INVERSES OF EUCLIDEAN BUNDLES

J. M. Kister

To Raymond L. Wilder on his seventieth birthday.

For each Euclidean bundle or microbundle it is useful to find another bundle of the same type, called an *inverse bundle*, such that the Whitney sum of the two is a trivial bundle. Milnor in [4] ingeniously showed how to construct an inverse to a microbundle over a finite-dimensional, locally finite, simplicial complex. Here we give a short and elementary proof of the existence of inverses for Euclidean bundles over paracompact spaces having a finiteness condition. This contains Milnor's result, since one may regard a microbundle as a Euclidean bundle [2]. Hirsch [1] has also developed a new proof of the existence of the inverse of a bundle over a polyhedron, in his work on the stable existence and stable isotopy of normal microbundles.

Terminology. By *Euclidean bundle* we mean a fibre bundle (in the sense of Steenrod [5]) whose fibre is Euclidean space R^n and whose structural group is $H_0(R^n)$, the group of all homeomorphisms of R^n leaving the origin fixed, and provided with the compact-open topology. Other bundle terminology will also be taken from [5]. For microbundle terminology, see [4]. The identity map on a space will be denoted by id, the unit interval by I.

Define maps c and p of $H_0(R^n) \times H_0(R^n)$ into $H_0(R^{2n})$ by

$$c(f, g) = (g \circ f) \times \mathrm{id}\colon R^n \times R^n \to R^n \times R^n \quad \text{and} \quad p(f, g) = f \times g.$$

LEMMA 1. p *is homotopic to* c.

Proof. Let θ_t (t in I) be in SO(2n), and suppose that $\theta_0 = \mathrm{id}$ and $\theta_1(x, y) = (-y, x)$ (x, y in R^n). Define $\phi_t\colon H_0(R^n) \times H_0(R^n) \to H_0(R^{2n})$ by

$$\phi_t(f, g) = \theta_t^{-1} \circ (\mathrm{id} \times g) \circ \theta_t \circ (f \times \mathrm{id}).$$

Then ϕ_t (t in I) is the desired homotopy with $\phi_0 = p$ and $\phi_1 = c$.

Remark 1. If the homomorphism p is restricted to $G \times G$, where G is a subgroup of $H_0(R^n)$, and if K is a subgroup of $H_0(R^{2n})$ containing both SO(2n) and $p(G \times G)$, then the homotopy constructed above assumes values in K. Examples of this occur when G and K are the orthogonal, rotation, or stable homeomorphism groups in dimensions n and 2n, respectively.

LEMMA 2. *Let* ξ^k *and* η^ℓ *be two Euclidean bundles (of dimension* k *and* ℓ, *respectively) over a space* B. *Suppose that* B *is the union of two open sets* U *and* V, *and that* ξ *and* η *are both trivial over* U *and* V. *Let the coordinate transformations for* ξ *and* η *be given by* $f\colon U \cap V \to H_0(R^k)$ *and* $g\colon U \cap V \to H_0(R^\ell)$, *respectively. Then the Whitney sum* $\xi \oplus \eta$ *is also trivial over* U *and* V, *and the coordinate transformation may be taken to be* $h\colon U \cap V \to H_0(R^{k+\ell})$, *where* $h(b) = f(b) \times g(b)$.

The proof is straightforward, and we omit it.

Received August 9, 1966.
This research was partially supported by NSF Grant GP4151.

LEMMA 3. *Let* ξ^n *be an* n-*dimensional Euclidean bundle over the union of two open sets* U *and* V *in a normal space* B, *and suppose* ξ *is trivial over* U *and* V. *Let* U' *be a set whose closure lies in* U. *Then there exists an* n-*dimensional Euclidean bundle* η^n *over* $U \cup V$ *such that* $(\xi \oplus \eta) \mid (U' \cup V)$ *is trivial.*

Proof. Let $k_U: U \times R^n \to E(\xi \mid U)$ and $k_V: V \times R^n \to E(\xi \mid V)$ be the coordinate functions with coordinate transformation $f: U \cap V \to H_0(R^n)$ defined by

$$f(b)(y) = pr \circ k_V^{-1} \circ k_U(b, y),$$

where $pr: V \times R^n \to R^n$ is the projection map. Next define $g: U \cap V \to H_0(R^n)$ by $g(b) = f(b)^{-1}$, and let η be the Euclidean bundle over $U \cup V$ having g as its coordinate transformation. More precisely, let $E(\eta)$ be the decomposition space obtained by taking the disjoint union of $U \times R^n$ and $V \times R^n$ and identifying (b, y) in $U \times R^n$ with $(b, [f(b)]^{-1}(y))$ in $V \times R^n$, for all b in $U \cap V$ and y in R^n. The projection map of η is ordinary projection onto the first factor.

By Lemma 2, $\xi \oplus \eta$ is trivial over U and V, and the coordinate transformation is $h: U \cap V \to H_0(R^{2n})$. Let

$$k'_U: U \times R^{2n} \to E((\xi \oplus \eta) \mid U) \quad \text{and} \quad k'_V: V \times R^{2n} \to E((\xi \oplus \eta) \mid V)$$

be the corresponding coordinate functions for $\xi \oplus \eta$, so that

$$h(b)(y) = (f(b) \times g(b))(y) = pr \circ k_V'^{-1} \circ k'_U(b, y)$$

for all b in $U \cap V$. By Lemma 1, there is a homotopy

$$\phi_t: H_0(R^n) \times H_0(R^n) \to H_0(R^{2n}) \qquad (t \text{ in } I)$$

such that

$$\phi_0(f(b), g(b)) = f(b) \times g(b) = h(b) \quad \text{and} \quad \phi_1(f(b), g(b)) = (g(b) \circ f(b)) \times id = id.$$

Define $h_t: U \cap V \to H_0(R^{2n})$ by $h_t(b) = \phi_t(f(b), g(b))$, so that $h_0 = h$ and $h_1(b) = id$ for all b. We need to change the coordinate function k'_V so as to realize the identity as the coordinate transformation of $(\xi \oplus \eta) \mid (U' \cup V)$.

This we do as follows. Let $\tau: B \to I$ be a Urysohn function with support contained in U and with $\tau(\overline{U}') = 1$. Define $k''_V: V \times R^{2n} \to E(\xi \oplus \eta)$ by

$$k''_V(b, y) = \begin{cases} k'_V(b, h(b) \circ [h_{\tau(b)}(b)]^{-1}(y)) & \text{for } b \in U \cap V, \\ k'_V(b, y) & \text{for } b \in V - U. \end{cases}$$

For b in U', we have $\tau(b) = 1$ and $k''_V(b, y) = k'_V(b, h(b)(y))$, and hence

$$\begin{aligned} pr \circ k_V''^{-1} \circ k'_U(b, y) &= pr \circ (k'_V \circ id \times h(b))^{-1} \circ k'_U(b, y) \\ &= pr \circ (id \times h(b))^{-1} \circ k_V'^{-1} \circ k'_U(b, y) \\ &= pr \circ (id \times h(b)^{-1})(b, h(b)(y)) = y. \end{aligned}$$

In other words, if we use $k'_U \mid U' \times R^{2n}$ and k''_V as coordinate functions for $(\xi \oplus \eta) \mid (U' \cup V)$, then the coordinate transformation takes on only the value id, and therefore $(\xi \oplus \eta) \mid (U' \cup V)$ is trivial.

We shall say that a bundle ξ is *finitary* if there exists a finite covering $\{U_0, U_1, \cdots, U_k\}$ of the base such that $\xi \mid U_i$ is trivial $(i = 0, 1, \cdots, k)$.

PROPOSITION. *If* ξ *is a bundle whose base space* B *has finite covering dimension and is paracompact, then* ξ *is finitary.*

Proof. Let $\mathscr{U} = \{U\}$ be an open covering of B such that $\xi \mid U$ is trivial for each U in $\mathscr{U}$. In view of the hypothesis on B, we may assume that $\mathscr{U}$ is locally finite and that no point of B lies in more than $k + 1$ sets in $\mathscr{U}$. Let $\{\phi_U\}$ be a partition of unity relative to $\mathscr{U}$, and let $N(\mathscr{U})$ be the nerve of $\mathscr{U}$, in this case a k-dimensional simplicial complex. Then $\{\phi_U\}$ determines a map $\phi: B \to N(\mathscr{U})$, in the usual manner, by letting the barycentric coordinate of $\phi(b)$ corresponding to U be $\phi_U(b)$. If S_U is the open-star neighborhood of the vertex in N(U) corresponding to U, then $\phi^{-1}(S_U) \subset U$.

Next, for each integer i $(0 \leq i \leq k)$, find a disjoint collection $\mathscr{V}_i$ of open subsets of $N(\mathscr{U})$, each containing the interior of an i-simplex and contained in the open star of some vertex of that simplex. Furthermore, every open i-simplex should lie in exactly one set in $\mathscr{V}_i$. For example, for the i-simplex whose vertices are $v_0, v_1, \cdots, v_i$, take the set of points in $N(\mathscr{U})$ each of whose barycentric coordinates in $v_0, \cdots, v_i$ is greater than any of its other barycentric coordinates.

Denote by V_i the union of the sets in $\mathscr{V}_i$. It follows from the disjointness condition that $\xi \mid \phi^{-1}(V_i)$ is trivial; hence ξ is finitary.

THEOREM. *Every finitary Euclidean bundle over a paracompact base has an inverse.*

Proof. Let ξ^n be a Euclidean bundle over B, and let $\{U_0, U_1, \cdots, U_k\}$ be an open covering of B such that $\xi \mid U_i$ is trivial $(i = 0, 1, \cdots, k)$. Shrink the covering to get another covering $\{U_0', U_1', \cdots, U_k'\}$ with $\overline{U_i'} \subset U_i$ $(i = 0, 1, \cdots, k)$. We shall proceed to construct the inverse inductively over the U_i' sets. By applying Lemma 3, we get a bundle over $U_0 \cup U_1'$ that is inverse for $\xi \mid (U_0 \cup U_1')$.

Assume we have a Euclidean bundle η^ℓ over $V = U_0 \cup U_1' \cup \cdots \cup U_j'$, so that $(\xi \mid V) \oplus \eta$ is $\varepsilon_V^{n+\ell}$, an $(n + \ell)$-dimensional trivial bundle over V. Let

$$U = U_{j+1}, \quad U' = U_{j+1}', \quad D = U \cap V, \quad W = U \cup V, \quad W' = U' \cup V.$$

We want to construct an inverse for $\xi \mid W'$.

Since $(\xi \mid D) \oplus (\eta \mid D)$ and $(\xi \mid D)$ are both trivial,

$$\varepsilon_D^n \oplus (\eta \mid D) = \varepsilon_D^{n+\ell} = (\varepsilon_V^n \oplus \eta) \mid D.$$

Therefore $\varepsilon_V^n \oplus \eta$ can be extended to a bundle $\eta'^{\,n+\ell}$ over W so that $\eta' \mid U$ is trivial. If we let ξ' be $(\xi \mid W) \oplus \eta'$, a bundle over W, we see that $\xi' \mid U$ is $(\xi \mid U) \oplus (\eta' \mid U)$, the sum of two trivial bundles, and that $\xi' \mid V$ is

$$(\xi \mid V) \oplus \varepsilon_V^n \oplus \eta \cong (\xi \mid V) \oplus \eta \oplus \varepsilon_V^n \cong \varepsilon_V^{n+\ell} \oplus \varepsilon_V^n,$$

another trivial bundle. Hence, applying Lemma 3 again to ξ', we obtain a bundle η'' over W' such that $(\xi' \mid W') \oplus \eta''$ is trivial. Thus

$$(\xi' \mid W') \oplus \eta'' = (\xi \mid W') \oplus (\eta' \mid W') \oplus \eta''$$

is trivial and $(\eta' \mid W') \oplus \eta''$ is an inverse for $\xi \mid W'$. This completes the induction and the proof.

Remark 2. The construction in the theorem also provides inverses of the same type to vector bundles, to orientable vector bundles, to bundles whose structural group is the group of stable homeomorphisms, and in fact to any bundle whose structural group contains the rotation group. See the remark after Lemma 1.

Remark 3. The hypothesis that the bundle in the theorem be finitary is necessary; this can be seen from the following example. There is a standard 1-dimensional nontrivial vector bundle ξ_n over real projective space P_n [3]. Its total Stiefel-Whitney class is $1 + \alpha$, where α is the nonzero element of $H^1(P_n; Z_2)$. It follows that $(1 + \alpha)^{-1} = 1 + \alpha + \alpha^2 + \cdots + \alpha^n$ in the ring $H^*(P_n; Z_2)$, and hence, by the Whitney product theorem, the nth S - W class of an inverse to ξ_n is nonzero and must have fibre dimension at least n. Thus, by taking B to be the disjoint union of $P_1, P_2, P_3, \cdots$, and ξ to be the 1-dimensional bundle such that $\xi \mid P_n = \xi_n$, we obtain a vector bundle with no inverse.

REFERENCES

1. M. Hirsch, *On normal microbundles* (to appear).
2. J. M. Kister, *Microbundles are fibre bundles*, Ann. of Math. (2) 80 (1964), 190-199.
3. J. Milnor, *Lectures on characteristic classes*, Princeton University notes, 1957.
4. ———, *Microbundles, Part I*, Topology 3 (1964-65), Supplement 1, 53-80.
5. N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, N. J., 1951.

The University of Michigan