MONOTONIC SINGULAR FUNCTIONS OF HIGH SMOOTHNESS

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This paper is concerned with the construction of monotonic functions that are *singular* (that is, have derivative zero almost everywhere) and possess good continuity properties as measured by the modulus of smoothness. We consider the operator Δ_h: $f(x) \to f(x+h) - f(x)$, and we recall that the *modulus of continuity* of f is the function

$$\omega(t) = \sup_x \sup_{0 \le h \le t} |\Delta_h f(x)|;$$

more generally, the (r^{th}-order) modulus of smoothness $\omega_r(t)$ is defined when Δ_h is replaced by the r^{th}-difference operator Δ_h^r. We shall refer to ω_2 simply as the *modulus of smoothness.*

Clearly, $\omega(t) = O(t)$ implies f is absolutely continuous. It is known that any bound on ω that does not imply $f \in \text{Lip } 1$ is compatible with the existence of an increasing singular function whose modulus of continuity does not exceed ω. This seems implicit in a construction of F. Hausdorff (see [4, p. 30], also our paper [8]). *Added in proof.* The result was proved by P. Hartman and R. Kershner, *The structure of monotone functions,* Amer. J. Math. 59 (1937), 809-822 (see p. 818). We are indebted to P. L. Duren for this reference.

On the other hand, it is remarkable that the (Zygmund) class Z of functions for which $\omega_2(t) = O(t)$ contains increasing singular functions. This was first deduced in [2] (it underlies a long-known counterexample in the theory of conformal mapping). The first direct construction was given by G. Piranian [6]. Another construction, due to J.-P. Kahane, is mentioned without detailed verification in [6]. Piranian also outlines a proof that there is a singular function with $\omega_2(t) = o(t)$. Our main result (see Theorem 2) is the construction of an increasing singular function with $\omega_2(t) = O(t\,|\log t|^{-1/2})$, and this is essentially an unimprovable result (see the following paragraph). Our method is an adaptation of the basic idea (selective successive modifications) underlying Piranian's construction. The main novelty in our construction is the choice, as a "basic building block," of a trigonometric polynomial that vanishes very smoothly at the end points. Our choice has the two-fold advantage over Piranian's cubic polynomial that all of our successive approximations are twice differentiable (this simplifies the estimation of $\omega_2(t)$), and that for the proof of singularity we are able to invoke a known theorem on the convergence of lacunary trigonometric series. This construction, together with the extension to higher-order moduli of smoothness, is given in Section 1.

From the other side, M. Weiss and A. Zygmund [10] have shown that if $\omega_2(t) = O(t\,|\log t|^{-c})$ for some $c > 1/2$, then f is absolutely continuous and in fact has a derivative of class L^p for every $p < \infty$. Their proof is based on the theory of trigonometric series. They showed that their theorem becomes false for $c = 1/2$, by exhibiting a function f for which f' exists almost nowhere and

$$\omega_2(t) = O(t\,|\log t|^{-1/2}).$$

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Thus, our construction shows the Weiss-Zygmund theorem to be essentially best possible even for *monotonic* functions. F. John and L. Nirenberg [3] deduced afresh the Weiss-Zygmund theorem in a sharper form, and they also generalized it to higher dimensions, from a general theorem on "mean oscillation" of functions. Various refinements and extensions of the Weiss-Zygmund theorem were obtained by E. Stein and Zygmund (see [9], where further references are given). In particular, it follows from their work that if

$$\int_0^1 t^{-3}[\omega_2(t)]^2\,dt < \infty,$$

then f is absolutely continuous. (The author is indebted to L. Carleson for bringing [9] to his attention, and for pointing out to him the very simple proof of this last proposition; for the reader's convenience we have included it in an appendix). It is easy to adapt the construction in Theorem 2 so that it yields an increasing singular function with modulus of smoothness $O(\omega(t))$, provided only that ω is monotonic and $\int_0^1 t^{-3}[\omega(t)]^2\,dt = \infty$.

We call the reader's attention to an analogous circle of problems concerning the functional relation $\frac{f(x+t) - f(x)}{f(x) - f(x-t)} = 1 + O(\psi(t))$, where f is continuous and increasing. This arises in the study of the boundary behavior of quasi-conformal mappings (see [5, pp. 85 ff.], and especially Carleson [1]).

In Section 2 we apply the reasoning of Section 1 to produce an example of "interference," whereby an absolutely continuous and a purely singular function can combine so that their sum is a function much smoother (as measured by the modulus of continuity) than either component. This solves a problem that had been communicated by the author to Kahane, who also obtained a solution. Kahane's method is more direct, and his result more general (see Section 2 for details). We wish here also to thank J.-P. Kahane for kindly sending us, upon request, the detailed verification of the example referred to above, outlined in [6].

Finally, in Section 3 we apply results of Section 1 to supplement the discussion in [2].

1. SINGULAR FUNCTIONS WITH SMALL MODULUS OF SMOOTHNESS

The following theorem is in essence due to Piranian [6]. We have added slight refinements to the formulation, which will be necessary for the discussion in Section 3. Actually, Theorem 1 is contained in Theorem 2 below; but because of its applications, it is desirable to give a direct proof, which can then be modified to give Theorem 2, and which avoids the rather deep theorem on lacunary trigonometric series needed for Theorem 2.

THEOREM 1. *Let* $f_0(x)$ *be a twice differentiable, nondecreasing function on* $[0, 2\pi]$ *such that* $|f_0''(x)| \leq 1$. *Then, for every* $\varepsilon > 0$, *there exists a nondecreasing singular function* $f(x)$ *on* $[0, 2\pi]$ *such that*

(1) $$|f_0(x) - f(x)| < \varepsilon,$$

(2) $$|f(x+h) - 2f(x) + f(x-h)| \leq 5h \qquad \textit{whenever } 0 \leq x - h < x + h \leq 2\pi.$$

Proof. By an inductive process, we shall define a sequence $\{f_n\}$ of nondecreasing functions that converges to the desired f. We shall use as our basic building block the function

$$g(x) = \frac{\sin x}{2} - \frac{\sin 2x}{4}.$$

Note that g vanishes, together with its first two derivatives, at $x = 0$ and at $x = 2\pi$, and that

$$|g(x)| \leq \frac{3}{4}, \qquad |g'(x)| \leq 1, \qquad |g''(x)| < \frac{3}{2}.$$

As we already remarked, the underlying idea in the following construction is due to Piranian, to whose paper we refer for motivation.

Let us suppose that $f_{n-1}(x)$ has either been given or constructed, for some $n \geq 1$, and that it is nondecreasing and twice differentiable. To define $f_n(x)$, we first divide $[0, 2\pi]$ into $n!$ equal intervals of length $\delta_n = 2\pi/n!$. For the i^{th} of these, we write $m_{n,i} = \min f'_{n-1}(x)$. On the i^{th} interval we now define

$$f_n(x) = f_{n-1}(x) + bm^*_{n,i}\,\phi_n(x),$$

where

$$m^*_{n,i} = \min(1, m_{n,i}) \quad \text{and} \quad \phi_n(x) = \frac{g(n!\,x)}{n!}.$$

(Here b is a positive number not exceeding 1, fixed throughout the discussion, and to be chosen later in terms of ε. If the reader is not interested in (1), he may simply take $b = 1$).

Denoting $f_n - f_{n-1}$ by u_n, we observe that $u_n(x)$ is twice differentiable on $[0, 2\pi]$ and that it vanishes with its first two derivatives at all points $j\delta_n$ $(j = 0, 1, \cdots, n!)$. For later purposes, it is essential to observe that for fixed n, j the sequence $\{f_i(j\delta_n)\}$ remains constant for $i \geq n - 1$, and that the same is true of $\{f'_i(j\delta_n)\}$ and $\{f''_i(j\delta_n)\}$. Further,

(3) $$|u_n(x)| \leq \frac{b}{n!},$$

(4) $$|u''_n(x)| < \frac{3}{2} bn!.$$

Moreover, the maximum of $|u'_n(x)|$ in the i^{th} subinterval does not exceed $m_{n,i}$, so that $f'_n(x) \geq 0$ on $[0, 2\pi]$ and f_n is nondecreasing.

We now claim that with the choice $b = \min\left(\frac{1}{14}, \frac{\varepsilon}{3}\right)$, the sequence $\{f_n(x)\}$ converges to a function f with the desired properties. Note first that because of (3), $\{f_n(x)\}$ converges uniformly on $[0, 2\pi]$, since $f_n = f_0 + \sum_{i=1}^n u_i$. Clearly, the limit function f is continuous and nondecreasing.

Proof of (2). Denoting by Δ_h^2 the operator $f \to f(x+h) - 2f(x) + f(x-h)$, we observe that

$$|\Delta_h^2 f_n| \leq |\Delta_h^2 f_0| + \sum_{i=1}^{n} |\Delta_h^2 u_i|.$$

By the mean-value theorem, $\Delta_h^2 f_0 = h^2 f_0''(\xi)$ for some ξ; hence $|\Delta_h^2 f_0| \leq 4h$. Similarly, using (4), we see that

$$|\Delta_h^2 u_i| \leq h^2 \max |u_i''(x)| < \frac{3bh^2 n!}{2}.$$

From (3), we also have the inequality $|\Delta_h^2 u_i| < 4b/n!$. Thus

$$|\Delta_h^2 f_n| < 4h + b \sum_{m=1}^{n} \min\left(\frac{3}{2}h^2 m!, \frac{4}{m!}\right).$$

Letting n go to ∞, we find that

$$\frac{|\Delta_h^2 f|}{h} \leq 4 + b \sum_{m=1}^{\infty} \min\left(\frac{3}{2}h\, m!, \frac{4}{hm!}\right).$$

To estimate the sum on the right, let p be the unique positive integer for which $p! \leq 4/h < (p+1)!$. Then the sum does not exceed

$$\frac{3}{2}h \sum_{m=1}^{p} m! + \frac{4}{h} \sum_{m=p+1}^{\infty} \frac{1}{m!} \leq \frac{3}{2} \cdot \frac{4}{p!} \cdot 2p! + (p+1)! \left(\frac{2}{(p+1)!}\right) = 14.$$

Thus, (2) holds if $b \leq 1/14$.

Proof of (1). If in addition $b < \varepsilon/2$, then (1) follows immediately from (3).

Proof that f *is singular.* Note first that, because of (4),

$$|f_{n-1}''(x)| \leq 1 + \frac{3b}{2} \sum_{1}^{n-1} i! \leq (n-1)! \qquad \text{for } n \geq 3 \text{ and } b \leq 1/14.$$

Hence, at the n^{th} stage, we have in the i^{th} subinterval of length $2\pi/n!$ the estimates

(5) $$m_{n,i} \leq f_{n-1}'(x) \leq m_{n,i} + (n-1)!\, 2\pi/n! < m_{n,i} + 7/n.$$

Let now t be any point such that $f'(t)$ exists. We claim that

(6) $$\lim_{n\to\infty} f_n'(t) = f'(t).$$

Indeed, following Piranian's argument, we observe that $f'(t)$ is the limit of the difference quotients of $f(x)$ formed with respect to a sequence of successively smaller intervals (of the basic subdivisions) containing t. That is, writing $\delta_n = 2\pi/n!$ and $i\delta_n \leq t < (i+1)\delta_n$, we know that for large n the derivative $f'(t)$ is approximated arbitrarily closely by

$$\frac{f((i+1)\delta_n) - f(i\delta_n)}{\delta_n} = \frac{f_{n-1}((i+1)\delta_n) - f_{n-1}(i\delta_n)}{\delta_n},$$

and that is equal to $f'_{n-1}(x_n)$ for some x_n in the interval $(i\delta_n, (i+1)\delta_n)$. Since, by (5), $|f'_{n-1}(x_n) - f'_{n-1}(t)| \leq \frac{7}{n}$, we see that (6) holds.

Finally, let E denote the set on which $f'(x) > 0$. Let us write $E = \bigcup_{k=0}^{\infty} E_k$, where

$$E_0 = \{x \mid f'(x) > 1\}, \qquad E_k = \left\{ x \,\middle|\, \frac{1}{k+1} < f'(x) \leq \frac{1}{k} \right\} \quad (k = 1, 2, \cdots).$$

To show that E has measure zero, it is enough to show that each E_k has measure zero. If $x \in E_k$, then from some n onward $f'_{n-1}(x) > 1/(k+1)$, and thus, for n large enough, (5) implies that $f'_{n-1}(y) > 1/(k+1)$ throughout the basic subinterval of the n^{th} subdivision [call it $I_n(x)$] containing x. Therefore, from the way in which f_n was defined, we have for $y \in I_n(x)$ and all $n > n_x$ the formula

$$f_n(y) = f_{n-1}(y) + C_n \phi_n(y), \tag{7}$$

where C_n is a constant not less than $b/(k+1)$. Differentiating (7) at the point $y = x$, we find that

$$f'_n(x) - f'_{n-1}(x) = C_n \phi'_n(x),$$

and since $\lim f'_n(x)$ exists, we deduce that

$$\lim_{n \to \infty} \phi'_n(x) = 0 \qquad \text{for all } x \in E_k. \tag{8}$$

Since $\phi'_n(x) = \frac{1}{2}(\cos n!\,x - \cos 2n!\,x)$, it follows from (8) that

$$\lim_{n \to \infty} \int_{E_k} (\cos n!x - \cos 2n!\,x)^2 \, dx = 0. \tag{9}$$

By a simple application of the Riemann-Lebesgue lemma, the left side of (9) equals the measure of E_k, which therefore vanishes, and Theorem 1 is proved.

Remark 1. Since every polynomial is the difference of two increasing polynomials, we deduce from Theorem 1 that the singular functions belonging to the Zygmund class Z are dense in $C[0, 2\pi]$.

Remark 2. We can modify the preceding construction so that it yields a function f of class Z, satisfying (1), and with $f'(x) = 0$ on an open set of measure 2π. (Kahane's construction also gave such a function.) The idea is to define $f_n(x)$ as before on subintervals where $\min f'_{n-1}(x) > 1$, but on each other subinterval *replace* $f_{n-1}(x)$ by a function that is constant on the middle third of the subinterval. It is possible to carry this out in such a way that $f_n(x)$ is twice differentiable and the new functions $u_n = f_n - f_{n-1}$ have the essential properties of the previously defined u_n. However, the details are complicated, and we shall not present them.

Remark 3. The following variant of the proof of Theorem 1 is perhaps somewhat simpler. Fix a positive number α, and at the n^{th} stage define f_n to be $f_{n-1} + \alpha\phi_n$ if $m_{n,i} \geq \alpha$, and otherwise $f_n = f_{n-1}$. Then we deduce that the limit function f satisfies the inequality $f'(x) \leq \alpha$ a.e. Thus, if $f(2\pi) - f(0) > 2\pi\alpha$, f has a nonvanishing singular part f_s. The absolutely continuous part has a derivative bounded a.e. by α, and therefore, if α is small enough, f_s satisfies the requirements of the theorem.

THEOREM 2. *Under the hypotheses of Theorem* 1, *there exists a nondecreasing singular function* f *that satisfies* (1) *and such that, for all* $h \leq 1/2$,

$$|f(x+h) - 2f(x) + f(x-h)| \leq C h \, |\log h|^{-1/2}, \tag{10}$$

where C *is a constant independent of* x *and* h.

Proof. The proof is similar to that of Theorem 1, in the variant suggested in Remark 3. Let $\{b_n\}$ be a decreasing sequence of positive numbers such that $b_n \to 0$ and

$$\sum_{1}^{\infty} b_n^2 = \infty. \tag{11}$$

We successively subdivide the base interval as before, but now, in the i^{th} interval of the n^{th} subdivision, we define

$$f_n(x) = \begin{cases} f_{n-1}(x) & \text{if } m_{n,i} \leq b_n, \\ f_{n-1}(x) + b_n \phi_n(x) & \text{if } m_{n,i} > b_n. \end{cases}$$

Just as before, we obtain a nondecreasing limit function f(x), and (1) holds if $b_1 \leq \varepsilon/3$. We now verify that f has the two crucial properties.

a) *Proof that* f *is singular.* Let x be a point where f'(x) exists and is positive. As before, $f_n'(x) \to f'(x)$. Now, from some n_x onwards, b_n is less than the minimum of $f_{n-1}'(y)$ on the interval $I_n(x)$ of the n^{th} subdivision containing x, and thus $f_n(y) = f_{n-1}(y) + b_n \phi_n(y)$ on $I_n(x)$, hence

$$f_n'(x) = f_{n-1}'(x) + b_n \phi_n'(x).$$

Since this holds from some n onward, and $\lim f_n'(x)$ exists, we deduce that the series

$$\sum_{1}^{\infty} b_n (\cos n!\, x - \cos 2n!\, x) \tag{12}$$

converges at each point x where $f'(x) > 0$. However, it is known from the theory of lacunary series (see [11, p. 203]) that if (11) holds, (12) cannot converge on a set of positive measure. Therefore $f'(x) = 0$ a.e.

b) *Verification of* (10). Once again, $u_n = f_n - f_{n-1}$ is twice differentiable, and now it satisfies the conditions

$$|u_n(x)| < \frac{b_n}{n!}, \qquad |u_n''(x)| < b_n n!\,.$$

Therefore, since $|\Delta^2 f_0| \leq h^2$,

$$|\Delta^2 f|/h \leq h + \sum_{n=1}^{\infty} b_n \min\left(\tfrac{3}{2} h n!, \frac{4}{h n!}\right).$$

To complete the proof, we now choose $b_n = \frac{\varepsilon}{3}[n \log(n+2)]^{-1/2}$, and we must verify that the right side in the last inequality is $O(|\log h|^{-1/2})$. Clearly,

$$(13) \qquad \sum_{n=1}^{\infty} b_n \min\left(\frac{3}{2}hn!, \frac{4}{hn!}\right) \leq \frac{3}{2}h \sum_{n=1}^{p} n!\, b_n + \frac{4}{h} \sum_{n=p+1}^{\infty} \frac{b_n}{n!},$$

where p is determined by the condition

$$(14) \qquad p! \leq \frac{1}{h} < (p+1)!\ .$$

The second sum in the right hand member of (13) is bounded by

$$\frac{4b_{p+1}}{h} \sum_{n=p+1}^{\infty} \frac{1}{n!} < \frac{8b_{p+1}}{h(p+1)!} < 8b_{p+1},$$

and the first sum is bounded by

$$\frac{3h}{2}\left(b_1 \sum_{n=1}^{p-1} n! + p!\, b_p\right) < \frac{3h}{2}(2b_1 (p-1)! + p!\, b_p) \leq \frac{3}{2}\left(\frac{2b_1}{p} + b_p\right);$$

hence the left side of (13) is $O((p \log p)^{-1/2})$, and this, in view of (14), is $O(|\log h|^{-1/2})$. This completes the proof.

Remark. The analogues of Theorem 2 for higher-order moduli of smoothness are also true. The only difference in the proofs is that in place of $g(x)$ we take trigonometric polynomials that vanish together with sufficiently many derivatives at 0 and at 2π. The estimates on $\Delta_h^r f$ are fully analogous, the only change being that we now use the inequality

$$|\Delta_h^r u_n| \leq A \min\{(n!)^{r-1} h^r, 1/n!\}$$

to find once again that $|\Delta_h^r f| \leq C h\, |\log h|^{-1/2}$.

2. INTERFERENCE PHENOMENA

Consider the following question: if a function of bounded variation has a certain smoothness, is this "inherited" by its absolutely continuous and singular parts separately, or is an "interference" possible whereby the oscillations of the two components cancel each other? The method of Section 1 enables us to show that such interference is indeed possible.

THEOREM 3. *There exists a function of bounded variation that belongs to the class* Lip α *for every* $\alpha < 1$, *but whose singular and absolutely continuous parts belong to no Lipschitz class.*

We note that the construction in the proof of Theorem 2 can be applied to an arbitrary increasing function f_0 having a continuous second derivative on the open interval $(0, 2\pi)$; the process gives a convergent sequence $\{f_n\}$, and the limit function f is once again increasing and singular. The proof of singularity still works, since $f_0''(x)$ is bounded on $(\varepsilon, 2\pi - \varepsilon)$ for each fixed $\varepsilon > 0$, so that, as before, the variation of $f_{n-1}'(x)$ tends to zero in the basic intervals of the n^{th} subdivision that lie in $(\varepsilon, 2\pi - \varepsilon)$. We can therefore make the following assertion.

THEOREM 4. *Let* f_0 *be any increasing function with a continuous second derivative on the open interval* $(0, 2\pi)$. *Then there exists a function* u, *continuous on* $[0, 2\pi]$, *such that*

(i) $|u(x+h) - 2u(x) + u(x-h)| < Kh\,|\log h|^{-1/2}$ $\quad (0 < h < 1/2,$

$0 \leq x - h < x + h \leq 2\pi)$,

(ii) $f = f_0 + u$ *is increasing and singular.*

Now we can give the proof of Theorem 3. Suppose f_0 is continuous and increasing on $[0, 2\pi]$ and twice differentiable in $(0, 2\pi)$, and suppose it belongs to no Lipschitz class (for instance, take $f_0(x) = |\log x/8|^{-1}$. Applying Theorem 4 to this f_0, we obtain a function u with the desired properties. Indeed, u belongs to Z, and hence to Lip α for every $\alpha < 1$, and $u = f - f_0$, where f is singular; therefore $-f_0$, which belongs to no Lipschitz class, is the absolutely continuous part of u.

Remark 1. As we stated in the introduction, J.-P. Kahane has proved a theorem stronger than Theorem 3, namely: suppose that p(t) is increasing, that $\lim_{t\to 0} p(t)/t = \infty$, and that q is any continuous increasing function with $q(0) = 0$; then there exists a function of bounded variation with modulus of continuity O(p(t)) such that the modulus of continuity of its absolutely continuous part is not O(q(t)) (private communication).

Remark 2. Of course, if an *increasing* function has a certain modulus of continuity, this *is* inherited by its absolutely continuous part. On the other hand, the corresponding question for the modulus of smoothness is not yet settled.

Remark 3. Similar interference phenomena can also occur with respect to the (Jordan) representation of a function as the difference of its positive and negative variations. Thus, Theorem 3 remains true if we replace "singular and absolutely continuous parts" with "positive and negative variations" (this was noted also by Kahane) and the proof is much easier than that of Theorem 3. We can construct an example by taking a function whose graph consists of triangular peaks of suitable shape and distribution.

An alternate example with a simple analytic representation was kindly furnished by the referee. Consider the function

$$f(x) = \frac{x e^{-1/x}}{(\log x)^2} \sin e^{1/x} \qquad (0 < x < 1/2e),$$

with $f(0) = 0$. For $x \neq 0$,

$$f'(x) = \frac{e^{-1/x}}{x(\log x)^3}\,[(x+1)\log x - 2x]\sin e^{1/x} - \frac{\cos e^{1/x}}{x(\log x)^2},$$

and we note at once that $f'(x) = O(1/x(\log x)^2)$. Moreover, if δ is small enough, then in the interval $(0, \delta)$ the inequality

$$|f'(x)| > 1/2x(\log x)^2$$

holds on an open set of measure greater than $\delta/4$. Therefore f is of bounded variation, but the functions $\int_0^x f'(t)^+\,dt$ and $\int_0^x f'(t)^-\,dt$ have the order of magnitude $1/|\log x|$, so that they satisfy no Lipschitz condition, at $x = 0$.

To show that $f \in \mathrm{Lip}\,\alpha$ for each α $(0 < \alpha < 1)$, let α be fixed, and let δ denote any positive number small enough so that $\delta^{1-\alpha} < 1/2e$. For $0 \leq x \leq \delta^{1-\alpha}$, the definition of f gives the estimate

$$|f(x+\delta) - f(x)| \leq |f(x+\delta)| + |f(x)| < C_1\, \delta^{1-\alpha} \exp(-1/(2\delta)^{1-\alpha}) = O(\delta^{\alpha}).$$

For $\delta^{1-\alpha} \leq x \leq 1/2e - \delta$, the bound on f' gives the estimate

$$|f(x+\delta) - f(x)| < \delta C_2/\delta^{1-\alpha} = O(\delta^{\alpha}).$$

This completes the proof.

3. A COUNTEREXAMPLE IN CONFORMAL MAPPING

For the background of this section, see [2] and [7]. A *pseudo-circle* is the image of $|z| = 1$ under a conformal mapping $w = F(z)$ of its interior such that $F'(z)$ is bounded and $|F'(e^{i\theta})| = C$ almost everywhere, for some positive number C (the designation "pseudo-circle" is explained in [7]). Pseudo-circles were first constructed by Keldyš and Lavrentiev to demonstrate that the Smirnov condition for domains with rectifiable boundaries is not vacuous. In [2], the existence of pseudo-circles (other than circles) was shown to follow from the existence of an increasing singular function of class Z. However, the Keldyš-Lavrentiev construction had another noteworthy feature, not discussed in [2]: it showed that for every $\varepsilon > 0$, there exists a pseudo-circle of perimeter 2π (that is, with $C = 1$) contained in a circle of radius ε. Theorem 1 enables us to supplement the discussion in [2] so as to obtain this feature also. Indeed, consider the function

$$F'(z) = \exp \int_0^{2\pi} \frac{z + e^{it}}{z - e^{it}}\, d\rho(t),$$

where ρ is an increasing singular function on $[0, 2\pi]$, and write $P = \rho(2\pi) - \rho(0)$. Then

$$\log|F'(re^{i\theta})| = -\int_0^{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}\, d\rho(t) < -\frac{1-r}{1+r} P \leq -\frac{1}{2}(1 - r)P .$$

Therefore

$$|F(e^{i\theta}) - F(0)| = \left| \int_0^1 F'(re^{i\theta})\, dr \right| \leq \int_0^1 e^{-(1-r)P/2}\, dr < 2/P,$$

and it is therefore sufficient (in view of [2]) to construct an increasing singular function ρ such that $\rho(2\pi) - \rho(0)$ is sufficiently large, while

$$|\rho(t+h) - 2\rho(t) + \rho(t-h)| \leq Ah,$$

where A is a preassigned constant. But from Theorem 1, we see that if our initial function is $f_0(x) = Bx$ and if we choose $\varepsilon = 1$, the constructed singular function $f(x)$ satisfies the condition $f(2\pi) - f(0) \geq B - 2$, and this implies the desired result. (We should note that for the purpose of constructing univalent functions the singular function $\rho(t)$ must remain in class Z when extended for $t > 2\pi$ by the definition

$\rho(2\pi + x) - \rho(x) = \rho(2\pi) - \rho(0)$. It is easy to verify that with the choice of the initial function $f_0(x) = Bx$, this follows automatically from our construction.)

Using the full strength of Theorem 1, one could perhaps deduce the existence of pseudo-circles that approximate an arbitrary smooth Jordan curve.

APPENDIX

THEOREM (Stein and Zygmund, [9]). *Suppose* f *is continuous and of period* 2π, *and*

$$|f(x+t) - 2f(x) + f(x-t)| \leq \omega(t),$$

where $\int_0^1 t^{-3}[\omega(t)]^2\,dt < \infty$. *Then* f *is absolutely continuous and* $f' \in L^2$.

Proof. Let f have the Fourier expansion $\sum_{-\infty}^{\infty} c_n e^{inx}$. The hypotheses imply that

$$\int_0^1 \int_0^{2\pi} \frac{|f(x+t) - 2f(x) + f(x-t)|^2}{t^3}\,dx\,dt < \infty,$$

and the integral is seen to equal

$$16 \sum_{-\infty}^{\infty} |c_n|^2 \int_0^1 t^{-3} \sin^4 \frac{nt}{2}\,dt.$$

Since $\int_0^1 t^{-3} \sin^4 \frac{nt}{2}\,dt = \frac{n^2}{4} \int_0^{n/2} u^{-3} \sin^4 u\,du \geq Bn^2$ for $n \geq 1$, we deduce the convergence of $\sum n^2 |c_n|^2$, which is equivalent to the stated result.

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