \times (-1, 1)$ given by $p \times (-1, 1)$. Since g_1 is the identity on

$$H^3 \times \left[-1, -\frac{1}{2}\right] \cup H^3 \times \left[\frac{1}{2}, 1\right],$$

the set $[p \times (-1, 1)] - g_1(\text{int } B_1 \times (-1, 1))$ is compact. The set $g_1(\hat{B}_1 \times (-1, 1))$ is closed in $H^3 \times (-1, 1)$ and lies in $g_1(\text{int } B_1 \times (-1, 1))$. Also, the pair

$$([H^3 \times (-1, 1)] - [g_1(\hat{B}_1 \times (-1, 1))], [g_1(\text{int } B_1 \times (-1, 1))] - [g_1(\hat{B}_1 \times (-1, 1))])$$

is 1-connected. Hence we can apply the engulfing theorem of [13]. That is, there exist a compact set $E \subset [H^3 \times (-1, 1)] - [g_1(\hat{B}_1 \times (-1, 1))]$ and a piecewise linear homeomorphism G_1 taking $H^3 \times (-1, 1)$ onto itself such that

$$G_1(g_1(\text{int } B_1 \times (-1, 1))) \supset p \times (-1, 1)$$

and G_1 = identity outside of E. Since E is compact and G_1 is the identity outside of E, it follows that G_1 can be extended by the identity to all of $H^3 \times [-1, 1]$ (we again denote the extended homeomorphism by G_1) and

$$G_1(g_1(\text{int } B_1 \times [-1, 1])) \supset (p \times [-1, 1]) \cup (H^3 - \text{int } B_2) \times 0.$$

By the last relation, there exist piecewise linear 3-cells $\tilde{B}_1$ and B_1^* such that

$$\tilde{B}_1 \subset \text{int } B_1 \subset B_1 \subset \text{int } B_1^* \subset B_1^* \subset \text{int}(H^3 - \text{int } B_2)$$

and $G_1(g_1(\text{int } B_1 \times [-1, 1])) \supset \tilde{B}_1 \times [-1, 1]$. Let $\hat{k}$ be a homeomorphism taking H^3 onto itself and B_1^* onto itself in such a way that $\hat{k}(\tilde{B}_1) = B_1$ and $\hat{k}$ = identity on $\{p\} \cup (H^3 - \text{int } B_1^*)$. Let k_1 be the homeomorphism of $H^3 \times [-1, 1]$ onto itself defined by $k_1(x, t) = (\hat{k}(x), t)$, where $x \in H^3$ and $t \in [-1, 1]$. Then

$$k_1 \circ G_1 \circ g_1(\text{int } B_1 \times [-1, 1]) \supset (B_1 \times [-1, 1]) \cup (H^3 - \text{int } B_2) \times 0.$$

Also, since G_1 and g_1 are the identity on $(H^3 \times -1) \cup (H^3 \times 1)$, it follows that

$$k_1 \circ G_1 \circ g_1(x, -1) = (\hat{k}(x), -1) \quad \text{and} \quad k_1 \circ G_1 \circ g_1(x, 1) = (\hat{k}(x), 1).$$

Hence, if π denotes the projection of $H^3 \times [-1, 1]$ onto $H^3 \times 0$, then

$$\pi \circ k_1 \circ G_1 \circ g_1(x, -1) = \pi \circ k_1 \circ G_1 \circ g_1(x, 1).$$

The homeomorphism $\phi_1 = k_1 \circ G_1 \circ g_1$ maps $H^3 \times [-1, 1]$ onto itself in such a way that

(1) $\phi_1(\text{int } B_1 \times [-1, 1]) \supset (B_1 \times [-1, 1]) \cup (H^3 - \text{int } B_2) \times 0$ and

(2) ϕ_1 carries each of $H^3 \times -1$ and $H^3 \times 1$ onto itself so that

$$\pi \circ \phi_1(x, -1) = \pi \circ \phi_1(x, 1).$$

Similarly, there is a homeomorphism ϕ_2 taking $H^3 \times [-1, 1]$ onto itself such that

(1) $\phi_2(\text{int } B_2 \times [-1, 1]) \supset (B_2 \times [-1, 1]) \cup (H^3 - \text{int } B_1) \times 0$ and

(2) ϕ_2 carries each of $H^3 \times -1$ and $H^3 \times 1$ onto itself so that

$$\pi \circ \phi_2(x, -1) = \pi \circ \phi_2(x, 1).$$

For $(x, t) \in H^3 \times [2i - 1, 2i + 1]$ $(i = 0, \pm 1, \pm 2, \cdots)$, let ϕ_{1i} be the homeomorphism taking $H^3 \times [2i - 1, 2i + 1]$ onto itself defined by

$$\phi_{1i}(x, t) = \tau_i \circ \phi_1(x, t - 2i),$$

where $\tau_i\colon H^3 \times [-1, 1] \to H^3 \times [2i - 1, 2i + 1]$ is the obvious map. For $(x, t) \in H^3 \times [2i, 2i + 2]$ $(i = 0, \pm 1, \pm 2, \cdots)$, let ϕ_{2i} be the homeomorphism taking $H^3 \times [2i, 2i + 2]$ onto itself defined by $\phi_{2i}(x, t) = \tau_i' \circ \phi_2(x, t - (2i + 1))$, where $\tau_i'\colon H^3 \times [-1, 1] \to H^3 \times [2i, 2i + 2]$ is also the obvious map. For

$$(x, t) \in H^3 \times [2i - 2 + j, 2i + j] \quad (j = 1, 2;\ i = 0, \pm 1, \pm 2, \cdots),$$

let $F_j(x, t) = \phi_{ji}(x, t)$. Each of F_1 and F_2 is a well-defined homeomorphism of $H^3 \times E^1$ onto itself (because of property (2) of the homeomorphisms ϕ_1 and ϕ_2, respectively). We note that for $i = 0, \pm 1, \pm 2, \cdots$,

(1) $F_1(\text{int } B_1 \times E^1) \supset (B_1 \times E^1) \cup \left\{ \bigcup_i (H^3 - \text{int } B_2) \times 2i \right\}$,

(2) F_1 carries each region of the form $H^3 \times [2i - 1, 2i + 1]$ onto itself,

(3) $F_2(\text{int } B_2 \times E^1) \supset (B_2 \times E^1) \cup \left\{ \bigcup_i (H^3 - \text{int } B_1) \times (2i + 1) \right\}$, and

(4) F_2 carries each region of the form $H^3 \times [2i, 2i + 2]$ onto itself.

From the definition of F_1 and F_2, it follows that there exist real numbers δ_1 and δ_2 $(0 < \delta_1 < 1/4,\ 0 < \delta_2 < 1/4)$ such that

(1)* $F_1(\text{int } B_1 \times E^1) \supset (B_1 \times E^1) \cup \left\{ \bigcup_i (H^3 - \text{int } B_2) \times [2i - \delta_1, 2i + \delta_1] \right\}$ and

(2)* $F_2(\text{int } B_2 \times E^1) \supset (B_2 \times E^1) \cup \left\{ \bigcup_i (H^3 - \text{int } B_1) \times [2i+1-\delta_2,\ 2i+1+\delta_2] \right\}$.

We now consider the regions of the form $H^3 \times [2i-1,\ 2i+1]$. Let γ_i be the homeomorphism carrying the interval $[2i-1,\ 2i+1]$ onto itself, by carrying the intervals

$$[2i-1,\ 2i-\delta_1],\quad [2i-\delta_1,\ 2i],\quad [2i,\ 2i+\delta_1],\quad [2i+\delta_1,\ 2i+1]$$

linearly onto the intervals

$$[2i-1,\ 2i-1+\delta_2],\quad [2i-1+\delta_2,\ 2i],\quad [2i,\ 2i+1-\delta_2],\quad [2i+1-\delta_2,\ 2i+1],$$

respectively. Let F_3 be the homeomorphism of $H^3 \times E^1$ onto itself defined by $F_3(x, t) = (x, \gamma_i(t))$ for $(x, t) \in H^3 \times [2i-1,\ 2i+1]$. Then

$$F_3\left((B_1 \times E^1) \cup \left\{ \bigcup_i (H^3 - \text{int } B_2) \times [2i-\delta_1,\ 2i+\delta_1] \right\} \right)$$
$$= (B_1 \times E^1) \cup \left\{ \bigcup_i (H^3 - \text{int } B_2) \times [2i-1+\delta_2,\ 2i+1-\delta_2] \right\}.$$

Since

$$(B_1 \times E^1) \cup \left\{ \bigcup_i (H^3 - \text{int } B_2) \times [2i-1+\delta_2,\ 2i+1-\delta_2] \right\} \cup (B_2 \times E^1)$$
$$\cup \left\{ \bigcup_i (H^3 - \text{int } B_1) \times [2i+1-\delta_2,\ 2i+1+\delta_2] \right\} = H^3 \times E^1,$$

it follows by properties (1)* and (2)* that

$$F_3 \circ F_1(\text{int } B_1 \times E^1) \cup F_2(\text{int } B_2 \times E^1) = H^3 \times E^1.$$

Hence, defining h to be the homeomorphism $F_2^{-1} \circ F_3 \circ F_1$, we see that $h(\text{int } B_1 \times E^1) \cup (\text{int } B_2 \times E^1) = H^3 \times E^1$, and the proof of Theorem 3 is now complete.

In [10] it was shown that if S^{n-1} $(n \geq 4)$ is embedded in S^n so as to be locally flat except perhaps for a subset C of a Cantor set such that C lies on a flat arc in S^{n-1} and a flat arc in S^n, then S^{n-1} is flat in S^n. That is, the closure of each complementary domain of S^{n-1} in S^n is a cell. In particular, if $S^{n-1} \subset S^n$ $(n \geq 4)$ is locally flat modulo two points, then S^{n-1} is flat in S^n. By means of this difficult result, we can improve Theorem 1 rather easily and get a result from which Theorem 2 follows trivially.

THEOREM 4. *Assume the* ΣH-*Conjecture; if* H^3 *is a homotopy* 3-*sphere and* F^3 *is a fake cube, then* $\Sigma F^3 = I^4$ *and* $\Sigma H^3 = S^4$.

Proof. We may suppose that $F^3 \subset H^3$ and that there exists a piecewise linear 3-ball B_2 in H^3 such that $H^3 - \text{int } B_2 = F^3$. That is, given H^3, let B_2 be some 3-simplex of H^3, and let $F^3 = H^3 - \text{int } B_2$. Given F^3, let B_2 be an arbitrary 3-ball, and let $H^3 = F^3 \cup B_2$, where the boundary of B_2 is identified with the boundary of F^3 by some piecewise linear homeomorphism.

It is now necessary to recall some of the ideas from the proof of Theorem 2. Taking B_1 to be a piecewise linear 3-cell in int F^3, we have (by the ΣH-Conjecture) a homeomorphism h of $H^3 \times E^1$ onto itself such that

$$h(\text{int } B_1 \times E^1) \cup (\text{int } B_2 \times E^1) = H^3 \times E^1 .$$

As in the proof of Theorem 2, we let ΣH^3 be the two-point compactification of $H^3 \times E^1$. Then $\Sigma F^3 \subset \Sigma H^3$, $\Sigma B_2 \subset \Sigma H^3$, and

$$C = h(B_1 \times E^1) \cup \{\omega\} \cup \{-\omega\} \subset \Sigma H^3,$$

so that $\Sigma F^3 \subset C$, $\Sigma F^3 - \{\omega\} - \{-\omega\} \subset \text{int } C$, and $\Sigma F^3 \cup \Sigma B_2 = \Sigma H^3$, where $\Sigma F^3 \cap \Sigma B_2 = \Sigma\partial F^3$ ($= \Sigma\partial B_2 = S^3$). Also, both C and ΣB_2 are 4-cells.

We now consider the 4-sphere 2C and one copy of ΣF^3 lying in 2C. The 3-sphere $\Sigma\partial F^3$ is embedded in the 4-sphere 2C so that it is locally flat except perhaps at the "suspension" points ω and $-\omega$. Hence, by the results of [10], it follows that $\Sigma\partial F^3$ is flat in 2C and that $\Sigma F^3 = I^4$. Since $\Sigma H^3 = \Sigma F^3 \cup \Sigma B_2$, it then follows that $\Sigma H^3 = S^4$.

4. A PARTIAL RESULT

Here we give a result that illustrates how some of the present techniques can be applied toward a solution of the ΣH-Conjecture. The result itself appears to lead to a dead end, but perhaps someone will be clever enough to obtain the desired proof by an appropriate modification or by a new approach. The result, already mentioned in the introduction, is as follows.

THEOREM 5. *Let* H^3 *be a homotopy* 3*-sphere, and let* B_1 *and* B_2 *be two disjoint piecewise linear* 3*-cells in* H^3 *under some combinatorial triangulation of* H^3. *If* D_1 *and* D_2 *are piecewise linear* 3*-cells in* int B_1 *and* int B_2*, respectively, and if* α, β, *and* ε *are real numbers* ($\alpha < \beta$, $\varepsilon > 0$), *then there exist homeomorphisms* h_1 *and* h_2*, taking* $H^3 \times E^1$ *onto itself, such that*

(1) $h_1(\text{int } B_1 \times E^1) \cup h_2(\text{int } B_2 \times E^1) \supset H^3 \times [\alpha, \beta]$,

(2) $h_1 = \text{identity}$ *on* $D_1 \times E^1 \cup H^3 \times (-\infty, \alpha - \varepsilon] \cup H^3 \times [\beta + \varepsilon, \infty)$, *and*

(3) $h_2 = \text{identity}$ *on* $D_2 \times E^1 \cup H^3 \times (-\infty, \alpha - \varepsilon] \cup H^3 \times [\beta + \varepsilon, \infty)$.

Proof. Let $\hat{B}_1$, $\hat{B}_2$, $\tilde{B}_1$, and $\tilde{B}_2$ be piecewise linear 3-cells such that

$$D_i \subset \text{int } \hat{B}_i \subset \hat{B}_i \subset \text{int } \tilde{B}_i \subset \tilde{B}_i \subset \text{int } B_i \qquad (i = 1, 2).$$

We suppose that H^3 is given a combinatorial triangulation K such that D_1, D_2, B_1, B_2, $\hat{B}_1$, $\hat{B}_2$, $\tilde{B}_1$, and $\tilde{B}_2$ are contained in K as subcomplexes. Let $m = (\alpha + \beta)/2$, and suppose γ is a real number such that $0 < \gamma < (\beta - \alpha)/2$. Let $\delta_1 = m - \gamma$ and $\delta_2 = m + \gamma$. Now $H^3 \times [\delta_1, \delta_2]$ has a natural combinatorial triangulation (the "product" triangulation) induced from the triangulation K of H^3 such that

$$K \times m, \quad \tilde{B}_1 \times [\delta_1, \delta_2], \quad \tilde{B}_2 \times [\delta_1, \delta_2], \quad \hat{B}_1 \times [\delta_1, \delta_2], \quad \text{and } \hat{B}_2 \times [\delta_1, \delta_2]$$

are contained as subcomplexes in the triangulation of $H^3 \times [\delta_1, \delta_2]$. Also, $M = H^3 \times (\delta_1, \delta_2)$ has a triangulation locally compatible with the triangulation of $H^3 \times [\delta_1, \delta_2]$, so that $K \times m$ is a subcomplex of the triangulation of M.

Let U and V be the open subsets of M, defined by

$$U = \operatorname{int} B_1 \times (\delta_1, \delta_2) \quad \text{and} \quad V = \operatorname{int} B_2 \times (\delta_1, \delta_2).$$

Let C_1 and C_2 be the closed subsets of M defined by $C_1 = \tilde{B}_1 \times (\delta_1, \delta_2)$ and $C_2 = \tilde{B}_2 \times (\delta_1, \delta_2)$. Then $C_1 \subset U$, $C_2 \subset V$, and each of the pairs $(M - C_1, U - C_1)$ and $(M - C_2, V - C_2)$ is 1-connected. Let $\tilde{K}$ denote the 1-skeleton of K, and let L be the subcomplex of K' maximal with respect to missing $\tilde{K}'$. Then both $\tilde{K}'$ and L are compact 1-dimensional subcomplexes of K'. Also, every simplex of K' is the join of a simplex of $\tilde{K}'$ and a simplex of L.

We now can apply the engulfing theorem of [14]. That is, there exist compact sets $Z_1 \subset M - C_1$ and $Z_2 \subset M - C_2$ and piecewise linear homeomorphisms f_1 and f_2 of M onto itself such that

$$f_1(U) \supset \tilde{K}' \times m, \quad f_2(V) \supset L \times m,$$

$$f_1 = \text{identity outside of } Z_1, \quad f_2 = \text{identity outside of } Z_2.$$

It follows that f_1 is the identity on a neighborhood of $\tilde{B}_1 \times (\delta_1, \delta_2)$ and f_2 is the identity on a neighborhood of $\tilde{B}_2 \times (\delta_1, \delta_2)$.

Let $\tilde{U} = f_1(U) \cap (K' \times m)$ and $\tilde{V} = f_2(V) \cap (K' \times m)$. Then $\tilde{U}$ and $\tilde{V}$ are open subsets of $K' \times m$ such that $\tilde{K}' \times m \subset \tilde{U}$ and $L \times m \subset \tilde{V}$. Let $\hat{U}$, $\hat{V}$, $\tilde{K}'$, and L be the corresponding sets in K'. By Lemma 8.1 of [14], there exists a piecewise linear homeomorphism $\hat{f}_3$ of K' onto itself "pushing" $\hat{U}$ towards L so that $\hat{f}_3(\hat{U}) \cup \hat{V} = K'$. Also, $\hat{f}_3$ can be defined so that it is the identity on $\tilde{K}' \cup L$ and carries each simplex K' onto itself. Furthermore, $\hat{f}_3$ is isotopic to the identity by an isotopy $\hat{F}_t$ such that for each t $(0 \leq t \leq 1)$ $\hat{F}_t$ carries each simplex K' onto itself, $\hat{F}_1$ = identity, and $\hat{F}_0 = \hat{f}_3$.

We now want to modify the isotopy $\hat{F}_t$ to obtain an isotopy F_t such that for each t $(0 \leq t \leq 1)$

$$F_t = \text{identity on } \hat{B}_1', \quad F_1 = \text{identity}, \quad F_0(\hat{U}) \cup \hat{V} = K'.$$

We note that since f_1 = identity on $\tilde{B}_1 \times (\delta_1, \delta_2)$ and f_2 = identity on $\tilde{B}_2 \times (\delta_1, \delta_2)$, the subcomplexes $\tilde{B}_1'$ and $\tilde{B}_2'$ of K' lie in $\hat{U}$ and $\hat{V}$, respectively. Also, since $\hat{F}_t$ carries each simplex of K' onto itself, for each t, it follows that $\hat{F}_t$ carries each of $\hat{B}_1'$, $\hat{B}_2'$, $\tilde{B}_1'$, and $\tilde{B}_2'$ onto itself for each t, and $\hat{F}_t(\hat{U}) \supset \tilde{B}_1'$ for each t. In particular, $\hat{F}_t$ carries $\partial\tilde{B}_1'$ onto itself for each t.

Since $\hat{B}_1' \subset \operatorname{int} \tilde{B}_1'$, we can suppose (by [9]) that we have a homeomorphism

$$k: \partial\tilde{B}_1' \times [0, 1] \rightarrow (\tilde{B}_1' - \operatorname{int} \hat{B}_1'),$$

where $k(w, 0) = w$ for $w \in \partial\tilde{B}_1'$ and $k(\partial\tilde{B}_1' \times 1) = \partial\hat{B}_1'$. Thus each point of $(\tilde{B}_1' - \operatorname{int} \hat{B}_1')$ is of the form k(w, s), where $w \in \partial\tilde{B}_1'$ and $s \in [0, 1]$. Let the isotopy F_t of K' onto itself be defined by the conditions

$$F_t = \begin{cases} \text{identity} & \text{on } \hat{B}_1', \\ \hat{F}_t & \text{on } K' - \operatorname{int} \tilde{B}_1', \\ k(\hat{F}_{(1-s)t+s}(w), s) & \text{on } k(w, s) \in \tilde{B}_1' - \operatorname{int} \hat{B}_1'. \end{cases}$$

As we noted above, $\hat{F}_t$ carries $\partial\tilde{B}_1'$ onto itself for all t. Hence, for each $s \in [0, 1]$, F_t carries $k(\partial\tilde{B}_1' \times s)$ onto itself. In particular, since $k(w, 0) = w$ for $w \in \partial\tilde{B}_1'$, it follows for $s = 0$ that

$$F_t(w) = F_t(k(w, 0)) = k(\hat{F}_t(w), 0) = \hat{F}_t(w).$$

Also, since $\hat{F}_1$ = identity and $k(\partial\tilde{B}_1' \times 1) = \partial\hat{B}_1'$, it follows for the special case $s = 1$ that $F_t(k(w, 1)) = k(\hat{F}_1(w), 1) = k(w, 1)$. Therefore, the various isotopies match up correctly, and F_t is well-defined.

Since $\hat{F}_1$ = identity, $F_1(k(w, s)) = k(\hat{F}_1(w), s) = k(w, s)$ and hence F_1 = identity. Clearly, F_t = identity on $\hat{B}_1'$ for all t. Finally, since F_t carries each of $\tilde{B}_1'$ and $K' - \text{int}\,\tilde{B}_1'$ onto itself for each t, since $\tilde{B}_1' \subset \hat{U}$ and $F_t = \hat{F}_t$ on $K' - \text{int}\,\tilde{B}_1'$, and since $\hat{F}_0(\hat{U}) \cup \hat{V} = K'$, it follows that $F_0(\hat{U}) \cup \hat{V} = K'$ also.

Let δ_3 be a real number $(0 < \delta_3 < \gamma)$. Then $\delta_1 = m - \gamma < m - \delta_3 < m + \gamma = \delta_2$. Let the homeomorphism f_3 of $H^3 \times (\delta_1, \delta_2)$ onto itself be defined by the rule

$$f_3 = \begin{cases} \text{identity} & \text{on } H^3 \times (\delta_1, m - \delta_3] \cup H^3 \times [m + \delta_3, \delta_2), \\ (F_t(x), (1-t)m + t(m + \delta_3)) & \text{on } (x, (1-t)m + t(m+\delta_3)) \in H^3 \times [m, m + \delta_3], \\ (F_t(x), (1-t)m + t(m - \delta_3)) & \text{on } (x, (1-t)m + t(m - \delta_3)) \in H^3 \times [m - \delta_3, m]. \end{cases}$$

Since $f_1(U) \supset \tilde{K}' \times m$, $f_2(V) \supset L \times m$, and $F_0(\hat{U}) \cup \hat{V} = H^3$, it follows that

$$f_3 \circ f_1(\text{int}\, B_1 \times (\delta_1, \delta_2)) \cup f_2(\text{int}\, B_2 \times (\delta_1, \delta_2)) \supset H^3 \times m.$$

Since f_1 is the identity on $\tilde{B}_1 \times (\delta_1, \delta_2)$ and F_t is the identity on $\hat{B}_1$ for all t, it follows that $f_3 \circ f_1$ = identity on $\hat{B}_1 \times (\delta_1, \delta_2)$. Since f_1 = identity outside Z_1, f_2 = identity outside Z_2, and f_3 = identity outside $H^3 \times [m - \delta_3, m + \delta_3]$, each can be extended by the identity to all of $H^3 \times E^1$. Let us denote the homeomorphism extending $f_3 \circ f_1$ by g_1, and the homeomorphism extending f_2 by g_2.

We now have homeomorphisms g_1, g_2 of $H^3 \times E^1$ onto itself such that

(1) $g_1(\text{int}\, B_1 \times E^1) \cup g_2(\text{int}\, B_2 \times E^1) \supset H^3 \times m$,

(2) g_1 = identity on $\hat{B}_1 \times E^1 \cup H^3 \times (-\infty, \delta_1] \cup H^3 \times [\delta_2, \infty)$,

(3) g_2 = identity on $\hat{B}_2 \times E^1 \cup H^3 \times (-\infty, \delta_1] \cup H^3 \times [\delta_2, \infty)$.

We are now ready to construct the homeomorphisms h_1, h_2 promised in Theorem 5. Since $g_1(\text{int}\, B_1 \times E^1) \cup g_2(\text{int}\, B_2 \times E^1) \supset H^3 \times m$, there is a real number δ_4 $(0 < \delta_4 < \gamma)$ such that

$$g_1(\text{int}\, B_1 \times E^1) \cup g_2(\text{int}\, B_2 \times E^1) \supset H^3 \times [m - \delta_4, m + \delta_4].$$

We first consider the interval $[\alpha - \varepsilon, \beta + \varepsilon]$. Let ϕ be the homeomorphism that takes $[\alpha - \varepsilon, \beta + \varepsilon]$ onto itself by sending the intervals

$$[\alpha - \varepsilon, m - \delta_4], \quad [m - \delta_4, m], \quad [m, m + \delta_4], \quad [m + \delta_4, \alpha + \varepsilon]$$

linearly onto the intervals $[\alpha - \varepsilon, \alpha]$, $[\alpha, m]$, $[m, \beta]$, and $[\beta, \beta + \varepsilon]$, respectively.

We consider the sets $\hat{B}_1 - \text{int}\, D_1$ and $\hat{B}_2 - \text{int}\, D_2$. By [9], we can suppose that for $i = 1, 2$, we have a homeomorphism

$$k_i: \partial \hat{B}_i \times [0, 1] \to \hat{B}_i - \operatorname{int} D_i$$

such that $k_i(w, 0) = w$ for $w \in \partial \hat{B}_i$ and $k_i(\partial \hat{B}_i \times 1) = \partial D_i$. For $i = 1, 2$, let G_i be the homeomorphism of $H^3 \times E^1$ onto itself defined as follows.

$$G_i = \begin{cases} \text{identity} & \text{on } H^3 \times (-\infty, \alpha - \varepsilon] \cup H^3 \times [\beta + \varepsilon, \infty), \\ \text{identity} & \text{on } D_i \times E^1, \\ (x, \phi(t)) & \text{on } (x, t) \in (H^3 - \operatorname{int} \hat{B}_i) \times [\alpha - \varepsilon, \beta + \varepsilon], \\ (k_i(w, s), (1 - s)\phi(t) + st) & \text{on } (k_i(w, s), t) \in (\hat{B}_i - \operatorname{int} D_i) \times [\alpha - \varepsilon, \beta + \varepsilon]. \end{cases}$$

Since

$$g_1(\operatorname{int} B_1 \times E^1) \cup g_2(\operatorname{int} B_2 \times E^1) \supset H^3 \times [m - \delta_4, m + \delta_4],$$

$$g_1 = \text{identity} \quad \text{on } (\hat{B}_1 \times E^1) \cup (H^3 \times (-\infty, \delta_1]) \cup (H^3 \times [\delta_2, \infty)),$$

$$g_2 = \text{identity} \quad \text{on } (\hat{B}_2 \times E^1) \cup (H^3 \times (-\infty, \delta_1]) \cup (H^3 \times [\delta_2, \infty)),$$

$$\phi([m - \delta_4, m + \delta_4]) = [\alpha, \beta],$$

it follows that

$$G_1 \circ g_1(\operatorname{int} B_1 \times E^1) \cup G_2 \circ g_2(\operatorname{int} B_2 \times E^1) \supset H^3 \times [\alpha, \beta],$$

$$G_1 \circ g_1 = \text{identity} \quad \text{on } (D_1 \times E^1) \cup (H^3 \times (-\infty, \alpha - \varepsilon]) \cup (H^3 \times [\beta + \varepsilon, \infty)),$$

$$G_2 \circ g_2 = \text{identity} \quad \text{on } (D_2 \times E^1) \cup (H^3 \times (-\infty, \alpha - \varepsilon]) \cup (H^3 \times [\beta + \varepsilon, \infty)).$$

Therefore, if we define $h_1 = G_1 \circ g_1$ and $h_2 = G_2 \circ g_2$, the proof of Theorem 5 is complete.

REFERENCES

1. R. H. Bing, *An alternative proof that* 3-*manifolds can be triangulated.* Ann. of Math. (2) 69 (1959), 37-65.

2. ———, *Some aspects of the topology of* 3-*manifolds related to the Poincaré conjecture.* Lectures on Modern Mathematics, Vol. II, 93-128; John Wiley and Sons, Inc., New York, 1964.

3. M. Brown, *A proof of the generalized Schoenflies theorem.* Bull. Amer. Math. Soc. 66 (1960), 74-76.

4. J. C. Cantrell, *Separation of the* n-*sphere by an* (n - 1)-*sphere.* Trans. Amer. Math. Soc. 108 (1963), 185-194.

5. ———, *Non-flat embeddings of* S^{n-1} *in* S^n. Michigan Math. J. 10 (1963), 359-362.

6. J. C. Cantrell and C. H. Edwards, Jr., *Almost locally polyhedral curves in Euclidean* n-*space.* Trans. Amer. Math. Soc. 107 (1963), 451-457.

7. L. C. Glaser, *Geometrical combinatorial topology.* Lecture notes, Rice University (revised 1967), 300 pages.

8. ———, *On the double suspension of certain homotopy* 3-*spheres.* Ann. of Math. (2) 85 (1967), 494-507.

9. J. F. P. Hudson and E. C. Zeeman, *On regular neighborhoods.* Proc. London Math. Soc. (3) 14 (1964), 719-745.

10. R. C. Kirby, *On the set of non-locally flat points of a submanifold of codimension one.* Ann. of Math. (to appear).

11. T. M. Price, *Equivalence of embeddings of* k-*complexes in* E^n *for* $n \leq 2k + 1$. Michigan Math. J. 13 (1966), 65-69.

12. J. R. Stallings, *Polyhedral homotopy-spheres.* Bull. Amer. Math. Soc. 66 (1960), 485-488.

13. ———, *The piecewise-linear structure of Euclidean space.* Proc. Cambridge Philos. Soc. 58 (1962), 481-488.

14. ———, *On topologically unknotted spheres.* Ann. of Math. (2) 77 (1963), 490-503.

15. J. H. C. Whitehead, *Simplicial spaces, nuclei and* m-*groups.* Proc. London Math. Soc. (2) 45 (1939), 243-327.

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