ARBITRARY FUNCTIONS DEFINED ON PLANE SETS

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By an *arbitrary function* f we mean a (single-valued) function whose domain is a subset of the complex plane Z and whose range is in the one-point compactification K of three-dimensional Euclidean space. (We are particularly interested in the case where the range of f is on the Riemann sphere. Actually, our proofs only require that the range space be a compact Hausdorff space possessing a countable base of open sets.) If J is a Jordan arc in Z with one endpoint z, then $J - \{z\}$ is an *arc at* z. If f is defined at every point of an arc A at z, then the *cluster set of* f *on* A *at* z, denoted by $C_A(f, z)$, is defined to be the set of points $P \in K$ for which there exists a sequence $\{z_n\}$ of points of A such that $z_n \to z$ and $f(z_n) \to P$. We say that the arcs $A_1, \cdots, A_n$ at z $(n = 2, 3, \cdots)$ are *separating arcs of* f *at* z provided they are contained in the domain of f and no two of the cluster sets $C_{A_j}(f, z)$ $(j = 1, \cdots, n)$ intersect.

THEOREM 1. *Let* $\mathfrak{F}$ *be a family of subsets of* Z *such that* $\{z\} \in \mathfrak{F}$ *for every* $z \in Z$, *such that* $\bigcup S_n \in \mathfrak{F}$ *if* $S_n \in \mathfrak{F}$ $(n = 1, 2, \cdots)$, *and such that if* $S \in \mathfrak{F}$ *and* $S_0 \subset S$, *then* $S_0 \in \mathfrak{F}$. (For example, $\mathfrak{F}$ might be the family of all countable subsets of Z.) *Let* f *be an arbitrary function whose domain is* Z, *and suppose that for each point* z *of a subset* E *of* Z *there exist separating arcs* A_1, A_2, *and* A_3 *of* f *at* z *such that* $A_j \cap E \in \mathfrak{F}$ $(j = 1, 2, 3)$. *Then* $E \in \mathfrak{F}$.

Proof. Let $\mathfrak{B}_0$ denote a countable base of open sets for the topology of K, and let $\mathfrak{B}$ denote the family of finite unions of open sets in $\mathfrak{B}_0$. Now consider an arbitrary point $z \in E$. Let A_j^* $(j = 1, 2, 3)$ be separating arcs of f at z such that $A_j^* \cap E \in \mathfrak{F}$ $(j = 1, 2, 3)$. Let U_j $(j = 1, 2, 3)$ be elements of $\mathfrak{B}$, no two of which intersect, such that $C_{A_j^*}(f, z) \subset U_j$ $(j = 1, 2, 3)$. We can find an open disc D, whose radius is rational and whose center has rational real and imaginary parts, such that

(i) $z \in D$,

(ii) each A_j^* intersects the circumference C of D,

(iii) $f(A_j) \subset U_j$ $(j = 1, 2, 3)$, where A_j denotes the subarc of A_j^* that contains the point z and lies in D except for one endpoint z_j on C.

Note that $A_j \cap E \in \mathfrak{F}$ $(j = 1, 2, 3)$. Let γ_j $(j = 1, 2, 3)$ be open arcs of C, no two of which intersect, such that $z_j \in \gamma_j$ $(j = 1, 2, 3)$ and the radii of D terminating in the endpoints of the arcs γ_j have rational slopes. We call the set

$$\{D, (\gamma_1, U_1), (\gamma_2, U_2), (\gamma_3, U_3)\}$$

a *collection for* z (or simply a *collection*). There exist only countably many collections.

Suppose that $E \notin \mathfrak{F}$. Then there exists a subset E_0 of E such that $E_0 \notin \mathfrak{F}$ and one collection $\{D, (\gamma_1, U_1), (\gamma_2, U_2), (\gamma_3, U_3)\}$ is a collection for each point of E_0. Suppose now that z_1 and z_2 are any two distinct points of E_0. Let A_j^k $(j = 1, 2, 3)$ be arcs at z_k $(k = 1, 2)$ such that $A_j^k \cap E \in \mathfrak{F}$, A_j^k lies in D except for one endpoint

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on γ_j, and $f(A_j^k) \subset U_j$ (k = 1, 2; j = 1, 2, 3). Then $A_j^1 \cap A_\ell^2 = \emptyset$ if $j \neq \ell$. Let $T_k = A_1^k \cup A_2^k \cup A_3^k$ (k = 1, 2). Then either $z_1 \in T_2$ or $z_2 \in T_1$: if, for example, $z_1 \notin T_2$ and z_1 is in the component of $D - (T_2 \cup \{z_2\})$ whose boundary contains $A_1^2 \cup A_2^2$, then $z_2 \in A_3^1$.

For each point $z_1 \in E_0$ there exists a set T_1 defined as above. Clearly $E_0 \not\subset T_1 \cup \{z_1\}$, because $(T_1 \cap E) \cup \{z_1\} \in \mathfrak{F}$. Hence there exists an open disc Δ, whose radius is rational and whose center has rational real and imaginary parts, such that $\Delta \cap (T_1 \cup \{z_1\}) = \emptyset$ and $\Delta \cap E_0 \neq \emptyset$. Therefore, we can find one such open disc Δ that intersects E_0, and a subset E^* of E_0 such that $E^* \notin \mathfrak{F}$ and for each $z_1 \in E^*$ there exists a set T_1, defined as above, that satisfies the relation $\Delta \cap (T_1 \cup \{z_1\}) = \emptyset$. Choose $z_2 \in \Delta \cap E_0$ and let T_2 be defined as above. Now choose $z_1 \in E^*$, and let T_1 be a set, defined as above, such that $z_2 \notin T_1$. Then $z_1 \in T_2$. We have shown that $E^* \subset T_2$, which is a contradiction, since $T_2 \cap E \in \mathfrak{F}$. The proof of Theorem 1 is complete.

We note a trivial example.

Example 1. Let f assume exactly three values and be constant on the (open) upper half-plane, on the lower half-plane, and on the real axis. Then there exist three separating arcs (which can be taken to be rectilinear segments) of f at each point of the real axis.

COROLLARY 1. *Let* S *be an arbitrary subset of* Z, *and let* f *be an arbitrary function whose domain is* S. *Let* E *be the set of points of* Z - S *at which there exist three separating arcs of* f. *Then* E *is countable.*

Remark 1. The first two paragraphs of the proof of Theorem 1 suffice to prove Corollary 1.

Remark 2. Bagemihl's ambiguous-point theorem [1] says, in our terminology, that if f is an arbitrary function whose domain is $\{|z| < 1\}$ and whose range is on the Riemann sphere Ω, and if E is the set of points of $\{|z| = 1\}$ at which there exist two separating arcs of f (such a point is an *ambiguous point* of f), then E is countable. To see that this theorem follows from Corollary 1, extend the definition of f to the complement of $\{|z| = 1\}$ by giving it on $\{|z| > 1\}$ a constant value that is in $K - \Omega$.

Remark 3. Part of the argument in the first paragraph of the proof of Theorem 1 was given by R. L. Moore [3] in the proof of a theorem on triods. Corollary 1 is readily seen to imply the following special case of Moore's theorem: A family of mutually exclusive triods whose rays are Jordan arcs is countable (for the terminology, see [3]).

Remark 4. It follows from Corollary 1 that if the domain of f is an open set U, and if E denotes the set of points of the boundary of U at which there exist three separating arcs of f, then E is countable..

Example 2. There exists a function f, whose domain is a subset S of Z, such that the boundary B of S is uncountable, and such that at each point of B there exist three separating arcs of f. To see this, let C be a Cantor set on the real axis, and let $\{z_n\}$ be a sequence of points of the (open) upper half-plane such that the set of cluster points of $\{z_n\}$ is C. Let $S = Z - \{z_n\}$. Let f have a constant value P_1 on the lower half-plane and a different constant value P_2 on the real axis, and in the part of S in the upper half-plane, let f be bounded away from P_1 and P_2 and be such that at each z_n there exist three separating arcs of f. Then f clearly has the desired property.

Since a Jordan arc in Z is nowhere dense, Theorem 1 has the following corollary.

COROLLARY 2. *Let* f *be an arbitrary function whose domain is* Z, *and let* E *be the set of points of* Z *at which there exist three separating arcs of* f. *Then* E *is a set of the first category.*

Since a rectifiable Jordan arc is a set of two-dimensional (Lebesgue) measure zero, we obtain a further corollary to Theorem 1.

COROLLARY 3. *Let* f *be an arbitrary function whose domain is* Z, *and let* E *be the set of points of* Z *at which there exist three rectifiable separating arcs of* f. *Then* E *is a set of two-dimensional measure zero.*

Example 3. Let J be a Jordan curve with positive two-dimensional measure, and let f assume exactly three values and be constant on the interior domain of J, on the exterior domain of J, and on J. Then at each point of J there exist three separating arcs of f.

Example 4. According to Bagemihl [2, Theorem 9], there exists a function f, defined on Z, with the property that every point of Z is a rectilinearly oppositely ambiguous point of f. This means that at each point of Z there exist two separating arcs of f that are oppositely directed rectilinear segments.

THEOREM 2. *Let* f *be an arbitrary function whose domain is* Z, *and let* E *be the set of points of* Z *at which there exist four separating arcs of* f. *Then* E *is countable.*

Proof. Suppose that E is uncountable. Define $\mathfrak{B}$ as in the proof of Theorem 1. By a routine argument, there exist open sets U_j (j = 1, 2, 3, 4) in $\mathfrak{B}$, no two of which intersect, such that at each point z of an uncountable subset E_0 of E there exist arcs A_j (j = 1, 2, 3, 4) satisfying the relations $C_{A_j}(f, z) \subset U_j$ and $f(A_j) \subset U_j$ (j = 1, 2, 3, 4). For each $z \in E_0$, at least three of the sets U_j (j = 1, 2, 3, 4) do not contain f(z). Therefore, there exist an uncountable subset E^* of E_0 and three of the sets U_j (j = 1, 2, 3, 4) that do not intersect $f(E^*)$. Let the notation be such that

$$f(E^*) \cap (U_1 \cup U_2 \cup U_3) = \emptyset.$$

Then at each point $z \in E^*$ there exist arcs A_1, A_2, and A_3 such that $A_j \subset Z - E^*$ and $C_{A_j}(f, z) \subset U_j$ (j = 1, 2, 3). By Corollary 1, we have a contradiction, and the proof of Theorem 2 is complete.

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