ON RECURSIVE SETS AND REGRESSIVE ISOLS

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1. INTRODUCTION

We shall assume that the reader is familiar with the concepts and main results of the papers listed as references. We let E denote the collection of all nonnegative integers *(numbers)*, Λ the collection of all isols, Λ^* the collection of all isolic integers, and Λ_R the collection of all regressive isols. It is known that $E \subseteq \Lambda_R \subseteq \Lambda$ and that each of the collections Λ_R - E and Λ - Λ_R has the cardinality of the continuum. In [7] and [8], A. Nerode associated with every recursive function f: $E \to E$ a function D_f: $\Lambda \to \Lambda^*$, and with every recursive set of numbers α a set α_Λ of isols. D_f is an extension of f from E to Λ, and $\alpha \subseteq \alpha_\Lambda$. In [9], Nerode proved the following result: let f be a recursive and eventually combinatorial function; then $D_f(\Lambda) \subseteq (f(E))_\Lambda$, and $D_f(\Lambda) = (f(E))_\Lambda$ if and only if there exists a number n such that f(n), f(n + 1), ··· is an arithmetic progression. This result motivated the problem considered in this paper. For a recursive set α, we define $\alpha_R = \Lambda_R \cap \alpha_\Lambda$. We are interested in comparing the two collections $D_f(\Lambda_R)$ and $(f(E))_R$ in the case where f is an eventually increasing recursive function.

A function f: $E \to E$ is *increasing* if $x < y$ implies $f(x) \leq f(y)$, and *eventually increasing* if there exists a number n such that the function $g(x) = f(x + n)$ is increasing. It was proved in [1] that if f is a recursive and eventually increasing function, then D_f: $\Lambda_R \to \Lambda_R$. The main result of this paper states that if f is a recursive and eventually increasing function, then $D_f(\Lambda_R) = (f(E))_R$.

2. EXTENSIONS

Let α be a set of numbers. The *characteristic* function of α, denoted by c_α, is defined by

$$c_\alpha(x) = \begin{cases} 0 & \text{if } x \in \alpha, \\ 1 & \text{if } x \notin \alpha. \end{cases}$$

If α is a recursive set, then c_α is a recursive function. Let C_α denote the extension of c_α to Λ. Then the extension of α to Λ can be characterized (see [7, Theorem 9.5] and [1, Section 5]) as the set

$$\alpha_\Lambda = \{X \mid X \in \Lambda \text{ and } C_\Lambda(X) = 0\}.$$

Combining this with the definition of α_R, we see that

$$(*) \qquad \alpha_R = \{X \mid X \in \Lambda_R \text{ and } C_\alpha(X) = 0\}.$$

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We state the next result without proof. It can be established by [1, Proposition 2] and by methods of infinite series of isols.

PROPOSITION 1. *Let* f *and* g *be eventually increasing recursive functions, and let* $f \circ g$ *denote the composition function* f(g(x)). *Then* $f \circ g$ *is also an eventually increasing recursive function, and*

$$D_{f\circ g}(T) = D_f(D_g(T)) \quad \text{for } T \in \Lambda_R .$$

PROPOSITION 2. *Let* f *and* g *be eventually increasing recursive functions such that* $f(E) \subseteq g(E)$. *Then there exists an eventually increasing recursive function* h *such that*

$$f(t) = g(h(t)) \quad \text{for } t \in E .$$

Proof. For $t \in E$, let the function h be defined by

$$h(t) = (\mu x)[g(x) = f(t)] .$$

We readily see that h is a recursive function and satisfies the desired identity of the proposition. Also, h is an eventually increasing function because each of the functions f and g is eventually increasing.

THEOREM 1. *Let* f *and* g *be eventually increasing recursive functions. Then*

(a) $$f(E) \subseteq g(E) \Rightarrow D_f(\Lambda_R) \subseteq D_g(\Lambda_R),$$

(b) $$f(E) = g(E) \Rightarrow D_f(\Lambda_R) = D_g(\Lambda_R).$$

Proof. Because (a) implies (b), we only prove part (a). Assume the left-hand side of (a). Then, by Proposition 2, there is an eventually increasing recursive function h such that

$$f(t) = g(h(t)) \quad \text{for } t \in E .$$

By Proposition 1, it follows from this identity that

(1) $$D_f(T) = D_g(D_h(T)) \quad \text{for } T \in \Lambda_R .$$

In addition, because h is eventually increasing, we know that $D_h\colon \Lambda_R \to \Lambda_R$. Combining this with (1), we see that

$$D_f(\Lambda_R) \subseteq D_g(\Lambda_R).$$

The next result is a special case of a well-known theorem of Nerode, and we state it without proof.

LEMMA. *Let* f, g, h, *and* k *be recursive functions. If*

$$(\forall t \in E)[f(t) = g(t) \Rightarrow h(t) = k(t)],$$

then

$$(\forall T \in \Lambda)[D_f(T) = D_g(T) \Rightarrow D_h(T) = D_k(T)].$$

Notation. Let $f\colon E \to E$ be a strictly increasing function. Then the function $\bar{f}$ is defined by

$$\bar{f}(t) = (\mu y)[f(y) \geq t] \quad \text{for } t \in E.$$

THEOREM 2. *Let* f *be a strictly increasing recursive function. Let* $\alpha = f(E)$, *let* c_α *be the characteristic function of* α, *and let* C_α *be the extension of* c_α *to* Λ. *Then* α *is a recursive set,* c_α *is a recursive function, and* $\bar{f}$ *is an increasing recursive function. In addition,*

(a) $$D_{\bar{f}}(D_f(T)) = T \quad \textit{for } T \in \Lambda_R,$$

(b) $$C_\alpha(T) = 0 \Leftrightarrow D_f(D_{\bar{f}}(T)) = T \quad \textit{for } T \in \Lambda_R,$$

(c) $$T \in (f(E))_R \Leftrightarrow D_f(D_{\bar{f}}(T)) = T \quad \textit{for } T \in \Lambda_R.$$

Proof. Each of the three statements, α is a recursive set, c_α is a recursive function, and $\bar{f}$ is an increasing recursive function, is easily seen to be true. Also it is readily proved that

(2) $$\bar{f} \circ f(t) = t \quad \text{for } t \in E.$$

In addition,

$$t \in \alpha \Leftrightarrow f \circ \bar{f}(t) = t \quad \text{for } t \in E,$$

and therefore also

(3) $$c_\alpha(t) = 0 \Leftrightarrow f \circ \bar{f}(t) = t \quad \text{for } t \in E.$$

The identity of (2) concerns only recursive functions, and therefore, by a well-known theorem of Nerode, yields

(4) $$D_{\bar{f} \circ f}(T) = T \quad \text{for } T \in \Lambda.$$

If we restrict the identity of (4) to regressive isols, then, by Proposition 1, it follows that

$$D_{\bar{f}}(D_f(T)) = T \quad \text{for } T \in \Lambda_R.$$

This proves part (a). Part (b) follows from (3), the lemma, and Proposition 1. Finally part (c) follows from (b) and the relation denoted by (*).

Remark. Part (a) of Theorem 2 was first proved by Sansone in [11].

3. THE PRINCIPAL THEOREM

PROPOSITION 3. *Let* f *be a strictly increasing recursive function. Then* $D_f(\Lambda_R) = (f(E))_R$.

Proof. Let T be a regressive isol. First assume that $T \in (f(E))_R$. Then, by Theorem 2(c),

(5) $$D_f(D_{\bar{f}}(T)) = T.$$

Because $\bar{f}$ is an increasing recursive function and T is a regressive isol, we know that $D_{\bar{f}}(T)$ is also a regressive isol. Combining this with (5), we see that $T \in D_f(\Lambda_R)$. This proves that $(f(E))_R \subseteq D_f(\Lambda_R)$.

To verify the inclusion in the other direction, let $T \in D_f(\Lambda_R)$, and let

(6) $$T = D_f(S), \quad \text{with } S \in \Lambda_R .$$

By Theorem 2(a), we then have the relation $D_{\bar{f}}(D_f(S)) = S$, and hence also the relation

(7) $$D_f(D_{\bar{f}}(D_f(S))) = D_f(S) = T .$$

Combining (6) and (7), we obtain

(8) $$D_f(D_{\bar{f}}(T)) = T .$$

Finally, in view of Theorem 2(c), it follows from (8) that $T \in (f(E))_R$. Therefore we also have the inclusion $D_f(\Lambda_R) \subseteq (f(E))_R$, and this completes the proof.

Combining Proposition 3 together with Theorem 2(c), one obtains the following corollary, proved first by Sansone [11, Section II, Proposition 2]. It is readily seen that the corollary and Theorem 2(c) together imply Proposition 3.

COROLLARY (Sansone). *Let* f *be a strictly increasing recursive function. Then*

$$T \in D_f(\Lambda_R) \iff D_f(D_{\bar{f}}(T)) = T \quad \textit{for } T \in \Lambda_R .$$

THEOREM 3. *Let* f *be an eventually increasing recursive function. Then* $D_f(\Lambda_R) = (f(E))_R$.

Proof. First assume that $f(E)$ is finite. Then

(9) $$(f(E))_R = f(E) .$$

In addition, f is an eventually constant function, and from this it follows that

(10) $$D_f(\Lambda_R) = D_f(E) = f(E) .$$

Combining (9) and (10), we obtain the desired result.

Assume now that $f(E)$ is an infinite set. Since f is an eventually increasing recursive function, $f(E)$ is an (infinite) recursive set. Let f^* be the strictly increasing recursive function that ranges over $f(E)$. Then

(11) $$f^*(E) = f(E),$$

and therefore

(12) $$(f^*(E))_R = (f(E))_R .$$

By Theorem 1(b), we also obtain from (11) the identity

(13) $$D_{f^*}(\Lambda_R) = D_f(\Lambda_R) .$$

Finally, by Proposition 3 it follows that

(14) $$D_{f^*}(\Lambda_R) = (f^*(E))_R .$$

Combining (12), (13), and (14), we see that $D_f(\Lambda_R) = (f(E))_R$, and this completes the proof.

It can readily be shown that if α is a finite set, then $\alpha_R = \alpha$. In addition, if α is a recursive set and $\alpha_R \subseteq E$, then α is finite and $\alpha_R = \alpha$. This last property is obtainable from Proposition 3, and by noting that if f is a strictly increasing recursive function, then $D_f(\Lambda_R) - E$ is not empty.

Let α and β be recursive sets. By [7, Theorem 3.1 (3.4)], one sees that

(15) $$(\alpha \cap \beta)_\Lambda = \alpha_\Lambda \cap \beta_\Lambda .$$

In view of the definition of the extension to Λ_R of a recursive set, it follows from (15) that

(16) $$(\alpha \cap \beta)_R = \alpha_R \cap \beta_R .$$

THEOREM 4. *Let* f *and* g *be eventually increasing recursive functions such that the intersection of their ranges is not empty. Let* h *be any eventually increasing recursive function with*

$$h(E) = f(E) \cap g(E) .$$

Then

$$D_h(\Lambda_R) = D_f(\Lambda_R) \cap D_g(\Lambda_R) .$$

Proof. Let $\alpha = f(E)$ and $\beta = g(E)$, and use (16) and Theorem 3.

Concluding remark. We consider an application of Theorem 4 to a particular arithmetic statement that is true in E, and was proved, in [9], to be false in Λ. It is the statement

($\triangle$) if x is both a square and a cube, then x is a sixth power .

We want to show that ($\triangle$) is a true statement in Λ_R. For this purpose, let the functions f, g, and h be defined by

$$f(t) = t^2 , \quad g(t) = t^3 , \quad h(t) = t^6 .$$

Each of the functions f, g, and h is strictly increasing and recursive. Also, it is readily seen that

(17) $$f(E) \cap g(E) = h(E) .$$

In view of Theorem 4, it follows from (17) that

(18) $$D_f(\Lambda_R) \cap D_g(\Lambda_R) = D_h(\Lambda_R) .$$

Also, because

(19) $$D_f(T) = T^2 , \quad D_g(T) = T^3 , \quad \text{and } D_h(T) = T^6 , \qquad \text{for } T \in \Lambda_R ,$$

we know that if $X \in \Lambda_R$ and X is both a square (of some regressive isol) and a cube (of some regressive isol) then

$$X \in D_f(\Lambda_R) \quad \text{and} \quad X \in D_g(\Lambda_R) .$$

In view of (18) and (19), this also means that

$$X \in D_h(\Lambda_R) ,$$

and therefore that X is the sixth power (of some regressive isol). It follows that (Δ) denotes a true arithmetic statement in Λ_R.

Added April 18, 1967. Since this paper was submitted, Matthew Hassett has obtained some other arithmetic statements that are true in E and Λ_R but not in Λ. One particular example is the statement

$$2^A = B^C \Rightarrow (\exists W)[B = 2^W],$$

which had also been shown to be false in Λ, by Nerode [9].

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