WILD POINTS OF CELLULAR SUBSETS OF 2-SPHERES IN S^3

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1. INTRODUCTION

In this note we prove that if W is the set of all wild points of a cellular arc that lies on a 2-sphere in S^3, then either W is empty, W is degenerate, or W contains an arc (Theorem 1). Thus there are two types of wild cellular arcs that lie on 2-spheres in S^3: those with exactly one wild point and those with an arc of wild points.

Examples of arcs belonging to each of these two categories have already been described. The arc A constructed on the boundary of a 3-cell C in S^3 by Alford [1] is wild at each of its points. Since the proof for Theorem 5 in [8] also shows that C is cellular, it follows from [12] that A is cellular. Fox and Artin [6] have given an example of a cellular arc on a 2-sphere in S^3 such that the arc has exactly one wild point. It follows from Theorem 2 that a cellular arc on a 2-sphere in S^3 cannot contain two isolated wild points.

In Section 3, we shall show that the results mentioned in the previous paragraphs also hold for cellular finite graphs on 2-spheres in S^3. Section 4 deals with sufficient conditions for certain subsets of 2-spheres to be tame modulo finite sets.

Recently, Burgess [4] gave sufficient conditions for 2-spheres in S^3 to be locally tame except at two points. We make strong use of his results here. In Section 4 of [11], we use the techniques of [4] to prove similar results in a slightly more general setting. Thus, wherever we refer to [4], we could have used [11] instead.

2. DEFINITIONS AND NOTATION

A subset G of S^3 is *cellular* (in S^3) if and only if there exists a sequence $C_1, C_2, \cdots$ of 3-cells in S^3 such that for each positive integer i, $C_{i+1} \subset \operatorname{Int} C_i$ and $G = \bigcap_{i=1}^{\infty} C_i$. A *finite graph* is the union of a finite collection of arcs such that if p is a point of intersection of two of these arcs then p is an endpoint of each of the two arcs.

A 2-sphere S in S^3 is *locally tame* at a point p if there exists a disk D on S and a homeomorphism h of S^3 onto itself such that $p \in \operatorname{Int} D$ and h(D) is a polyhedron. Furthermore, S is *tame* in S^3 if there exists a homeomorphism h of S^3 onto itself such that h(S) is a polyhedron. We define a subset X of a 2-sphere to be *tame* if X lies on a tame 2-sphere. Also, we say that X is *locally tame* at a point p of X if X lies on a 2-sphere that is locally tame at p. A set X is *locally tame* modulo K if and only if it is locally tame at each point of X - K. We say that a point p of X is a *wild point* of X (or that X is wild at p) if and only if X is not locally tame at p.

A wild point p of K is called an *isolated wild point* of a set K if it lies in an open subset O of K such that K is locally tame at each point of $O - \{p\}$.

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3. CELLULAR FINITE GRAPHS

Because the following lemma is well known, we do not include a detailed proof. Harrold, Griffith, and Posey outline a proof for it in the second and third paragraphs of their proof for Lemma 5.1 of [9].

LEMMA 1. *Suppose that* $\varepsilon > 0$, *that* α *is the boundary of a disk* D *on a* 2-*sphere* S *in* S^3 *such that* D *is locally polyhedral at its interior points, and that* V *is a component of* S^3 - S; *then there exists a* 3-*cell* C *in* S^3 *such that*

(1) D $\subset$ Bd C,

(2) C - D $\subset$ V,

(3) Bd C *is locally polyhedral at every point of* Bd C - α, *and*

(4) C *lies in an* ε-*neighborhood of* D.

For the sake of completeness, we indicate briefly how the 3-cell C in Lemma 1 can be obtained. Since D is locally polyhedral at its interior points, one can push Int D slightly into V to obtain a crumpled cube (a 2-sphere together with one of its complementary domains) satisfying all four conditions in the conclusion of Lemma 1. The only difficulty lies in showing that, under suitable restrictions on the "pushing", the crumpled cube is a 3-cell.

LEMMA 2. *If a cellular finite graph* G *on a* 2-*sphere in* S^3 *is locally tame modulo a* 0-*dimensional set, then* G *has at most two wild points.*

Proof. Let S be a 2-sphere containing G such that S is locally polyhedral except at points of G [2], let K be the 0-dimensional set of wild points of G, and let p be a point of G - K. Since K is closed, there exists an open set O containing p such that O $\cap$ K = $\emptyset$ and O $\cap$ G is a tame finite graph. Since S is locally tame at every point of O $\cap$ S - G and O $\cap$ G is a tame finite graph, it follows (see [3] or [5]) that S is locally tame at each point of O. This implies that S is locally tame modulo K. It follows from Corollary 2.4 of [13] that each component of S^3 - S is an open 3-cell. Together with Corollary 1 of [4], this implies that S has at most two wild points. Since G $\subset$ S, the graph G has at most two wild points.

LEMMA 3. *If a cellular finite graph* G *on a* 2-*sphere in* S^3 *is locally tame modulo a* 0-*dimensional set* W, *then there exists a cellular* 3-*cell* C *such that*

(1) G $\subset$ Bd C *and*

(2) Bd C *is locally tame modulo* W.

Proof. Lemma 2 asserts that W has at most two points; let p and q be points such that G is locally tame modulo $\{p, q\}$. Let S be a 2-sphere containing G and such that S - G is locally polyhedral [2], and let D' be a disk on S such that G $\subset$ D'. Using the fact that G is cellular, we can obtain a subdisk D of D' such that G $\subset$ D and p and q both lie on Bd D. Using Theorem 8.5 of [3], we see that S is locally tame at each point of Int D; therefore we may assume without loss in generality that Int D is locally polyhedral. Now we apply Lemma 1 to obtain a 3-cell C such that D $\subset$ Bd C and Bd C is locally polyhedral modulo Bd D. Since p and q are the only possible wild points of Bd D, we use [3] again and deduce that Bd C is locally tame modulo $\{p, q\}$. Thus we may assume that Bd C is locally polyhedral modulo W and hence locally polyhedral modulo G; therefore Corollary 2.4 of [13] is applicable, and we conclude that S^3 - C is an open 3-cell. Thus C is a cellular 3-cell satisfying (1) and (2) in Lemma 3.

THEOREM 1. *If* G *is a cellular finite graph on a* 2-*sphere in* S^3 *and* W *is the set of wild points of* G, *then either*

(1) $W = \emptyset$,

(2) W *is degenerate, or*

(3) W *contains a cellular arc.*

Proof. Suppose W contains no cellular arc. Then it follows from Theorem 3.1 of [13] that W contains no arc. Thus W is totally disconnected and hence 0-dimensional. Applying Lemma 3, we see that G lies on the boundary of a cellular 3-cell C such that Bd C is locally tame modulo W. Now we apply Corollary 2 of [4] and see that Bd C has at most one wild point. Thus either $W = \emptyset$ or W consists of a single point. This completes the proof.

THEOREM 2. *A cellular finite graph on a* 2-*sphere in* S^3 *cannot have two isolated wild points.*

Proof. Suppose that G is a cellular finite graph on a 2-sphere in S^3 and that G has two isolated wild points p and q. Let S be a 2-sphere that contains G and is locally polyhedral modulo G [2], and let A be an arc in G that contains both p and q. In the next paragraph, we shall show that p and q are wild points of A.

There exists a disk D on S such that p lies in Int D, p is the only wild point of G in D, and $G \cap D$ consists of a finite collection H of arcs each having p as an endpoint. If A were locally tame at p, it would follow from Theorem 5 of [5] that each arc of H is tame. Then we could apply Theorem 2 of [7] and Theorems 21 and 6 of [10] to show that $D \cap G$ is tame, which contradicts the fact that p is a wild point of G. Thus p and q are isolated wild points of A.

By Lemma 1, there exists a 3-cell C such that $A \subset \text{Bd } C$ and Bd C is locally polyhedral modulo A. It follows that C is cellular [13] and that p and q are isolated wild points of Bd C. This contradicts Corollary 2 of [4].

4. LOCALLY CELLULAR SETS

A finite graph G is *locally cellular at a point* p of G if and only if there exists an open subset O of G such that p belongs to O and the closure of O is a cellular finite graph. We say that G is *locally cellular* if G is locally cellular at each point of G.

THEOREM 3. *If a locally cellular finite graph* G *on a* 2-*sphere in* S^3 *contains no arc of wild points, then it has at most a finite number of wild points.*

Proof. For each point p in G, let O_p be an open subset of G such that p is in O_p and $\overline{O}_p$ is a cellular finite graph G_p. Let $\{G_{p_1}, G_{p_2}, \cdots, G_{p_n}\}$ be a finite subcollection of the set $\{G_p \mid p \text{ is in } G\}$ such that the set $\{O_{p_1}, O_{p_2}, \cdots, O_{p_n}\}$ covers G. By Theorem 1, each G_{p_i} has at most one wild point. Let q_i be a point of G_{p_i} such that G_{p_i} is locally tame modulo $\{q_i\}$. If x is a point of $G - \bigcup_{i=1}^{n} \{q_i\}$, then x belongs to some $O_{p_i} - \{q_i\}$, and so G is locally tame at x. Thus the q_i are the only possible wild points of G.

THEOREM 4. *Suppose* G *is a finite graph on a* 2-*sphere in* S^3. *If* G *contains no arc of wild points and each interior point of* G *lies in the interior of a cellular arc in* G, *then* G *has at most a countable number of wild points.*

Proof. Let $A_1, A_2, \cdots$ be a countable collection of cellular arcs in G such that the union of the interiors of the A_i covers the interior of G. It follows from Theorem 1 that each A_i has at most one wild point. Let p_i be a point of A_i such that A_i is locally tame modulo $\{p_i\}$, and let H be the set of all the p_i together with all the vertices of G. Then H is a countable set, and if x is a point of G - H, then x lies in the interior of a tame arc in some A_i; therefore G is locally tame at x. Thus G is locally tame modulo H.

Example. We shall construct an example to show that Theorem 4 does not remain true when "countable" is replaced by "finite" in the conclusion. Let $E_1, E_2, \cdots$ be a countable collection of disjoint disks on a polyhedral 2-sphere S in S^3, and let the disks E_i converge to a point p. By "swelling" the arc described by Fox and Artin in Example 1.2 of [6], we obtain a disk D that is locally polyhedral modulo a point of Int D. We replace each E_i with a copy D_i of D such that D_i is locally polyhedral modulo a point p_i in Int D_i. The D_i can be constructed so that they are disjoint and converge to p; the set

$$\left(S - \bigcup E_i\right) \cup \left(\bigcup D_i\right)$$

is then a 2-sphere S' that is locally tame modulo $\{p, p_1, p_2, \cdots\}$. Let T be an arc in S' containing all the p_i and having p as an endpoint. Then T is wild at each p_i and locally tame at each other interior point. To show that T satisfies the hypothesis of Theorem 4, we let x be an interior point of T. We need to show that x lies in the interior of a cellular arc in T. This is obvious if x is not a p_i, because T is locally tame at such points. If $x = p_j$ for some j, we let T_j be a subarc of T in D_j such that x is an interior point of T_j. It follows from [6] that $(S - E_j) \cup D_j$ is a 2-sphere S_j such that each component of $S^3 - S_j$ is an open 3-cell. Since S_j is locally polyhedral modulo T_j, it follows (see [12] or [13]) that T_j is cellular.

THEOREM 5. *If the hypothesis of Theorem 4 is satisfied and G is locally tame at each of its vertices, then G has at most finitely many wild points.*

Proof. By Theorem 4, G has at most countably many wild points. Suppose $\{p_1, p_2, \cdots\}$ is a countable point set in G such that G is wild at each p_i. Since G is compact, some point p is a limit point of the set of p_i. There exists a cellular arc A in G such that p is a point of A. In fact, since p cannot be a vertex of G, we may choose A so that p lies in the interior of A. But now it follows from Theorem 1 that there exists an open subset O of G such that p belongs to O and G is locally tame at every point of O - $\{p\}$. This is a contradiction.

THEOREM 6. *If a connected finite graph G on a 2-sphere in S^3 contains no arc of wild points, and if every arc in G is cellular, then G has at most one wild point.*

Proof. Suppose G contains two wild points p and q, and let A be an arc in G containing both p and q. As we showed in the proof of Theorem 2, A is wild at both p and q. Since A is cellular and contains no arc of wild points, it follows from Theorem 1 that A is locally tame modulo some point. This is a contradiction.

THEOREM 7. *If G is a cellular subset of a 2-sphere S in S^3, and if S is locally tame modulo G and W is the set of wild points of S, then either*

(1) W *is degenerate,*

(2) W *contains exactly two points, or*

(3) W *contains a nondegenerate continuum.*

Proof. Since each component of S^3 - S is an open 3-cell [13], Theorem 7 follows from Corollary 2 of [4].

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