GEOMETRIC CHARACTERIZATION OF DIFFERENTIABLE MANIFOLDS IN EUCLIDEAN SPACE, II

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1. INTRODUCTION

In the predecessor [1] to this paper, I described a geometric characterization of one- and two-dimensional differentiable manifolds of class C^1 in Euclidean space. The general case will be given in this paper, which can be read independently of [1].

The one-dimensional result is the following.

PROTOTYPE. *Let* M *be a one-dimensional topological manifold in* R^n. *Then* M *is a* C^1*-manifold in* R^n *if and only if the secant map*

$$\Sigma\colon M \times M - \Delta \to P^{n-1}$$

admits a continuous extension over all of $M \times M$.

Here Δ denotes the diagonal $\{(x, x)\colon x \in M\}$ of $M \times M$, and the secant map Σ assigns to each pair (x, y) of distinct points of M the line through the origin in R^n (and thus an element of projective space P^{n-1}) that is parallel to the secant line through x and y.

The direct generalization of the secant map to the case of a k-dimensional manifold M in R^n would be a map Σ that assigns to each $(k+1)$-tuple $(x_0, x_1, \cdots, x_k)$ of linearly independent points of M the k-plane through the origin in R^n (and thus an element of the Grassmann manifold $G_{n,k}$) parallel to the secant k-plane through $x_0, x_1, \cdots, x_k$. This is fine.

The direct generalization of the prototype theorem would then say that a k-dimensional topological manifold M in R^n is a C^1-manifold if and only if the map Σ admits a continuous extension over the diagonal $\Delta = \{(x, x, \cdots, x)\colon x \in M\}$ of $M \times M \times \cdots \times M = (M)^{k+1}$. But this is incorrect!

The difficulty is already apparent in the two-dimensional case. Look at three linearly independent points x_0, x_1, x_2 that approach a single point x on a two-sphere. If the approach is "conventional" then the secant plane through x_0, x_1 and x_2 approaches the tangent plane to the two-sphere at x. But if x_0, x_1, x_2 and x all lie on an equator, then the secant plane through x_0, x_1 and x_2 contains the equator, and is therefore orthogonal to the tangent plane at x. So there is no hope for continuously extending the generalized secant map over the diagonal, even when the manifold is known to be differentiable.

What goes wrong? As the three points x_0, x_1, x_2 converge to x on the two-sphere, the three edges of the triangle $x_0 x_1 x_2$ approach the tangent plane at x. But the mere fact that the three edges of a triangle make small angles with a given plane does *not* imply that the triangle itself makes a small angle with the plane.

Received July 25, 1966.

Research on this paper was partially supported by a grant GP 4433 from the National Science Foundation. The author holds an Alfred P. Sloan Foundation Research Fellowship.

It is precisely this sort of behavior which makes the study of surface area so much more difficult than that of arc length. Schwarz's example of a sequence of polyhedral approximations to a finite cylinder, in which the surface areas of the approximating polyhedra grow without bound, is based on just this phenomenon of small triangles inscribed in a surface making a large angle with nearby tangent planes to the surface.

This peculiar behavior can nevertheless be controlled. An easy calculation shows that if the three edges of an equilateral triangle make small angles with a given plane, then the triangle itself makes a small angle with the plane. The worse the shape of a triangle, the more eccentric its behavior can be. What is needed is a preliminary restriction on the shape of triangles (or more generally, k-simplexes). Definition 2.1 introduces a *shape function* σ for simplexes, which will be zero if and only if the simplex is degenerate. When the shape of simplexes under consideration is bounded away from zero, I will show that if each edge of such a k-simplex makes a sufficiently small angle with a k-plane, then the simplex itself makes a small angle with the k-plane (Theorems 4.1 and 4.4).

If M is a k-dimensional topological manifold in R^n, let $(M)^{k+1}$ denote the $(k+1)$-fold cartesian product $M \times M \times \cdots \times M$. If σ_0 is any real number, let $(M)^{k+1}_{\sigma_0}$ denote the set of all $(k+1)$-tuples $(x_0, x_1, \cdots, x_k)$ of points of M that form a k-simplex whose shape function is greater than σ_0. If σ_0 is negative, then $(M)^{k+1}_{\sigma_0}$ is simply all of $(M)^{k+1}$, while $(M)^{k+1}_0$ consists of those $(k+1)$-tuples whose points are linearly independent in R^n, and which therefore form nondegenerate k-simplexes.

The goal of this paper is to prove the following characterization of C^1-manifolds in R^n.

MAIN THEOREM. *Let* M *be a* k-*dimensional topological manifold in* R^n, *and let* $\sigma_0 > 0$ *be a real number (which for technical reasons must not exceed* $1/\sqrt{k}$). *Then* M *is a* C^1-*manifold in* R^n *if and only if the generalized secant map*

$$\Sigma\colon (M)^{k+1}_{\sigma_0} \to G_{n,k}$$

admits a continuous extension over $(M)^{k+1}_{\sigma_0} \cup \Delta$.

The necessity of the condition persists without the restriction that $\sigma_0 \leq 1/\sqrt{k}$.

The remainder of this paper is divided into two parts. Part I considers the shape of a simplex and the distribution of simplexes of good shape on a topological manifold in R^n. Part II uses the geometrical information from Part I to derive the Main Theorem.

I have benefited greatly from several conversations with L. V. Toralballa, who has used the technique of controlling the shape of triangles in his study of surface area.

I. SIMPLEXES IN EUCLIDEAN SPACE

2. THE SHAPE OF A SIMPLEX

If $x_0, x_1, \cdots, x_k$ are linearly independent points in Euclidean n-space R^n, their convex hull Δ^k is called a k-*simplex.* If these points are arbitrarily chosen in R^n, I shall still call their convex hull Δ^k, together with the distinguished elements $x_0, x_1, \cdots, x_k$, a k-*simplex.* Distinguishing the vertices permits their recovery in the degenerate case. I shall always assume that $k \geq 1$.

Degenerate or not, Δ^k lies in some k-plane in R^n provided $n \geq k$, and applying a rigid motion to R^n, we may assume that $\Delta^k \subset R^k$. Suppose for the moment that not all the vertices of Δ^k coincide. For each pair of distinct vertices x_i and x_j, let L_{ij} denote the line through the origin in R^k parallel to the edge $x_i x_j$. Then L_{ij} is an element of the projective space P^{k-1}, in which we use the angle between lines as a metric.

The shape of a simplex depends on the distribution of the directions of its edges, or what is the same, on the distribution of the L_{ij} in P^{k-1}. Each point $L \in P^{k-1}$ lies at a certain (minimum) distance from the set $\{L_{ij}: x_i \neq x_j\}$. The maximum of these distances, taken over all L in P^{k-1}, measures the distribution of the points L_{ij} in P^{k-1}. This maximum, $\theta(\Delta^k)$, is actually assumed and is bounded:

$$0 \leq \theta(\Delta^k) \leq \pi/2.$$

There is a line L through the origin in R^k which makes an angle at least $\theta(\Delta^k)$ with each edge of the simplex Δ^k, and $\theta(\Delta^k)$ is the largest number for which such a statement is true.

A k-simplex Δ^k not all of whose vertices coincide is degenerate if and only if $\theta(\Delta^k) = \pi/2$. If all of the vertices coincide, $\{L_{ij}: x_i \neq x_j\}$ is an empty set and we then arbitrarily put $\theta(\Delta^k) = \pi/2$.

If Δ^k is degenerate, it is contained in more than one k-plane in R^n. But any two of these planes can be related by a rigid motion that is the identity on Δ^k; therefore none of the above measurements depends on such a choice.

Definition 2.1. If Δ^k is a (possibly degenerate) k-simplex in R^n $(n \geq k \geq 1)$, then the quantity

$$\sigma(\Delta^k) = \cos\theta(\Delta^k)$$

will be called the *shape* of the simplex Δ^k.

This definition assigns the shape 0 to degenerate simplexes (and only to these), and the "better" a simplex, the larger its shape function. If all the vertices $x_0, x_1, \cdots, x_k$ of Δ^k are linearly independent in R^n, then $\sigma(\Delta^k)$ depends continuously on the variables $x_0, x_1, \cdots, x_k$. This is no longer true if Δ^k is degenerate, for a degenerate simplex can be approached by nondegenerate simplexes of reasonably good shape.

The above definition of shape is somewhat different from others appearing in the literature (in fact, it is more liberal). A tall, skinny isosceles triangle, for example, is acute and therefore has shape $\sigma > 1/\sqrt{2}$. An alternate definition, such as the inradius over the length of the longest side, would give such a triangle a shape measure close to zero. Thus such a triangle would have roughly the same shape measure as one with an angle close to 180°. For our purposes, the tall skinny isosceles triangle, while not as good as an equilateral triangle, is nevertheless much better than a very obtuse triangle, because the directions of its edges are better distributed. This distinction is crucial.

3. ANGLES BETWEEN k-PLANES IN R^n

In this section I collect some useful facts about angles in Euclidean space. The discussion is purposely sketchy, because the material is straightforward; more details may be found in [2].

Let P and Q be k-planes in R^n, which for our purposes may be assumed to pass through the origin. Then there exist orthonormal bases $\{u_1, u_2, \cdots, u_k\}$ and $\{v_1, v_2, \cdots, v_k\}$ for P and Q such that

$$\langle u_i, v_j \rangle = 0 \qquad \text{for } i \neq j$$

and

$$\langle u_1, v_1 \rangle \geq \langle u_2, v_2 \rangle \geq \cdots \geq \langle u_k, v_k \rangle \geq 0 .$$

Such bases are said to be in *normal form*. The k angles

$$0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_k \leq \pi/2$$

such that

$$\cos \phi_i = \langle u_i, v_i \rangle$$

are called the *principal angles* between P and Q. They depend only on P and Q, and they completely characterize the relative position of these two k-planes in R^n. Unlike the principal angles, the bases in normal form for P and Q are not completely determined by the above conditions. For example, u_i and v_i may simultaneously (or individually, if $\phi_i = \pi/2$) be replaced by their negatives for assorted values of i. This is the only ambiguity possible if all the principal angles are distinct, but more variations can occur if some of the successive principal angles coincide.

The largest principal angle ϕ_k between P and Q is also the largest angle that any line in P makes with Q, the line through u_k making such an angle with Q. It is similarly the largest angle that any line in Q makes with P, the line through v_k making such an angle with P.

If we desire to find a *single* angle ϕ that might reasonably be called *the* angle between the k-planes P and Q, then the appropriate choice is the angle between 0 and $\pi/2$ whose cosine is the reduction factor for p-dimensional measure under orthogonal projection of P into Q, or equivalently of Q into P. The correctness of such a choice is borne out by two facts:

(1) The definition extends the usual definitions for the case $k = 1$ and also for the case in which P and Q intersect in a $(k - 1)$-plane.

(2) The k-planes P and Q correspond to one-dimensional subspaces L(P) and L(Q) in the exterior product $\wedge^k R^n$, and when $\wedge^k R^n$ is given the natural inner product structure induced from R^n, the angle between L(P) and L(Q) is precisely the angle ϕ defined above. Thus ϕ is also the distance between P and Q in the metric on the Grassmann manifold $G_{n,k}$ induced by the "inclusion" of $G_{n,k}$ in the projective space $P^{\binom{n}{k}-1}$ of $\wedge^k R^n$.

The angle ϕ between P and Q is related to the principal angles by the formula

$$\cos\phi = (\cos\phi_1)(\cos\phi_2)\cdots(\cos\phi_k). \tag{3.1}$$

The largest principal angle ϕ_k between P and Q provides a crude metric on $G_{n,k}$ that is nevertheless topologically equivalent to the natural one mentioned above, because formula (3.1) implies that

$$\frac{\phi}{k} \leq \phi_k \leq \phi. \tag{3.2}$$

4. ANGLES BETWEEN A k-SIMPLEX AND A k-PLANE

THEOREM 4.1. *Let* Δ^k *be a nondegenerate* k-*simplex in* R^n *each of whose edges makes an angle at most* ε $(\varepsilon \leq \pi/2)$ *with some unknown* k-*plane* Q *in* R^n. *Then every line* L *in the* k-*plane* P *spanned by* Δ^k *makes an angle* $\phi(L)$ *with* Q *subject to the inequality*

$$\sin\phi(L) \leq \frac{\sin\varepsilon}{\sigma(\Delta^k)}.$$

We may assume that both the k-plane P spanned by Δ^k and the k-plane Q pass through the origin. Let $\{u_1, u_2, \cdots, u_k\}$ and $\{v_1, v_2, \cdots, v_k\}$ be bases in normal form for P and Q. The principal angles

$$\phi_1 \leq \phi_2 \leq \cdots \leq \phi_k$$

between P and Q are then given by the formulas

$$\cos\phi_i = \langle u_i, v_i \rangle.$$

The largest angle that any line L in P makes with Q is precisely the largest principal angle ϕ_k; to prove the theorem we must therefore show that

$$\sin\phi_k \leq \frac{\sin\varepsilon}{\sigma(\Delta^k)}.$$

Recall that $\sigma(\Delta^k) = \cos\theta(\Delta^k)$, and that according to the construction of $\theta(\Delta^k)$ in Section 2, some edge E of the simplex Δ^k must make an angle not greater than $\theta(\Delta^k)$ with the vector u_k in P. Let u be a unit vector in P parallel to E, chosen so that u also makes an angle at most $\theta(\Delta^k)$ with u_k. Since $\{u_1, u_2, \cdots, u_k\}$ is an orthonormal basis for P, we can write

$$u = \sum_1^k (\cos\theta_i) u_i, \tag{4.2}$$

where

$$\sum_1^k \cos^2\theta_i = 1 \tag{4.3}$$

and $\theta_k \leq \theta(\Delta^k)$.

According to hypothesis, the edge E of Δ^k makes an angle $\alpha \leq \varepsilon$ with Q, or what is the same, with its orthogonal projection on Q. Thus the vector u also makes an angle $\alpha \leq \varepsilon$ with its orthogonal projection v on Q. Since the given bases for P and Q are in normal form, the orthogonal projection of u_i on Q is $(\cos \phi_i) v_i$. Now project both sides of equation (4.2) orthogonally onto Q; this gives the equation

$$v = \sum_1^k (\cos \theta_i \cos \phi_i) v_i .$$

The angle α between u and v is given by

$$\cos^2 \alpha = \langle v, v \rangle = \sum_1^k \cos^2 \theta_i \cos^2 \phi_i .$$

Since $\alpha \leq \varepsilon \leq \pi/2$,

$$\cos^2 \varepsilon \leq \cos^2 \alpha = \sum_1^k \cos^2 \theta_i \cos^2 \phi_i .$$

Furthermore,

$$\sum_1^k \cos^2 \theta_i \cos^2 \phi_i \leq \sum_1^{k-1} \cos^2 \theta_i + \cos^2 \theta_k \cos^2 \phi_k$$

$$= \sin^2 \theta_k + \cos^2 \theta_k \cos^2 \phi_k ,$$

by virtue of (4.3). Thus

$$\cos^2 \varepsilon \leq \sin^2 \theta_k + \cos^2 \theta_k \cos^2 \phi_k .$$

This is equivalent to the inequality

$$\cos^2 \theta_k \sin^2 \phi_k \leq \sin^2 \varepsilon .$$

Since $\theta_k \leq \theta(\Delta^k) < \pi/2$ (because Δ^k is nondegenerate), $\cos \theta_k \neq 0$ and the above inequality is therefore equivalent to

$$\sin \phi_k \leq \frac{\sin \varepsilon}{\cos \theta_k} .$$

Since $\theta_k \leq \theta(\Delta^k) < \pi/2$, $\cos \theta_k \geq \cos \theta(\Delta^k) = \sigma(\Delta^k)$, and the above inequality implies that

$$\sin \phi_k \leq \frac{\sin \varepsilon}{\sigma(\Delta^k)} ;$$

as we have already observed, this implies the theorem. The following is an easy corollary.

THEOREM 4.4. *Let the* k*-plane* $Q \subset R^n$ *and the real number* $\sigma_0 > 0$ *be given. For every* ε $(0 < \varepsilon \leq \pi/2)$ *there exists a* $\delta > 0$ *such that if* Δ^k *is a* k*-simplex in*

R^n *with shape* $\sigma(\Delta^k) > \sigma_0$, *each of whose edges makes an angle at most* δ *with* Q, *then the* k-*plane* P *spanned by* Δ^k *makes an angle less than* ε *with* Q.

Simply choose $\delta > 0$ so small that

$$\frac{\sin \delta}{\sigma_0} \leq \sin(\varepsilon/k).$$

Now, if each edge of Δ^k makes an angle at most δ with Q, then by Theorem 4.1, the largest principal angle ϕ_k between P and Q satisfies

$$\sin \phi_k \leq \frac{\sin \delta}{\sigma(\Delta^k)} < \frac{\sin \delta}{\sigma_0} \leq \sin(\varepsilon/k).$$

Since $\varepsilon \leq \pi/2$,

$$\phi_k < \varepsilon/k.$$

But then, by formula (3.2), the angle ϕ between P and Q satisfies the condition

$$\phi \leq k\phi_k < \varepsilon;$$

this proves the theorem.

5. ORTHOGONAL PARTURITION

In order to show that inscribed k-simplexes of good shape are well distributed on a k-dimensional topological manifold in R^n, I need some techniques for building simplexes of good shape from others of smaller dimension.

Let Δ^r be a nondegenerate r-simplex in R^n $(n > r)$. Single out any one of its vertices, and name the vertices $x_0, x_1, \cdots, x_r$ so that x_r is the distinguished one. Let P^{n-r} be the $(n - r)$-plane through x_r that is orthogonal to the r-plane P^r spanned by Δ^r. No vertex of Δ^r other than x_r lies in P^{n-r}.

Now let x_{r+1} be any point of P^{n-r} other than x_r. Then the points $x_0, x_1, \cdots, x_r, x_{r+1}$ are the vertices of a nondegenerate $(r + 1)$-simplex Δ^{r+1} in R^n that has Δ^r as a face. This construction of Δ^{r+1} from Δ^r will be referred to as *orthogonal parturition* (applied to the vertex x_r). The problem for this section is to compare the shape $\sigma(\Delta^{r+1})$ of Δ^{r+1} (which depends only on Δ^r and the length of the edge $x_r x_{r+1}$) with the shape $\sigma(\Delta^r)$ of Δ^r.

THEOREM 5.1. *If the edge* $x_r x_{r+1}$ *of* Δ^{r+1} *is orthogonal to the* k-*simplex* Δ^r *spanned by* $x_0, x_1, \cdots, x_r$, *then*

$$\sigma(\Delta^{r+1}) \geq \frac{\sigma(\Delta^r)}{\sqrt{1 + \sigma(\Delta^r)^2}},$$

and this inequality is best possible, with the given information.

Using a preliminary rigid motion of R^n, if necessary, we may assume that $\Delta^r \subset R^r$ and $\Delta^{r+1} \subset R^{r+1}$. For each pair of distinct indices $i \neq j$ between 0 and $r + 1$, let L_{ij} denote the line through the origin in R^{r+1} parallel to the edge $x_i x_j$ of Δ^{r+1}. The angle $\theta(\Delta^{r+1})$ measures the distribution of these lines in the sense of Section 2, and $\sigma(\Delta^{r+1}) = \cos\ \theta(\Delta^{r+1})$.

According to the information given, the only lines about which we have definite knowledge are the lines L_{ij} $(0 \leq i, j \leq r)$ corresponding to the edges of Δ^r, and the "vertical" line $L_{r,r+1}$ corresponding to the edge $x_r x_{r+1}$ of Δ^{r+1} which is orthogonal to the face Δ^r. Let the angle θ_{r+1} measure the distribution of these lines (again in the sense of Section 2), and let $\sigma_{r+1} = \cos\ \theta_{r+1}$. Since all of the edges of Δ^{r+1} are better distributed than some of them,

$$\theta(\Delta^{r+1}) \leq \theta_{r+1},$$

and therefore

$$\sigma(\Delta^{r+1}) = \cos\ \theta(\Delta^{r+1}) \geq \cos\ \theta_{r+1} = \sigma_{r+1}.$$

We shall prove the inequality of the theorem by establishing the *equality*

$$\sigma_{r+1} = \frac{\sigma(\Delta^r)}{\sqrt{1 + \sigma(\Delta^r)^2}}. \tag{5.2}$$

Let L be a line through the origin in R^{r+1}, other than the vertical line $L_{r,r+1}$, and let the line M be its orthogonal projection onto R^r. If L makes an angle α with the vertical line $L_{r,r+1}$, then L makes an angle $\pi/2 - \alpha$ with M. Suppose M in turn makes an angle β with a line N through the origin in R^r. Then the angle γ between L and N is automatically determined by the formula

$$\cos\gamma = \sin\alpha\ \cos\beta, \tag{5.3}$$

as may be seen by a trivial geometric argument. If, among all lines in R^{r+1}, L makes the largest possible angle θ_{r+1} with the set

$$\{L_{ij}\colon 0 \leq i, j \leq r \text{ or } i = r,\ j = r+1\},$$

then

$$\alpha = \gamma = \theta_{r+1} \quad \text{and} \quad \beta = \theta(\Delta^r),$$

in which case N is parallel to some edge of Δ^r. Inserting these values into formula (5.3), we get the formula

$$\cos\ \theta_{r+1} = \sin\ \theta_{r+1} \cos\ \theta(\Delta^r),$$

or equivalently,

$$\cos\ \theta_{r+1} = \frac{\cos\ \theta(\Delta^r)}{\sqrt{1 + \cos^2\ \theta(\Delta^r)}},$$

which is the same as (5.2) and therefore proves the theorem.

If the edge $x_r x_{r+1}$ is very short compared with the edges of Δ^r, then the edges of Δ^{r+1} disregarded in the computation of θ_{r+1} and σ_{r+1} have approximately the same directions as those of Δ^r, which were included in the computation. Similarly, if the edge $x_r x_{r+1}$ is very long compared with the edges of Δ^r, then the edges of Δ^{r+1} disregarded in the computation all have directions approximately the same as that of $x_r x_{r+1}$, which was included. Thus if $x_r x_{r+1}$ is either very short or very long, then $\theta(\Delta^{r+1})$ is approximately equal to θ_{r+1}, and $\sigma(\Delta^{r+1})$ is approximately equal to σ_{r+1}. In such cases the inequality of the theorem is practically an equality, and therefore the result is best possible.

Note that when $r = 1$, the inequality of the theorem is an equality because the shape of an edge is 1, while the shape of a right triangle is $1/\sqrt{2}$.

Note also that if orthogonal parturition is iterated, beginning at stage one with a one-simplex, then the k-simplex Δ^k obtained at stage k will have shape $\sigma(\Delta^k) \geq 1/\sqrt{k}$. The appearance of the number $1/\sqrt{k}$ in the statement of the Main Theorem is related to this.

Finally, we observe that if the nondegenerate r-simplex $\Delta^r \subset R^n$ $(n > r)$ is given and we wish to add a vertex x_{r+1} to produce an $(r+1)$-simplex Δ^{r+1} with as good shape as possible, and if the edge $x_r x_{r+1}$ is either very short or very long compared with the edges of Δ^r, then orthogonal parturition is approximately the most efficient way of doing this.

6. ORTHOGONAL FISSION

Here I give a variation on the theme of the preceding section.

As before, let Δ^r be a nondegenerate r-simplex in R^n $(n > r)$ with a distinguished vertex, and name the vertices $x_0, x_1, \cdots, x_r$ so that x_r is the distinguished one. Let U be a small neighborhood of x_r in R^n, not containing any of the other vertices of Δ^r. Let x'_r and x'_{r+1} be two distinct points of U such that the edge $x'_r x'_{r+1}$ is orthogonal to the r-plane P^r spanned by Δ^r. Then

$$x_0, x_1, \cdots, x_{r-1}, x'_r, x'_{r+1}$$

are the vertices of an $(r+1)$-simplex Δ^{r+1} in R^n, and if U is sufficiently small, then Δ^{r+1} is nondegenerate. I call this new procedure *orthogonal fission* (applied to the vertex x_r).

Observe the difference between orthogonal fission and orthogonal parturition. In orthogonal parturition we add one new vertex to Δ^r to produce Δ^{r+1}; thus Δ^r is a face of Δ^{r+1}. In orthogonal fission we *replace* a vertex of Δ^r by *two* new vertices to produce Δ^{r+1}; thus Δ^r is generally not a face of Δ^{r+1}.

If U is chosen sufficiently small, then the simplex Δ^{r+1} obtained from Δ^r by orthogonal fission can be made arbitrarily close in shape to one obtained from Δ^r by orthogonal parturition, in which the new vertex x_{r+1} is very close to x_r. Then by the preceding section we have the following result.

THEOREM 6.1. *Given Δ^r with a distinguished vertex x_r, and given $\varepsilon > 0$, there exists a $\delta > 0$ such that if Δ^{r+1} is obtained from Δ^r by orthogonal fission applied to x_r, and if x'_r and x'_{r+1} are both within δ of x_r, then*

$$\frac{\sigma(\Delta^r)}{\sqrt{1+\sigma(\Delta^r)^2}} - \varepsilon < \sigma(\Delta^{r+1}) < \frac{\sigma(\Delta^r)}{\sqrt{1+\sigma(\Delta^r)^2}} + \varepsilon .$$

7. ORTHOGONAL FISSION OF A SIMPLEX INSCRIBED IN A TOPOLOGICAL MANIFOLD IN EUCLIDEAN SPACE

Both orthogonal parturition and orthogonal fission are valid procedures for building simplexes of good shape from simplexes of smaller dimension sitting alone in Euclidean space. But when the simplexes must be inscribed in a topological manifold in Euclidean space, then orthogonal fission succeeds where orthogonal parturition fails. To see this, consider one nappe of a cone, with vertex-angle less than 90°. Let x_0 be any point on the cone, other than the vertex, and let x_1 be the vertex. Then the line segment x_0x_1 in Euclidean space is a one-simplex Δ^1 inscribed in the cone. It is impossible to extend this inscribed one-simplex to an inscribed two-simplex by applying orthogonal parturition to the point x_1, regardless of where x_0 may be located. But it is evident that orthogonal fission can be applied successfully to x_1 to yield an inscribed two-simplex.

THEOREM 7.1. *Let* M *be a topological* k-*manifold in* R^n, *and let* Δ^r *be a nondegenerate* r-*simplex inscribed in* M $(r < k)$. *For any vertex* x_r *of* Δ^r *and any* $\varepsilon > 0$, *one can apply orthogonal fission to* x_r *so as to obtain a nondegenerate* $(r + 1)$-*simplex* Δ^{r+1} *inscribed in* M, *such that*

$$\frac{\sigma(\Delta^r)}{\sqrt{1 + \sigma(\Delta^r)^2}} - \varepsilon < \sigma(\Delta^{r+1}) < \frac{\sigma(\Delta^r)}{\sqrt{1 + \sigma(\Delta^r)^2}} + \varepsilon.$$

Given Δ^r, the distinguished vertex x_r, and $\varepsilon > 0$, first use Theorem 6.1 to find a $\delta > 0$ such that if orthogonal fission is applied to x_r and if both x'_r and x'_{r+1} are within δ of x_r, then $\sigma(\Delta^{r+1})$ will satisfy the desired inequality. To prove the theorem, it remains to show that x'_r and x'_{r+1} can in addition be selected on M.

Let X be a small neighborhood of x_r in M, lying within the δ-neighborhood of x_r in R^n. Consider the orthogonal projection of X onto the r-plane P^r spanned by Δ^r in R^n. Since $r < k$, this projection mapping cannot be one-to-one on X. Thus there exist two points x'_r and x'_{r+1} in X that are projected orthogonally onto the same point of P^r. Since this means that the edge $x'_r x'_{r+1}$ is orthogonal to P^r, all the conditions for orthogonal fission are satisfied, and in addition the resulting Δ^{r+1} is inscribed in M. This proves the theorem.

THEOREM 7.2. *Let* M *be a topological* k-*manifold in* R^n, x_0 *a preassigned point of* M, *and* X *a preassigned neighborhood of* x_0 *in* M. *Then for each* $r \leq k$ *and each* $\varepsilon > 0$, *there exists an* r-*simplex* Δ^r, *inscribed in* X *with one vertex at* x_0, *such that*

$$\frac{1}{\sqrt{r}} - \varepsilon < \sigma(\Delta^r) < \frac{1}{\sqrt{r}} + \varepsilon.$$

Consider only X and disregard the rest of M. Choose a second point x_1 in X. Now x_0x_1 is a one-simplex inscribed in X. Use Theorem 7.1 and iterate orthogonal fission (applied to the last vertex) $r - 1$ times. The validity of the specific inequality given in the statement of the theorem can either be seen directly, or established by induction with the aid of the inequality of Theorem 7.1.

Observe that the only feature of M used in the proofs of Theorems 7.1 and 7.2 is the fact that no open set in M can be mapped in a one-to-one manner into an r-plane, for $r < k$. The theorems are therefore valid for any M with this property (for example, for any locally compact M having dimension at least k at each point).

8. DISTRIBUTION OF INSCRIBED k-SIMPLEXES OF GOOD SHAPE IN A TOPOLOGICAL k-MANIFOLD IN EUCLIDEAN SPACE

If M is a topological k-manifold in R^n, then Theorem 7.2 guarantees the existence of a k-simplex of good shape inscribed in any open set $X \subset M$, with *one* vertex preassigned. I will need the same sort of guarantee, but with *two* vertices preassigned, in order to show that a manifold satisfying the hypothesis of the Main Theorem has a tangent plane at each point. The orthogonal fission procedure of Sections 6 and 7 is by itself inadequate for obtaining such a result, because if two vertices are preassigned, fission will destroy one of them while producing an inscribed triangle. We need an independent guarantee that two preassigned vertices can be extended to an inscribed triangle of good shape. Orthogonal fission (Theorem 7.1) can then finish the job, because there will always be at least one extra vertex that can be split by fission.

THEOREM 8.1. *Let* M *be a topological* k*-manifold in* R^n $(k \geq 2)$. *Let* X *be a preassigned connected open subset of* M, *and let* x_0 *and* x_1 *be preassigned points of* X *such that* $|x_0 - x_1| <$ diam X. *Then there exists a third point* x_2 *in* X *such that the triangle* Δ^2 *with vertices* x_0, x_1, *and* x_2 *is acute, in other words, such that* $\sigma(\Delta^2) > 1/\sqrt{2}$.

The conditions on X, x_0, and x_1 are necessary. If $M = X$ consisted of two disjoint two-spheres in R^3, and x_0 and x_1 were closest points, one on each two-sphere, then the conclusion of the theorem would be false. It would similarly be false if $M = X$ were a single two-sphere in R^3, and x_0 and x_1 were antipodal points.

Now let M, X, x_0, and x_1 be given as in the hypotheses. I will construct the set A of all points x in R^n, distinct from x_0 and x_1, such that x_0x_1x is an acute triangle. To prove the theorem, I will show that X must meet A.

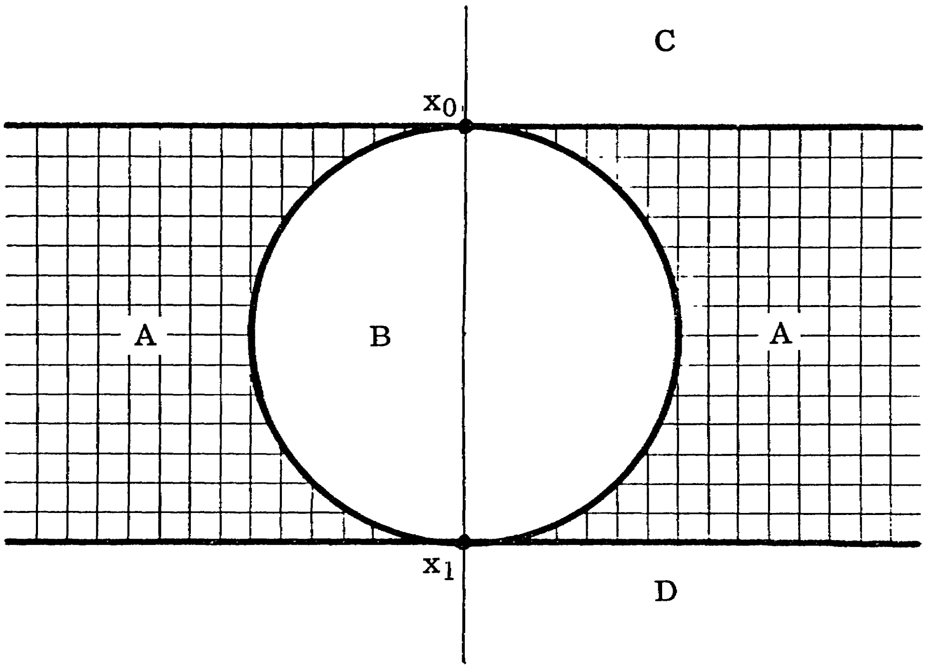

Figure 1

Through the point x_0 we draw the $(n - 1)$-dimensional hyperplane in R^n orthogonal to the segment x_0x_1, as in the figure. The third vertex x must lie on the same side of this hyperplane as x_1, if the triangle x_0x_1x is to have an acute angle at x_0. Similarly, we draw the $(n - 1)$-dimensional hyperplane through x_1 orthogonal to x_0x_1. Then x must lie on the same side of this hyperplane as x_0, if the triangle x_0x_1x is to have an acute angle at x_1. Finally, x must lie outside the $(n - 1)$-sphere of radius $|x_0 - x_1|/2$ with center at $(x_0 + x_1)/2$ if the triangle x_0x_1x is to have an acute angle at x. The combination of these three restrictions yields the open set A, as shown in the figure. B denotes the closed ball bounded by the $(n-1)$-sphere, C denotes the closed half-space of the $(n-1)$-plane through x_0 that does not contain x_1, and D denotes the closed half-space of the $(n-1)$-plane through x_1 that does not contain x_0.

To show that X meets A, suppose to the contrary that $X \cap A = \emptyset$. Then $X \subset B \cup C \cup D$. X cannot be entirely contained in B, because the diameter of B is precisely $|x_0 - x_1|$, while the diameter of X is by hypothesis larger than this. Furthermore, X cannot be entirely contained in $C \cup D$, because x_0 cannot be connected to x_1 in $C \cup D$, whereas it can be connected to x_1 in X since X is connected.

The assumption that $X \cap A = \emptyset$ therefore implies that X is contained entirely in $B \cup C \cup D$, partially in B and partially in $C \cup D$. But then removal of either x_0 or x_1 from X disconnects X, whereas this is patently impossible for a connected open set in a manifold of dimension at least 2. Thus X must meet A, and the theorem is proved.

Putting Theorem 8.1 to its intended use, we get the following result.

THEOREM 8.2. *Let* M *be a topological* k*-manifold in* R^n, X *a connected open subset of* M, *and* x_0 *and* x_1 *two points of* X *such that* $|x_0 - x_1| <$ diam X. *Then for each integer* $r \leq k$, *there exist points* $x_2, x_3, \cdots, x_r$ *in* X *such that the* r-*simplex* Δ^r *with vertices* $x_0, x_1, x_2, \cdots, x_r$ *has shape* $\sigma(\Delta^r) > 1/\sqrt{r}$.

The hypotheses are tailored to those of Theorem 8.1. If $k = 1$, the conclusion is automatically satisfied. If $k \geq 2$, apply Theorem 8.1 to get a third vertex x_2 in X such that the triangle Δ^2 with vertices x_0, x_1 and x_2 has shape $\sigma(\Delta^2) > 1/\sqrt{2}$. If $k = 2$, stop here. If $k > 2$, we are now in a position to apply orthogonal fission to x_2, and in each successive case to the last vertex. The result now follows if we iterate this procedure with the help of Theorem 7.1.

Note that Theorem 8.2 improves Theorem 7.2 not only in the sense that we can now preassign two vertices rather than one, but also in that the resulting r-simplex Δ^r has shape $\sigma(\Delta^r) > 1/\sqrt{r}$, rather than $1/\sqrt{r} - \varepsilon$.

THEOREM 8.3. *Let* M *be a topological* k*-manifold in* R^n. *If* $\sigma_0 \leq 1/\sqrt{k}$, *then the closure of* $(M)^{k+1}_{\sigma_0}$ *in* $(M)^{k+1}$ *contains the diagonal* Δ.

Use Theorem 8.2, with X an arbitrarily small neighborhood of a given point in M.

Without Theorem 8.3 it would not be reasonable to talk about the continuous extendability of the generalized secant map over the diagonal, as was done in the statement of the Main Theorem.

II. CHARACTERIZATION OF DIFFERENTIABLE MANIFOLDS IN EUCLIDEAN SPACE

9. SOME DEFINITIONS

Let M be any subset of R^n, x_0 a point of M, and P a k-plane in R^n through x_0. Then P is said to be a *tangent* k*-plane to* M *at* x_0 if

$$\operatorname*{Limit}_{\substack{x \in M - x_0 \\ x \to x_0}} \frac{d(x, P)}{d(x, x_0)} = 0,$$

where d is the ordinary Euclidean distance in R^n. Equivalently, P is a tangent k-plane to M at x_0 if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in M$ and $0 < d(x, x_0) < \delta$, then the vector $x - x_0$ makes an angle less than ε with P.

If M has a tangent (k - 1)-plane at x_0, then each k-plane containing it is a tangent k-plane to M at x_0. On the other hand, if M has no tangent (k - 1)-plane at x_0, then it has at most one tangent k-plane at x_0.

If P is a k-plane in R^n, then P_0 will denote the parallel k-plane through the origin. Thus P_0 will be an element of the Grassmann manifold $G_{n,k}$.

Topological manifolds in Euclidean space have already appeared in Sections 7 and 8. A subset $M \subset R^n$ is said to be a *k-dimensional topological* or *C^0-manifold in R^n* if for each $x_0 \in M$ there exist a neighborhood X of x_0 in M, an open set U in R^k, and a homeomorphism $F: U \to X$.

Suppose, in addition, that F can always be chosen to be a C^1-immersion of U into R^n, meaning that F is a map of class C^1 from $U \subset R^k$ to $X \subset R^n$ whose differential dF is nonsingular at each point of U. Then M is said to be a *k-dimensional C^1-manifold in R^n*.

When we parametrize a manifold, it is sometimes useful to allow the domain U of the parametrization F to be an open set in some k-plane $P(x_0)$ in R^n, possibly depending on x_0, rather than in R^k itself. It still makes sense to speak of the differentiability of F, and if F is differentiable, then the differential of F at $u \in U$ is a linear map

$$dF(u): P_0(x_0) \to R^n$$

whose domain is the k-dimensional subspace $P_0(x_0)$ parallel to the k-plane $P(x_0)$. If for each $x_0 \in M$, such an F exists which is of class C^1 with dF nonsingular at each point of U, then one can easily show that M also satisfies the strict definition of C^1-manifold in R^n given above. This permissiveness with the domain of F is particularly convenient because it allows direct local parametrization of k-dimensional C^1-manifolds in R^n by their tangent k-planes.

10. A PRELIMINARY CHARACTERIZATION OF C^1-MANIFOLDS IN R^n

THEOREM 10.1. *Let* M *be a* k-*dimensional topological manifold in* R^n. *Then* M *is a* C^1-*manifold in* R^n *if and only if*

(1) M *has a tangent* k-*plane* P(x) *at each point* $x \in M$;

(2) *the map* $M \to G_{n,k}$, *sending* x *to* $P_0(x)$, *is continuous;*

(3) *for each* $x \in M$, *the orthogonal projection* $\pi_x: R^n \to P(x)$ *is one-to-one on some neighborhood of* x *in* M.

This theorem is in itself a characterization of C^1-manifolds in R^n, though not particularly satisfying, since it revolves around the existence and properties of tangent planes (which seems a little like begging the question). It is in any case a useful tool for the proof of the Main Theorem.

Before the proof begins, observe that properties (1) and (2) alone are not enough to characterize C^1-manifolds, since they do not prevent M from having cusp-like folds.

That the conditions are necessary is easily verified, and we omit this. Suppose then that M is a topological manifold in R^n satisfying conditions (1), (2), and (3). If M is a C^1-manifold, then the local parametrizations obtained by inversion of the orthogonal projections onto tangent planes are automatically C^1-immersions when restricted to sufficiently small neighborhoods. Our program will be to prove this directly from conditions (1), (2), and (3).

Let x_0 be a fixed point of M, and using (1), let $P(x_0)$ be the tangent k-plane to M at x_0. According to (3), the orthogonal projection

$$\pi_{x_0}: R^n \to P(x_0)$$

is one-to-one on some neighborhood X_0 of x_0 in M. Let $U = \pi_{x_0}(X_0) \subset P(x_0)$. Then

$$\pi_{x_0}: X_0 \to U$$

is a homeomorphism; so let

$$F_0: U \to X_0 \subset M$$

be its inverse. To prove the theorem, I will show that F_0 is a C^1-immersion on some neighborhood of x_0 in $U \subset P(x_0)$; note that it need not be a C^1-immersion on all of U. The special feature of the parametrization F_0 to keep in mind is that if $u \in U$, then $F_0(u) - u$ is orthogonal to $P(x_0)$.

Properties (1) and (3) have already been used in the construction of F_0. Next I show that property (1) implies that F_0 is differentiable at $x_0 \in U$, by showing that F_0 is tangent at x_0 to the inclusion map

$$I_0: P(x_0) \subset R^n,$$

equivalently, that

$$\underset{\substack{u \in U - x_0 \\ u \to x_0}}{\text{Limit}} \frac{|F_0(u) - I_0(u)|}{|u - x_0|} = 0.$$

Since I_0 is certainly differentiable at x_0, the map F_0 will be, too.

Since $I_0(u) = u$ and $F_0(u) - u$ is orthogonal to $P(x_0)$, the fraction

$$\frac{|F_0(u) - I_0(u)|}{|u - x_0|} = \frac{|F_0(u) - u|}{|u - x_0|}$$

is the tangent of the angle between the vector

$$F_0(u) - x_0 = x - x_0$$

and the k-plane $P(x_0)$. Since $P(x_0)$ is a tangent k-plane to M at x_0, the tangent of this angle goes to zero as $x = F_0(u) \to x_0$, or equivalently, as $u \to x_0$. Thus F_0 is differentiable at x_0, and furthermore, $dF_0(x_0) = dI_0(x_0)$, which is the inclusion $P_0(x_0) \subset R^n$. Thus $dF(x_0)$ is nonsingular.

Since x_0 was arbitrary in M, it now follows that all the local parametrizations by tangent k-planes are differentiable at the distinguished points of contact, and have nonsingular differentials there. Using this, together with property (2), I can now show that F_0 is a differentiable immersion on some neighborhood of x_0 in U.

Consider the following diagram.

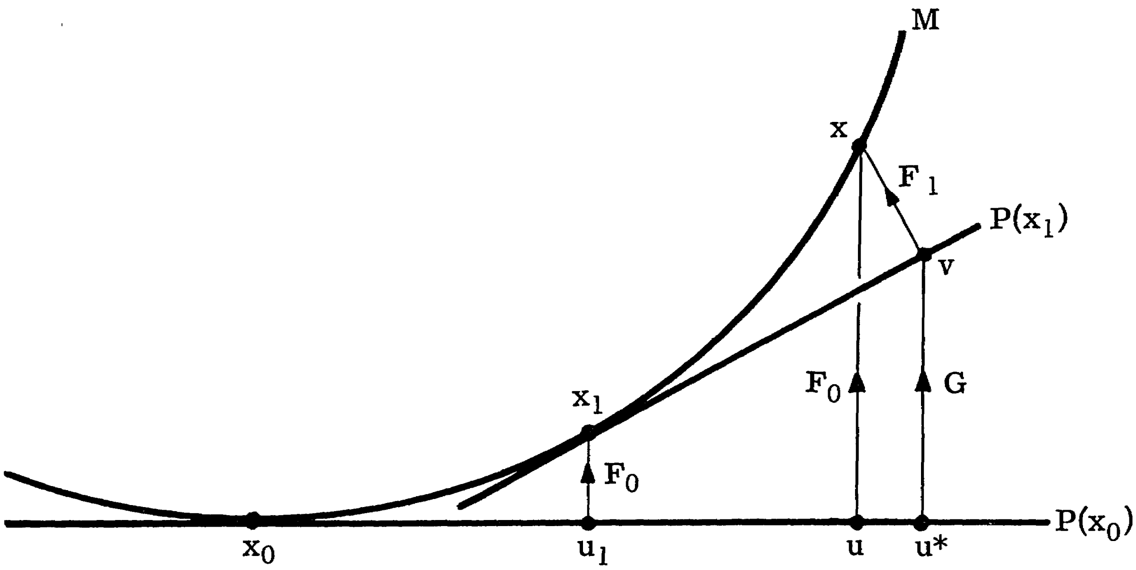

Figure 2

We already have the parametrization F_0: $U \to X_0 \subset M$. Using property (2), we make U smaller, if necessary, so that for each $u_1 \in U$ the tangent k-plane $P(x_1)$ to M at $x_1 = F_0(u_1)$ makes an angle less than $\pi/2$ with $P(x_0)$. The orthogonal projection π_{x_1}: $R^n \to P(x_1)$ has a local inverse

$$F_1: V \to X_1 \subset M,$$

and we know that F_1 will be differentiable at x_1, with $dF_1(x_1)$ the inclusion $P_0(x_1) \subset R^n$.

Since $P(x_1)$ makes an angle less than $\pi/2$ with $P(x_0)$, the orthogonal projection

$$\pi_{x_0}: P(x_1) \to P(x_0)$$

is an affine homeomorphism with inverse

$$G: P(x_0) \to P(x_1).$$

Since G is also an affine homeomorphism, it is differentiable at every point of $P(x_0)$, and $dG(u)$ is a fixed linear isomorphism of $P_0(x_0) \to P_0(x_1)$ (the linear part of G) for all $u \in P(x_0)$.

Since u_1 is an arbitrary point of U, I shall show that F_0 is a differentiable immersion on U by showing that F_0 is differentiable at u_1 and that $dF_0(u_1) = dG(u_1)$. This can be done by showing that F_0 is tangent to the composition $F_1 G$ at u_1, for we already know that G is differentiable at u_1 and that F_1 is differentiable at x_1, so that $F_1 G$ is differentiable at u_1, with

$$d(F_1 G)(u_1) = dF_1(x_1)dG(u_1) = dG(u_1).$$

Actually, it will be more convenient to show that $F_0^{-1} F_1 G$ is tangent to the inclusion I_0: $P(x_0) \subset R^n$ at u_1.

I must show that

$$\lim_{u^* \to u_1} \frac{|F_0^{-1} F_1 G(u^*) - I_0(u^*)|}{|u^* - u_1|} = 0. \tag{10.2}$$

According to the notation of Figure 2, $G(u^*) = v$, $F_1(v) = x$, and $F_0(u) = x$. Thus $F_0^{-1} F_1 G(u^*) = u$. Furthermore, $I_0(u^*) = u^*$, and therefore (10.2) can be written

$$\lim_{u^* \to u_1} \frac{|u - u^*|}{|u^* - u_1|} = 0.$$

Notice that $u = \pi_{x_0}(x)$ and $u^* = \pi_{x_0}(v)$. Since π_{x_0} is an orthogonal projection, it is norm-decreasing, and therefore

$$|u - u^*| \leq |x - v|.$$

Next, since G is affine, it satisfies a Lipschitz condition for some Lipschitz constant r. Since $G(u^*) = v$ and $G(u_1) = x_1$,

$$|v - x_1| \leq r\,|u^* - u_1|.$$

Thus

$$\frac{|u - u^*|}{|u^* - u_1|} \leq r \frac{|x - v|}{|v - x_1|}.$$

As $u^* \to u_1$ (or equivalently, as $v \to x_1$, or as $x \to x_1$), the quotient

$$\frac{|x - v|}{|v - x_1|},$$

which is the tangent of the angle between the vector $x - x_1$ and the k-plane $P(x_1)$ tangent to M at x_1, must also approach zero. Thus the limit appearing in (10.2) is zero, hence F_0 is indeed a differentiable immersion on U.

To complete the proof of the theorem, I must show that F_0 is of class C^1 on U, in other words, that the (nonsingular) linear transformation

$$dF_0(u)\colon P_0(x_0) \to R^n$$

varies continuously with $u \in U$. This is equivalent to showing that the vector

$$dF_0(u)(u') \in R^n$$

is jointly continuous in $u \in U$ and $u' \in P_0(x_0)$.

If $x = F_0(u)$, then the vector $dF_0(u)(u')$ is the unique point lying in the intersection of the k-dimensional subspace $P_0(x)$ with the $(n - k)$-plane through u' and orthogonal to $P_0(x_0)$. Now $x = F_0(u)$ varies continuously with u, and $P_0(x)$ varies continuously with x. Thus $P_0(x)$ varies continuously with u. The $(n - k)$-plane through u' and orthogonal to $P_0(x_0)$ varies continuously with u'. It then follows from the explicit formulas of Cramer's Rule that the intersection $dF_0(u)(u')$ of these two planes varies continuously with u and u'.

There is a slight catch in the above argument. In order to rely on Cramer's Rule, we must understand the continuous variability of a (linear or affine) subspace in terms of the continuous variability of an appropriate set of linear equations for it. This is all right for the $(n-k)$-plane through u' orthogonal to $P_0(x_0)$. But the continuous variability of $P_0(x)$ is understood in terms of angles, not equations. It is a trivial geometric argument to show that as the equations for a k-dimensional subspace vary continuously, so does the subspace in terms of angles. Indeed, this is how one might locally parametrize the Grassmann manifold $G_{n,k}$. But a one-to-one continuous parametrization of a manifold by an open subset of the same-dimensional Euclidean space is automatically a homeomorphism into. Thus, as a k-dimensional subspace of R^n varies continuously, it may also be assigned continuously varying linear equations (locally). This completes the argument, and with it the proof of Theorem 10.1.

11. PROOF OF THE MAIN THEOREM

I now prove the Main Theorem stated in Section 1.

Necessity. This will be proved without any upper bound on σ_0; if σ_0 is too large, then $(M)^{k+1}_{\sigma_0}$ is empty, and a map from the empty set can be continuously extended (in many ways) over $\triangle$.

Let M be a k-dimensional C^1-manifold in R^n, and consider the generalized secant map

$$\Sigma\colon (M)^{k+1}_{\sigma_0} \to G_{n,k}.$$

Extend Σ over the diagonal $\triangle$ of $(M)^{k+1}$ by defining $\Sigma(x, x, \cdots, x)$ to be $P_0(x)$, the k-dimensional subspace of R^n parallel to the tangent k-plane to M at x. The map Σ was already continuous on $(M)^{k+1}_{\sigma_0}$, which is open in $(M)^{k+1}_{\sigma_0} \cup \triangle$, and the extended Σ is continuous on $\triangle$ alone because $P_0(x)$ varies continuously with x (Theorem 10.1).

Let x be a point of M, and let $0 < \varepsilon \leq \pi/2$. According to Theorem 4.4, there exists a $\delta > 0$ such that if $\triangle^k$ is a k-simplex with shape $\sigma(\triangle^k) > \sigma_0$, each of whose edges makes an angle at most δ with $P_0(x)$, then the k-plane spanned by $\triangle^k$ makes an angle less than ε with $P_0(x)$. Let X be a neighborhood of x in M such that if y and z are distinct points of X, then the line through y and z makes an angle at most δ with $P_0(x)$ (mean-value theorem). If $(x_0, x_1, \cdots, x_k) \in (M)^{k+1}_{\sigma_0}$ and each x_i lies in the neighborhood X, then $\Sigma(x_0, x_1, \cdots, x_k)$ makes an angle less than ε with $\Sigma(x, x, \cdots, x) = P_0(x)$. Hence the extended Σ is indeed continuous.

Sufficiency. Now let M be a k-dimensional topological manifold in R^n such that the generalized secant map

$$\Sigma\colon (M)^{k+1}_{\sigma_0} \to G_{n,k} \qquad (0 < \sigma_0 \leq 1/\sqrt{k})$$

admits a continuous extension over $(M)^{k+1}_{\sigma_0} \cup \triangle$. To show that M is a C^1-manifold, I will show that M satisfies conditions (1), (2), and (3) of Theorem 10.1, and then invoke that theorem.

For each $x \in M$, let $P(x)$ be the k-plane in R^n through x and parallel to $\Sigma(x, x, \cdots, x)$. To show that M satisfies condition (1), I must show that $P(x)$ is a tangent k-plane to M at x. Let X be a connected open neighborhood of x in M such that if $(x_0, x_1, \cdots, x_k) \in (M)^{k+1}_{\sigma_0}$ with each x_i in X, then $\Sigma(x_0, x_1, \cdots, x_k)$ makes an angle less than ε with $\Sigma(x, x, \cdots, x)$, and hence with $P(x)$, where $\varepsilon > 0$ is some preassigned real number. I now claim that if y is any point of X such that $|y - x| <$ diam X, then the line through x and y makes an angle less than ε with $P(x)$. Choose any such y. Let $x_0 = x$ and $x_1 = y$. According to Theorem 8.2, there exist points $x_2, x_3, \cdots, x_k$ in X such that the k-simplex with vertices $x_0, x_1, x_2, \cdots, x_k$ has shape greater than $1/\sqrt{k}$. Since $\sigma_0 \leq 1/\sqrt{k}$, $(x_0, x_1, \cdots, x_k) \in (M)^{k+1}_{\sigma_0}$. By the initial choice of X, $\Sigma(x_0, x_1, \cdots, x_k)$ makes an angle less than ε with $P(x)$. *A fortiori,* the line through $x_0 = x$ and $x_1 = y$ makes an angle less than ε with $P(x)$, which therefore satisfies the definition of a tangent k-plane to M at x. This verifies condition (1).

Condition (2), that $P_0(x) = \Sigma(x, x, \cdots, x)$ varies continuously with x, follows because Σ is already continuous on $(M)^{k+1}_{\sigma_0} \cup \Delta$, *a fortiori* on Δ alone.

To verify condition (3), let X be a connected open neighborhood of x in M such that if $(x_0, x_1, \cdots, x_k) \in (M)^{k+1}_{\sigma_0}$ with each x_i in X, then $\Sigma(x_0, x_1, \cdots, x_k)$ makes an angle less than $\pi/2$ with $\Sigma(x, x, \cdots, x)$, and hence with $P(x)$. Let $X' \subset X$ be a neighborhood of x in M such that diam $X' <$ diam X. Let x_0 and x_1 be any two distinct points of X'. Since $|x_0 - x_1| \leq$ diam $X' <$ diam X, Theorem 8.2 guarantees the existence of points $x_2, x_3, \cdots, x_k$ in X such that $(x_0, x_1, \cdots, x_k) \in (M)^{k+1}_{\sigma_0}$. By choice of X, $\Sigma(x_0, x_1, \cdots, x_k)$ makes an angle less than $\pi/2$ with $\Sigma(x, x, \cdots, x)$, and hence with $P(x)$. In particular, the line through x_0 and x_1 makes an angle less than $\pi/2$ with $P(x)$. The orthogonal projection $\pi_x: R^n \to P(x)$ is one-to-one on X', hence condition (3) is also satisfied.

Now, by Theorem 10.1, M must be a C^1-manifold in R^n, and the proof of the Main Theorem is complete.

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