ALMOST COMBINATORIAL MANIFOLDS AND THE ANNULUS CONJECTURE

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Dedicated to Professor Raymond L. Wilder on his seventieth birthday.

In this paper we investigate a class of conjectures related to the conjectures $\beta(n, m, k)$ in [5]. We prove (Corollary 3.5) that each member of a fairly large class of these conjectures is equivalent to the annulus conjecture. This equivalence leads to an example of a set in R^4 that could conceivably be wild, in which case the 4-dimensional annulus conjecture would be false. As a by-product of Section 3, some positive results are obtained; for example, in S^8, the union of a tame 6-cell D_1 and a tame 4-cell D_2 meeting in a 2-cell D is tame, provided D is locally flat in both Bd D_1 and Bd D_2.

In the first two sections we establish some results needed for the main "equivalence" theorems (Theorems 3.2 and 3.3). However, the results in these first sections are proved in greater generality than is needed later, and they are of independent interest.

Definitions. An n-manifold N is said to be an *almost-combinatorial* n-*manifold* (abbreviated AC n-*manifold*) provided that both Bd N and Int N support locally finite combinatorial structures; in this case, N will be assumed to be already equipped with such structures. The most notorious examples of almost-combinatorial manifolds are the fake annuli; a *fake* n-*annulus* is a manifold that is homeomorphic to the closure of a region in S^n bounded by a nonintersecting pair of locally flat (n - 1)-spheres in S^n. If one can triangulate a given fake annulus A, then it follows that A is homeomorphic to $S^{n-1} \times [0, 1]$. Thus, if one could triangulate every fake n-annulus, the annulus conjecture would be settled affirmatively in dimension n.

As usual, an *isotopy* of a space X is a collection h_t $(0 \le t \le 1)$ of homeomorphisms of X onto itself such that the map H: $X \times I \to X$ defined by $H(x, t) = h_t(x)$ is continuous. If f, g: $X \to Y$ are embeddings and $B \subset Y$, then f and g are *ambient isotopic leaving* B *fixed* if there exists an isotopy h_t $(0 \le t \le 1)$ of Y such that h_0 is the identity, $h_t \mid B$ is the identity for each t, and $h_1 f = g$.

Our definition of *locally tame* is the same as that used by Gluck in [9]. The concepts of *locally flat embedding* and *submanifold* are well known.

1. EMBEDDINGS INTO ALMOST-COMBINATORIAL MANIFOLDS

In the following, I denotes the unit interval [0, 1], and B stands for the set $\{0, 1\}$. If X is a space and F is an embedding of $X \times I$ into the manifold N such that $F(X \times B) \subset$ Bd N, we say that F *agrees with a collar structure for* Bd N *in a neighborhood of* $X \times B$ provided there exists a collaring G: Bd $N \times [0, 1/2) \to N$ (that is, an embedding such that $G(x, 0) = x$ for $x \in$ Bd N) and an $\varepsilon > 0$ such that

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$$G(f(x, 0), t) = f(x, t) \quad \text{and} \quad G(f(x, 1), t) = f(x, 1 - t)$$

for $0 \leq t \leq \varepsilon$.

THEOREM 1.1. *Let* N *be an* AC n-*manifold, and let* K *be a (finite)* k-*complex* ($n \geq 2k + 4$). *Suppose that* $f: K \times B \to \text{Bd } N$ *is a locally tame embedding such that* $f \mid K \times 0$ *and* $f \mid K \times 1$ *are homotopic as maps of* K *into* N. *Then* f *can be extended to an embedding* $F: K \times I \to N$ *such that* $F \mid K \times (0, 1)$ *is a piecewise linear embedding of* $K \times (0, 1)$ *into* Int N. F *may be taken to agree with a collar structure for* Bd N *in a neighborhood of* $K \times B$.

Before proving this existence theorem, we state the corresponding uniqueness theorem and a corollary.

THEOREM 1.2. *Let* N *be an* AC n-*manifold, and let* K *be a (finite)* k-*complex* ($n \geq 2k + 4$). *Suppose that* $f, g: K \times I \to N$ *are embeddings such that* $f \mid K \times B = g \mid K \times B$ *maps* $K \times B$ *into* Bd N *and such that* $f \mid K \times (0, 1)$ *and* $g \mid K \times (0, 1)$ *are locally tame embeddings of* $K \times (0, 1)$ *into* Int N. *If* f *and* g *are homotopic through maps* $\phi_t: K \times I \to N$ $(0 \leq t \leq 1)$ *such that*

$$\phi_t \mid K \times B = f \mid K \times B \quad \textit{and} \quad \phi_t(K \times (0, 1)) \subset N - f(K \times B)$$

for each t, *then* f *and* g *are ambient isotopic leaving* Bd N *fixed.*

COROLLARY 1.3. *Let* A *be a fake* n-*annulus, and* K *a* k-*complex* ($n \geq 2k + 4$), *and let* $f: K \times B \to \text{Bd } A$ *be a locally tame embedding. Then* f *has an extension* $F: K \times I \to A$ *such that* $F \mid K \times (0, 1)$ *is a locally tame embedding of* $K \times (0, 1)$ *into* Int A; *any two such extensions of* f *are ambient isotopic leaving* Bd A *fixed; moreover, each of the extensions agrees with a collar structure for* Bd A *in a neighborhood of* $K \times B$.

The corollary follows immediately from the theorems, since A has the homotopy type of S^{n-1}.

Proof of Theorem 1.1. First we extend f to an embedding

$$g: K \times [0, 1/3] \cup K \times [2/3, 1] \to N$$

by making g agree with a collar for Bd N. Thus $g \mid (K \times (0, 1/3] \cup K \times [2/3, 1))$ is a locally tame embedding of $K \times (0, 1/3] \cup K \times [2/3, 1)$ into Int N. By Theorem 9.1 of [9], there is a homeomorphism h of N onto itself such that $h \mid \text{Bd } N$ is the identity and $hg \mid K \times (0, 1/3] \cup K \times [2/3, 1)$ is piecewise linear.

Let $\bar{g} = hg \mid K \times \{1/3, 2/3\}$. By the hypothesis that $f \mid K \times 0$ and $f \mid K \times 1$ are homotopic, it is easy to construct an extension $\bar{G}: K \times [1/3, 2/3] \to \text{Int } N$ of $\bar{g}$ such that $\bar{G}$ is piecewise linear on $K \times [1/3, 2/3]$. Letting $G = \bar{G}$ on $K \times [1/3, 2/3]$ and $G = hg$ on $K \times [0, 1/3] \cup K \times [2/3, 1]$, we see that G is an extension of f that is an embedding on $K \times [0, 1/3] \cup K \times [2/3, 1]$ and a piecewise linear map of $K \times (0, 1)$ into Int N. Since $n \geq 2(k + 1) + 2$, a general position argument completes the proof.

Proof of Theorem 1.2. Let C be a collar for Bd N in N; we identify C with $\text{Bd } N \times [0, 1/2)$, where $(x, 0) = x$ is a point of Bd N (see [2]).

For each $t \in [0, 1/2]$, let $X_t = K \times [0, t] \cup K \times [1 - t, 1]$. Thus $X_{1/2} = K \times [0, 1]$ and $X_0 = K \times \{0, 1\} = K \times B$. By Theorem 3.1 of [12], we may assume that $f \mid X_{1/3} = g \mid X_{1/3}$, that

$$f(x, t) = (f(x, 0), t) = g(x, t) = (g(x, 0), t) \in C \quad \text{for } 0 \leq t \leq 1/3,$$

and that

$$f(x, t) = (f(x, 1), 1 - t) = g(x, t) = (g(x, 1), 1 - t) \in C \qquad \text{for } 2/3 \leq t \leq 1.$$

We use the collar C to construct a homotopy ψ_t: $K \times I \to N$ $(0 \leq t \leq 1)$ between f and g such that

$$\psi_t \mid X_{1/6} = f \mid X_{1/6} \quad \text{and} \quad \psi_t(K \times (0, 1)) \subset \operatorname{Int} N \quad \text{for } 0 \leq t \leq 1 .$$

(See the next paragraph for a construction of ψ.) Then we let N_0 be a compact combinatorial submanifold of Int N that contains the image of $K \times [1/6, 5/6]$ under the homotopy ψ. By Theorem 9.1 or [9], it may be assumed that $f \mid K \times (0, 1)$ and $g \mid K \times (0, 1)$ are piecewise linear, so that $f^{-1}(N_0)$ and $g^{-1}(N_0)$ are not only the same set, but this set is a subpolyhedron P of $K \times I$. Applying Theorem 3.1 of [12], we see that the embeddings $f \mid P$ and $g \mid P$ are ambient isotopic leaving $\overline{N - N_0}$ fixed. Since f and g agree on $K \times I - P$, this completes the proof, except for the construction of the homotopy ψ.

Let p: $N \to \operatorname{Int} N$ be a map such that $p \mid N - C$ is the identity and $p(x, 0) = (x, 1/6)$ for $x \in \operatorname{Bd} N$. Define ψ_t on $K \times I$ by

$$\psi_t(x, s) = \begin{cases} f(x, s) & (0 \leq s \leq 1/6), \\ p\phi_t(x, 3s/2 - 1/4) & (1/6 \leq s \leq 5/6), \\ f(x, s) & (5/6 \leq s \leq 1). \end{cases}$$

It is easily verified that ψ meets the above requirements.

2. WEAK ISOTOPIES

In Section 3 we need an assumption about the extendability of certain homeomorphisms. The most convenient method of stating the assumption involves the concept of a *weak isotopy*, given by the following definition.

Definition. If X is a space, a *weak isotopy* of X is a homeomorphism H: $X \times I \approx X \times I$ such that $H(X \times 0) = X \times 0$ and $H(X \times 1) = X \times 1$. If H is a weak isotopy of X, we let H_0 and H_1 denote the homeomorphisms of X defined by $H(x, i) = (H_i(x), i)$ $(i = 0, 1)$, and we say that H_0 and H_1 are *weakly isotopic*. There is a similar concept of *weakly ambient isotopic* embeddings.

Crowell [7] showed that an isotopy h_t $(0 \leq t \leq 1)$ on a locally compact Hausdorff space X defines a weak isotopy H, by setting $H(x, t) = (h_t(x), t)$ $(x \in X,\ 0 \leq t \leq 1)$. Thus, in the case of manifolds, weak isotopy is a weaker concept than isotopy.

Digressing slightly, we point out that Theorem 1.2 above can be used, along with the proof of Theorem 4.1 of [8], to prove the following corollary.

COROLLARY 2.1. *Let* N *be a* $(k + 1)$-*connected combinatorial* n-*manifold, and let* X *be a tame, finite* k-*polyhedron in* Int N $(n \geq 2k + 3)$. *Suppose that* f *and* g *are homeomorphisms of* N *onto itself and that* f *and* g *are the identity on* X. *If* f *and* g *are weakly isotopic, then there is a weak isotopy* H: $N \times I \approx N \times I$ *such that* $H_0 = f$, $H_1 = g$, *and* $H \mid X \times I$ *is the identity. In particular,* $f \mid N - X$ *and* $g \mid N - X$ *are weakly isotopic as homeomorphisms of* N - X.

Corollary 2.1 can be used, for example, to prove the following statement. *Let* Δ_1^n, Δ_2^n, *and* Δ^k *be simplexes such that* $\Delta_2 \subset \Delta_1$, Δ *is a face of* Δ_1 *and of* Δ_2, *and* $\Delta_2 - \Delta \subset \text{Int}\, \Delta_1$. *Let* h *be a homeomorphism of* Bd Δ_1 *onto itself that is the identity on* Δ *and is (weakly) isotopic to the identity on* Bd Δ_1. *If* $n \geq 2k + 4$, *then* h *can be extended to a homeomorphism* $H: \Delta_1 \approx \Delta_1$ *such that* $H \mid \Delta_2 = \text{identity}$.

The following lemma states the relationship between weak isotopy and extension of homeomorphisms that we will need.

LEMMA 2.2. *Let* S *be a sphere that bounds a locally flat cell* D *in the interior of the topological manifold* N. *Let* h *be a homeomorphism of* S *onto itself that is weakly isotopic to the identity on* S, *and let* U *be a neighborhood of* D *in* N. *Then* h *can be extended to a homeomorphism* g *of* N *such that* $g(D) = D$ *and* $g \mid (N - U) = \text{identity}$.

Since D is contained in an open n-cell, it suffices to prove the lemma under the assumption that $N = R^n$. The proof is elementary; it makes extensive use of the "extension by radial projection" method of extending to the interior a homeomorphism on the boundary of a cell.

3. UNIONS OF FLAT CELLS

The remainder of this paper is devoted to relationships among the α, β, and σ-conjectures (stated below), and it may be considered a generalization of [5].

Definition. Let D_1 and D_2 be cells in S^n such that $D_1 \cap D_2 = \text{Bd}\, D_1 \cap \text{Bd}\, D_2$ is a cell. D_1 and D_2 are *simultaneously flat in* S^n if there exists a homeomorphism h of S^n such that $h(D_i)$ is a simplex and $h(D_1 \cap D_2)$ is a face of $h(D_i)$ $(i = 1, 2)$.

For appropriate integers n, m_1, m_2, and k, let $\beta(n, m_1, m_2, k)$ and $\sigma(n, k)$ denote the following conjectures.

$\beta(n, m_1, m_2, k)$. *If* D_1 *and* D_2 *are locally flat cells in* S^n, *of dimension* m_1 *and* m_2, *respectively, and if* $D_1 \cap D_2 = \text{Bd}\, D_1 \cap \text{Bd}\, D_2$ *is a* k-*cell that is locally flat in* Bd D_1 *and in* Bd D_2, *then* D_1 *and* D_2 *are simultaneously flat in* S^n.

$\sigma(n, k)$. *Let* $f: B^k \to S^n$ *be a locally flat embedding. Then there exists a weak isotopy* H *of* S^n *such that* H_0 *is the identity and* $H_1 \mid B^k = f$.

In $\beta(n, m_1, m_2, k)$ we always assume that $n \geq m_1 \geq m_2 > k \geq -1$. The above statement of $\sigma(n, k)$ is used whenever $n > k$; conjecture $\sigma(n, n)$ is the usual weak isotopy conjecture in dimension n. We note that the annulus conjecture in dimension n, denoted by $\alpha(n)$, is the same as conjecture $\beta(n, n, n, -1)$.

The following is a corollary to Corollary 3.2 of [12].

COROLLARY 3.1. $\beta(n, m_1, m_2, k)$ *is true whenever* $n \geq 2m_2 + 2$.

THEOREM 3.2. $\beta(n, m_1, m_2, k) \Rightarrow \beta(n, m_1, m_2, -1)$ *whenever* $n \geq 2k + 4$.

Proof. Let D_1 and D_2 be locally flat cells in S^n, of dimension m_1 and m_2, respectively, such that $D_1 \cap D_2 = \emptyset$. Without loss of generality we may assume that D_2 is a simplex.

We construct a $(k+1)$-cell E that "spans" D_1 and D_2 as follows. Let E_1 be a k-cell that is locally flat in Bd D_1, and let E_2 be a k-face of D_2. Construct nonintersecting n-cells Q_1 and Q_2 in S^n such that Bd Q_i is locally flat in S^n, $D_i \subset Q_i$, and Bd D_i is locally flat in Bd Q_i $(i = 1, 2)$. It follows from Corollary 1.3

that there exists a $(k+1)$-cell E in S^n such that E is locally flat in S^n, $E \cap D_i = E_i \subset \mathrm{Bd}\, E$ $(i = 1, 2)$, and $\mathrm{Bd}\, E_i$ is locally flat in $\mathrm{Bd}\, E$ $(i = 1, 2)$.

Now, by Corollary 3.1, some homeomorphism of S^n takes D_1 and E onto simplexes and takes E_1 onto a common face. Thus it is clear that there exists an m_1-cell D in S^n such that D is locally flat in S^n, $D_1 \subset \mathrm{Int}\, D$, $D - \mathrm{Int}\, D_1$ is an annulus, E_2 is locally flat in $\mathrm{Bd}\, D$, and $D \cap D_2 = E_2$. By the assumption that $\beta(n, m_1, m_2, k)$ is true, there exists a homeomorphism $h: S^n \approx S^n$ such that $h(D)$ and $h(D_2)$ are simplexes in S^{m_1}. (Note that $m_1 \geq m_2$.) But then $h(D_1)$ is a cell with locally flat boundary in S^{m_1}, and $h(\mathrm{Bd}\, D)$ and $h(\mathrm{Bd}\, D_1)$ bound an annulus in S^{m_1}. Hence $h(D_1)$ can be moved onto a simplex in S^{m_1} by a homeomorphism $g: S^n \approx S^n$ that is the identity on $S^{m_1} - \mathrm{Int}\, D$. The composition gh then moves both D_1 and D_2 onto simplexes; this completes the proof.

THEOREM 3.3. $[\beta(n, m_1, m_2, -1), \sigma(m_1 - 1, k), \sigma(m_2 - 1, k)] \Rightarrow \beta(n, m_1, m_2, k)$ *whenever* $n \geq 2k + 4$.

Proof. Let D_1, D_2 be a pair of locally flat cells in S^n ($\dim D_1 = m_1$, $\dim D_2 = m_2$) such that $D_1 \cap D_2 = \mathrm{Bd}\, D_1 \cap \mathrm{Bd}\, D_2 = D$ is a k-cell that is locally flat both in $\mathrm{Bd}\, D_1$ and $\mathrm{Bd}\, D_2$. We may assume that D_2 is an m_2-simplex in S^{m_2} and that D is a face of D_2. In order to straighten D_1 and D_2 simultaneously, we first construct an m_2-cell D_0 in $\mathrm{Int}\, D_2$, so that $\beta(n, m_1, m_2, -1)$ may be applied.

If $\hat{D}_2$ denotes the barycenter of D_2, let D_0 be the simplex whose vertices are the midpoints of the segments joining $\hat{D}_2$ with the vertices of D_2, and let E be the k-face of D_0 determined by D in this way. Construct the embedding $F: D \times I \to D_2$ by requiring that $F(x, 0) = x$, that $F(x, 1)$ be the midpoint of the segment joining x with $\hat{D}_2$, and that F be linear on each interval $x \times I$ $(x \in D)$. Thus $F(D \times 0) = D$ and $F(D \times 1) = E$.

Suppose that h is a homeomorphism of S^n such that

(1) $h(D_0) = D_0$ (D_0 is a simplex),

(2) $h(D_1)$ is a simplex,

(3) $h \mid D \cup E$ is the identity,

(4) $h(D_0 \cup D_1) \cap F(D \times I) = D \cup E$,

(5) D is a face of $h(D_1)$,

(6) E is a face of $h(D_0)$.

We shall complete the proof, assuming that h exists, and then return to the construction of h.

Since $F(D \times (0, 1)) \cup hF(D \times (0, 1))$ lies in the complement of $h(D_0) \cup h(D_1)$, it follows from Corollary 1.3 that we may strengthen condition (3) on h by the condition

(7) $h \mid F(D \times I)$ is the identity.

Let p be a mapping of S^n onto itself such that

(8) $p(D_0) = D_2$,

(9) $p \mid D_1 \cup h(D_1)$ is the identity,

(10) the nondegenerate inverse sets under p are precisely the sets $F(x \times I)$ $(x \in D)$.

Define $g = php^{-1}$. It follows easily from conditions (1) to (10) that g is a homeomorphism of S^n that maps D_1 onto $h(D_1)$ and D_2 onto itself. Thus the proof is complete, except for the construction of h. Notice that the β- and σ-conjectures in the hypothesis of the theorem have yet to be used.

Under the assumption that $\beta(n, m_1, m_2, -1)$ is true, there exists a homeomorphism h_1 of S^n such that $h_1(D_0)$ and $h_1(D_1)$ are simplexes. By making use of Theorem 7.1 of [3, Part I], we may assume that $h_1(D_0) = D_0$, that $h_1(D_1)$ meets $F(D \times I)$ only in D, and that D is a face of $h_1(D_1)$; the theorem cited in [3] can be applied, since any two simplexes of the same dimension in S^n are stably equivalent. The σ-hypothesis can now be used to unknot $h_1(D)$ and $h_1(E)$ in $h_1(\text{Bd } D_1)$ and $h_1(\text{Bd } D_0)$, respectively.

Notice that both D and $h_1(D)$ are locally flat k-cells in $h_1(\text{Bd } D_1)$. $h_1(D)$ is locally flat by hypothesis, and D is a face of $h_1(D_1)$. It follows from $\sigma(m_1 - 1, k)$ that there exists a weak isotopy G of $h_1(\text{Bd } D_1)$ such that G_0 is the identity and $G_1 \mid D = h_1 \mid D$. Hence, by Lemma 2.2, we can extend G_1^{-1} to a homeomorphism h_2 of S^n such that $h_2 h_1(D_1) = h_1(D_1)$, $h_2 \mid h_1(D_0)$ is the identity, and $h_2 h_1 \mid D$ is the identity. (The case $k = m_1 - 1$ is slightly more complicated; however, this case may be ignored, by Corollary 3.1.) A similar argument involving $\sigma(m_2 - 1, k)$ shows the existence of a homeomorphism h_3 of S^n such that $h_3 h_1(D_0) = h_1(D_0)$, $h_3 \mid h_1(D_1)$ is the identity, and $h_3 h_1 \mid E$ is the identity. Let $h = h_3 h_2 h_1$. A perusal of the last two paragraphs shows that h satisfies conditions (1) to (6), and the proof is complete.

COROLLARY 3.4. $\beta(n, m_1, m_2, k) \Leftrightarrow \beta(n, m_1, m_2, -1)$ *whenever* $n \geq 2k + 4$ *and* $m_2 \geq 3k/2 + 3$.

Proof. $\sigma(m, k)$ is true whenever $m \geq 3k/2 + 2$, by Theorem 1' of [6].

COROLLARY 3.5. $\beta(n, n, n, k) \Leftrightarrow \alpha(n)$ *whenever* $n \geq 2k + 4$.

COROLLARY 3.6. $\beta(n, m_1, m_2, k)$ *is true whenever* $n \geq 3m_2/2 + 3/2$ *and* $m_2 \geq 3k/2 + 3$.

Proof. The conditions imply that $n \geq 2k + 4$; therefore, by Corollary 3.4, it suffices to prove $\beta(n, m_1, m_2, -1)$ for $n \geq 3m_2/2 + 3/2$. But this follows from [13] and Theorem 7.1 of [3, Part I].

Remarks. 1. Corollary 3.5 is a generalization of the main result of [10] and [11], since $\beta(n, n, n, 0)$ is easily seen to be equivalent to the slab conjecture in dimension n, $n \geq 4$.

2. Although Corollary 3.6 is not very esthetic, it does give some previously unknown facts. For example, $\beta(20, m, 12, 6)$ is true for $12 \leq m \leq 20$.

3. Since $\sigma(m, k)$ is true for $m \leq 3$, one can use Theorem 3.3 to prove a few other low-dimensional cases. For example, $\beta(8, 4, 4, 2)$ and $\beta(8, 6, 4, 2)$ follow, but $\beta(8, 5, 4, 2)$ remains undecided.

It is well known that $\beta(n, n - 1, n - 1, -1)$ implies $\alpha(n)$. Hence, by Corollary 3.5, $\beta(n, n - 1, n - 1, -1)$ implies $\beta(n, n, n, k)$ for $n \geq 2k + 4$. The proof of this last assertion can be modified to show the following version of the "flat spot" theorem.

THEOREM 3.7. *Let* D_1 *and* D_2 *satisfy the hypothesis in* $\beta(n, n, n, k)$ $(n \geq 2k + 4)$. *Suppose further that there exist* $(n - 1)$*-cells* E_1 *and* E_2, *with* $E_i \subset \text{Bd } D_i$ $(i = 1, 2)$ *such that* E_1 *and* E_2 *are simultaneously flat in* S^n. *Then* D_1 *and* D_2 *are simultaneously flat in* S^n.

Proof. Let Δ_1 and Δ_2 be n-simplexes in S^n that meet in a k-face, and let P and X denote the closures of $S^n - \Delta_1 - \Delta_2$ and $S^n - D_1 - D_2$, respectively. The theorem follows if we show that $X \approx P$.

Let $S_i = \text{Bd } D_i$ (i = 1, 2). By the flat-spot hypothesis, there exists a locally flat (n - 1)-sphere $S_0 \subset \text{Int } D_1$ such that $S_0 \cup S_1$ and $S_0 \cup S_2$ bound annuli A_1 and A_2 in S^n. The annulus A_1 can be spanned by a (k + 1)-cell $E \times I$, where

$$E \times I \cap S_0 = E \times 0 \quad \text{and} \quad E \times I \cap S_1 = E \times 1 = S_1 \cap S_2 .$$

Shrinking the arcs $x \times I$ ($x \in E$) to points, we pull S_0 onto a locally flat (n - 1)-sphere S_3 in S^n such that $S_3 \cap S_1 = S_3 \cap S_2 = S_1 \cap S_2$. Moreover, by Corollary 1.3, $S_3 \cup S_1$ and $S_3 \cup S_2$ bound sets P_1 and P_2 each of which is homeomorphic to P. Thus $P_2 = P_1 \cup X$ and $P_1 \cap X = S_1$. Since clearly $P_1 \cup X \approx X$, we see that $X \approx P_2 \approx P$, and the proof is complete.

Theorem 3.7 can be used to reduce the annulus conjecture to a local problem as follows.

THEOREM 3.8. *Let* D_1 *and* D_2 *satisfy the hypothesis of* $\beta(n, n, n, k)$ ($n \geq 2k + 4$, $k \geq 0$). *If* $D_1 \cup D_2$ *is locally tame at some point of* $D_1 \cap D_2$, *then* D_1 *and* D_2 *are simultaneously flat in* S^n.

Proof. Clearly, local tameness at a point of $D_1 \cap D_2$ implies that there exist (n - 1)-cells $E_i \subset \text{Bd } D_i$ (i = 1, 2) that are simultaneously flat.

4. AN EXAMPLE

One might hope to use Corollary 3.5 to find a counterexample to $\alpha(n)$, if one exists. The example described below is not obviously tame; however, there is reason to believe that it is tame.

To begin, select a disjoint pair Q_1, Q_2 of 3-cells in R^3 and an arc A in R^3 such that $A \cap Q_i$ is an endpoint of A (i = 1, 2). Then construct 4-cells D_1 and D_2 in R^4, depending on Q_1, Q_2, and A, as follows.

Let f be the quotient map of R^3 onto R^3/A; the only nondegenerate inverse set under f is A. Thus $f(Q_1)$ and $f(Q_2)$ are 3-cells. By a result of Andrews and Curtis [1], $(R^3/A) \times R^1$ is homeomorphic to R^4. It is not difficult to see that the arc $f(A) \times I$ is cellular in R^4, so that there exists a mapping g of R^4 onto itself whose only nondegenerate inverse set is $f(A) \times I$. Let $E_i = g(f(Q_i) \times I)$ (i = 1, 2). Then E_1 and E_2 are 4-cells in R^4 having only the point $p = g(f(A) \times I)$ in common. Finally, form D_i by pulling E_i into $p \cup \text{Int } E_i$ along a collar that is pinched at p.

LEMMA 4.1. *If* D_1 *and* D_2 *are as constructed above, then* D_1 *and* D_2 *are 4-cells with locally flat boundaries in* R^4 *and* $D_1 \cap D_2$ *is a point.*

Proof. Clearly Bd D_1 and Bd D_2 are locally flat except possibly at the point $p = D_1 \cap D_2$. But then D_1 and D_2 are flat by [4].

LEMMA 4.2. *If there exist a disjoint pair* Q_1, Q_2 *of 3-cells in* R^3 *and an arc A spanning* Q_1 *and* Q_2 *in* R^3 *such that* $D_1 \cup D_2$ *is wild in* R^4 *at the point of intersection, then* $\alpha(4)$ *is false.*

Proof. According to Corollary 3.5, $\alpha(4)$ is false if and only if $\beta(4, 4, 4, 0)$ is false.

Remark. One can probably use Theorem 3.7 to show that $D_1 \cup D_2$ is tame for any Q_1, Q_2, A.

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