CRITERIA FOR A 2-SPHERE IN S^3 TO BE TAME MODULO TWO POINTS

C. E. Burgess

Dedicated to Professor Raymond L. Wilder on his seventieth birthday.

1. INTRODUCTION

Examples have been described by Fox and Artin [12] of wild 2-spheres in a 3-sphere S^3 that are locally tame except at a finite number of points. Harrold and Moise [14] have shown that at each of its points such a 2-sphere S must be locally tame from at least one of its complementary domains. Furthermore, if S has at most one point where it is wild from the component U of $S^3 - S$, then U is an open 3-cell [10, Theorem 1], [19, Corollary 2.4]. Sikkema [20] has studied a duality between spheres and arcs in E^3 that are locally tame except at one point. In this paper, we present some conditions which imply that a 2-sphere in S^3 has at most two wild points.

In Section 4, we use Theorem 1 to investigate the following question raised by Bing in [6]: Is a 2-sphere S in S^3 tame if it can be pierced along each arc in it by a tame disk? We do not answer this question, but we obtain an affirmative answer (Theorem 5) with the additional hypothesis that each component of $S^3 - S$ be an open 3-cell.

Hempel [15] has raised the following question: Is a 2-sphere S tame in S^3 if, for each $\varepsilon > 0$ and each component U of $S^3 - S$, there exists a map of S into U that moves no point more than a distance ε? He recently obtained an affirmative answer under an additional hypothesis [16]. In Section 4 we impose the alternative additional hypothesis that S can be pierced with a disk on each tame arc on S, and we observe (Theorem 6 and Corollary 3) that with this alternative additional hypothesis an affirmative answer follows from a combination of the Sphere Theorem [18], Theorem 4, and one of Hempel's recent results [16, Corollary 2].

In Section 5, we indicate how some of these results for a 2-sphere in S^3 can be adapted to a 2-manifold in a 3-manifold.

2. DEFINITIONS AND NOTATION

Let S be a 2-sphere in S^3, and let U be a component of $S^3 - S$. We define S to be *locally tame from* U *at the point* $p \in S$ if there exist a 3-cell K and a disk D such that

$$K \cap S = D, \qquad p \in \operatorname{Int} D, \qquad D \subset \operatorname{Bd} K, \qquad K - D \subset U.$$

The sphere S is defined to be *tame from* U if S is locally tame from U at each point of S. This is equivalent to requiring that $S \cup U$ be a 3-manifold with boundary, and to requiring that $S \cup U$ be a 3-cell [1], [8], [17]. Furthermore, S is *tame* if it

Received August 15, 1966.
This work was supported by the National Science Foundation under GP-3882.

is tame from each component of $S^3 - S$. If S is not locally tame from U at the point p of S, then S is said to be *wild from* U *at* p.

We say that a subset X of S can be *collared from* U if there exists a homeomorphism h of $X \times [0, 1]$ into $X \cup U$ such that, for each $x \in X$,

$$h(x \times 0) = x \qquad \text{and} \qquad h(x \times t) \subset U \text{ for } 0 < t \leq 1 .$$

Thus S is locally tame from U at a point $p \in S$ if there exists an open set V such that $p \in V$ and $V \cap S$ can be collared from U [8]. We say that S can be *locally peripherally collared from* U at a point $p \in S$ if for each $\varepsilon > 0$ there exists a disk D such that

$$p \in \text{Int } D \subset S,$$

$$\text{Diam } D < \varepsilon, \text{ and}$$

$$\text{Bd } D \text{ can be collared from } U .$$

We say that S can be *locally peripherally collared from* U if it can be locally peripherally collared from U at each point, and S can be *locally peripherally collared* if it can be locally peripherally collared from each component of $S^3 - S$. If an arc K in S can be collared from each component of $S^3 - S$, we say that S can be *pierced on* K *with a disk*. As in [9], we say that S can be *locally spanned* from U at a point $p \in S$ if for each $\varepsilon > 0$ there exist disks D and R such that

$$p \in \text{Int } D \subset S,$$

$$\text{Bd } D = \text{Bd } R,$$

$$\text{Int } R \subset U, \text{ and}$$

$$\text{Diam } (D \cup R) < \varepsilon .$$

Various other definitions used here can be found in some of the references, particularly in [1], [2], [4], and [9].

If K is a set in a metric space and $\varepsilon > 0$, then $N(K, \varepsilon)$ denotes the set of all points within a distance ε of K. If a and b are two points, then ab denotes an arc whose end points are a and b.

3. SPHERES THAT CAN BE LOCALLY PERIPHERALLY COLLARED

We present in this section some conditions which imply that a 2-sphere S in S^3 is locally tame, except possibly at one point, from the component U of $S^3 - S$. The converses of both Theorems 1 and 2 readily follow from known results cited in Section 1.

THEOREM 1. *If*

(i) S *is a* 2-*sphere in* S^3,

(ii) *the component* U *of* $S^3 - S$ *is an open* 3-*cell, and*

(iii) S *can be locally peripherally collared from* U, *then there exists a point* p *such that* S *is locally tame from* U *at each point of* S - p.

Proof. Let p_1 and p_2 be any two points of S. Our main task will be to show that S can be locally spanned from U at one of these two points. This will imply that there exists a point p such that S can be locally spanned from U at each point of S - p. Then it will follow from [9, Theorem 10] that S is locally tame from U at each point of S - p.

Let ε and δ be positive numbers such that

$$7\delta < \varepsilon. \tag{1}$$

From the hypothesis that S can be locally peripherally collared from U, it follows that there exist two disjoint disks D_1 and D_2 and two disjoint annuli A_1 and A_2 such that for each i (i = 1, 2)

$$p_i \in \text{Int } D_i \subset S,$$

$$A_i \cap S = \text{Bd } D_i \subset \text{Bd } A_i,$$

$$A_i - \text{Bd } D_i \subset U, \text{ and}$$

$$\text{Diam}(D_i \cup A_i) < \delta. \tag{2}$$

Furthermore, by using Bing's Approximation Theorem [2, Theorem 7], [7, Theorem 1.1], we may assume that each A_i is locally polyhedral at each point of $A_i \cap U$. We let J_i denote Bd A_i - Bd D_i and let a_i denote a point of J_i. There exists an arc $a_1 a_2$ such that

$$a_1 a_2 - (a_1 \cup a_2) \subset U - (A_1 \cup A_2).$$

From the hypothesis that U is an open 3-cell, it follows that there exists a polyhedral 2-sphere S_1 in U such that S and $J_1 \cup a_1 a_2 \cup J_2$ are in different components of $S^3 - S_1$ and

$$S_1 \subset N(S, \delta). \tag{3}$$

For each i (i = 1, 2), let A_i' denote an annulus in A_i such that $A_i' \cap S = \text{Bd } D_i$ and

$$\rho(x, S) < \rho(S_1, S) \quad \text{for each } x \in A_i'. \tag{4}$$

As above, the hypothesis implies that there exists a polyhedral 2-sphere S_2 such that S and $S_1 \cup (A_i - A_i')$ are in different components of $S^3 - S_2$ and

$$\rho(x, S) < \rho(S_1, S) \quad \text{for each } x \in S_2. \tag{5}$$

We assume that S_2 and A_i - Bd D_i are in general position, so that each component of $S_2 \cap A_i$ is a simple closed curve in A_i'. Since Bd D_i and J_i are in different components of $S^3 - S_2$, it follows that some component of $A_i \cap S_2$ separates Bd D_i from J_i in A_i. Thus there exists a disk D' in S_2 such that every component of $(A_1 \cup A_2) \cap \text{Int } D'$ is the boundary of a disk in $A_1' \cup A_2'$, no disk in $A_1 \cup A_2$ has Bd D' as its boundary, and Bd D' is a subset of one of the annuli A_1' and A_2', say of A_1'. Now, by using methods described by Bing [4, p. 297], we obtain a disk D such that

$$\mathrm{Bd}\ D = \mathrm{Bd}\ D',$$

$$\mathrm{Int}\ D \subset U - (A_1 \cup A_2 \cup S_1), \text{ and}$$

$$D \subset D' \cup N(A_1' \cup A_2', \delta).$$

Furthermore, in view of (4) and (5), this can be done so that

(6) $$\rho(x, S) < \rho(S_1, S) \quad \text{for each } x \in D.$$

We wish now to show that $\mathrm{Diam}\ D < 6\delta$. Suppose that

(7) $$\mathrm{Diam}\ D \geq 6\delta .$$

Following methods described in [21, p. 66], we construct an arc $p_1 a_1$ such that

(8) $$p_1 a_1 - (a_1 \cup p_1) \subset [U - (a_1 a_2 \cup A_1 \cup A_2)] \cap N(D_1 \cup A_1, \delta)$$

and no point of D is between a_1 and a point of S_1 on $p_1 a_1$. Let $a_2 p_2$ denote an arc in $D_2 \cup A_2$, let $p_2 p_1$ denote an arc such that

$$p_2 p_1 - (p_1 \cup p_2) \subset S^3 - (S \cup U),$$

and let J denote the simple closed curve $p_1 a_1 \cup a_1 a_2 \cup a_2 p_2 \cup p_2 p_1$. Let S' denote the 2-sphere that is a subset of $D \cup D_1 \cup A_1'$. Since $J \cap (D_1 \cup A_1') = p_1$ and J pierces S' at p_1, it follows that J intersects D and thus that D intersects the arc $p_1 a_1$. Let a denote a point of $D \cap p_1 a_1$. Now (7) implies that there is a point b in D such that

(9) $$\rho(a, b) \geq 3\delta .$$

It follows from (3) and (6) that there exists a point c of S such that

(10) $$\rho(b, c) < \delta$$

and the rectilinear arc bc does not intersect S_1. Let $p_1 a$ denote a subarc of $p_1 a_1$, let ab denote an arc in D such that $p_1 a \cup ab \cup bc$ is an arc, and let cp_1 denote an arc such that

$$cp_1 - (c \cup p_1) \subset S^3 - (S \cup U).$$

Let J' denote the simple closed curve $p_1 a \cup ab \cup bc \cup cp_1$. We have constructed J' in $S^3 - S_1$ so that the arc $cp_1 \cup p_1 a \cup ab$ pierces D_1 at p_1 and does not intersect $D_1 \cup (A_1 - p_1)$. Furthermore, it follows from (2), (8), (9), and (10) that the arc bc does not intersect $D_1 \cup A_1$, so J' links the simple closed curve J_1. But this is impossible, since S_1 separates J_1 from J' in S^3. Thus our supposition (7) has led to a contradiction; we conclude that

(11) $$\mathrm{Diam}\ D < 6\delta .$$

Now (1), (2), and (11) imply that

$$\mathrm{Diam}\,(D \cup A_1) < \varepsilon ;$$

hence, there exists a disk R in $D \cup A_1$ such that

$$\text{Diam } R < \varepsilon, \quad \text{Bd } R = \text{Bd } D_1, \quad \text{Int } R \subset U.$$

Thus we have shown that S can be locally spanned from U at one of the points p_1 and p_2. As we indicated in the first paragraph of this proof, it follows from [9, Theorem 10] that there is a point p such that S is locally tame from U at each point of S - p.

THEOREM 2. *If*

(i) S *is a* 2*-sphere in* S^3,

(ii) *the components* U_1 *and* U_2 *of* S^3 - S *are open* 3*-cells, and*

(iii) S *can be locally peripherally collared,*

then there exist two distinct points p_1 *and* p_2 *of* S *such that, for each* i (i = 1, 2), S *is locally tame from* U_i *at each point of* S - p_i.

Proof. It follows from Theorem 1 that, for each i (i = 1, 2), there exists a point p_i such that S is locally tame from U_i at each point of S - p_i. Harrold and Moise have shown [14] that a 2-sphere in S^3 cannot be wild from each of its complementary domains at one of its isolated wild points. Thus we can require that $p_1 \neq p_2$.

COROLLARY 1. *If* S *is a* 2*-sphere in* S^3 *such that each component of* S^3 - S *is an open* 3*-cell and the set* W *of wild points of* S *is* 0*-dimensional, then* W *consists of at most two points.*

THEOREM 3. *If*

(i) V *is an open subset of the* 2*-sphere* S *in* S^3,

(ii) *the component* U *of* S^3 - S *is an open* 3*-cell, and*

(iii) V *can be locally peripherally collared from* U,

then there exists a point p *such that* S *is locally tame from* U *at each point of* V - p.

With only slight changes, the above proof of Theorem 1 can be used as a proof of Theorem 3.

COROLLARY 2. *If*

(i) S *is a* 2*-sphere in* S^3,

(ii) *the component* U *of* S^3 - S *is an open* 3*-cell, and*

(iii) W *is the set of all points of* S *where* S *is wild from* U,

then W *has at most one isolated point. Furthermore, either* W = $\emptyset$, *or* W *consists of a point, or there exists a nondegenerate continuum in* W.

Remark. With reference to the hypotheses of Theorems 1 and 2, it is interesting to notice two 2-spheres, described by Bing [3] and Gillman [13], that are wild at every point and have the additional property that each arc in them is tame. The 2-sphere described by Bing can be locally peripherally collared, but one of its complementary domains in S^3 is not an open 3-cell. Gillman modified Bing's example to obtain a 2-sphere S such that each component of S^3 - S is an open 3-cell. However, Theorem 1 implies that there is a component of S^3 - S from which S cannot be locally peripherally collared.

4. SPHERES THAT CAN BE PIERCED WITH TAME DISKS

In Theorem 4, we require that each tame arc on S can be collared from U. Bing's theorem on the existence of tame arcs on S [5] enables us to obtain, in the proof of the lemma, two small tame arcs H_1 and H_2 that intersect and cross at two points such that a given point x is in the small component of $S - (H_1 \cup H_2)$. In the proof of Theorem 4, we do not use the tameness of H_1 and H_2, except to get two such arcs that can be collared from U. Clearly, the hypothesis of Theorem 4 could be changed accordingly.

THEOREM 4. *If*

(i) S *is a* 2-*sphere in* S^3,

(ii) *the component* U *of* $S^3 - S$ *is an open* 3-*cell, and*

(iii) *each tame arc on* S *can be collared from* U,

then there exists a point p *such that* S *is locally tame from* U *at each point of* S - p.

Proof. This theorem follows from Theorem 1 and the following lemma.

LEMMA. *If each tame arc on the* 2-*sphere* S *in* S^3 *can be collared from the component* U *of* $S^3 - S$, *then* S *can be locally peripherally collared from* U.

Proof. Let x be a point of S, and let V be an open set containing x. From Bing's theorem on the existence of tame Sierpiński curves in S [5], it follows that there exist two tame arcs H_1 and H_2 and a disk D such that

$$H_1 \cup H_2 \subset V \cap S,$$

$$H_1 \cap H_2 = a \cup b, \text{ where } a \text{ and } b \text{ are points,}$$

$$H_1 \text{ crosses } H_2 \text{ on } S \text{ at each of the points } a \text{ and } b,$$

$$x \in \text{Int } D \subset V \cap S, \text{ and}$$

$$\text{Bd } D \subset H_1 \cup H_2.$$

For each i (i = 1, 2), let H_i' denote the subarc of H_i with end points a and b. Clearly, Bd D = $H_1' \cup H_2'$. In the following two paragraphs, we complete the proof of our lemma by showing that $H_1' \cup H_2'$ can be collared from U.

By hypothesis, there exist two disks D_1 and D_2 such that

$$D_i \cap S = H_i \subset \text{Bd } D_i \quad \text{and} \quad D_i - H_i \subset U \quad \text{for each } i \ (i = 1, 2).$$

We assume that each D_i is locally polyhedral at each point of $D_i - H_i$ [2, Theorem 7]. With slight adjustments of $D_1 - H_1$ and $D_2 - H_2$, we can require that the closure of each component of $D_1 \cap D_2 - (a \cup b)$ be either a simple closed curve or a spanning arc of one of the disks D_1 and D_2. By using methods described by Bing [4, p. 297], we can further require that the closure of each such component contain exactly one of the points a and b. There exist two disjoint arcs K_1 and K_2 in $D_1 \cap D_2$ and a disk D_1' in D_1 such that

$$D_1' \cap S = H_1',$$

$$K_1 \cup K_2 \subset \text{Bd } D_1',$$

each of K_1 and K_2 is the closure of a component of $D_1' \cap D_2 - (a \cup b)$,

no component of $D_1' \cap D_2 - (K_1 \cup K_2)$ has an arc as its closure, and

$$a \in K_1 \text{ and } b \in K_2.$$

Let Y_1 denote the component of $D_1' - D_2$ that contains $\text{Bd } D_1' - (K_1 \cup K_2)$. No more than a finite number of components of $D_2 \cap \text{Cl } Y_1 - (K_1 \cup K_2)$ have closures that separate $K_1 - a$ from $K_2 - b$ in D_2. Thus there exist two disjoint arcs K_1' and K_2' in $D_2 \cap \text{Cl } Y_1$ and a disk D_2' in D_2 such that

$$D_2' \cap S = H_2',$$

$$K_1' \cup K_2' \subset \text{Bd } D_2',$$

K_1' and K_2' are closures of components of $D_1' \cap D_2' - (a \cup b)$,

no component of $D_2 \cap \text{Cl } Y_1 - (K_1' \cup K_2')$ has an arc as its closure, and

$$a \in K_1' \text{ and } b \in K_2'.$$

From the requirement that the arcs K_1' and K_2' are subsets of $D_2 \cap \text{Cl } Y_1$, it follows that no component of $D_1' \cap D_2' - (a \cup b)$ has a closure that separates $K_1' - a$ from $K_2' - b$ in D_1'. Thus there exists a subdisk D_1'' of D_1' such that

$$K_1' \cup K_2' \cup H_1' \subset \text{Bd } D_1''$$

and the closure of each component of $\text{Int } D_1'' \cap D_2' \cap \text{Cl } Y_1$ is a simple closed curve containing one of the points a and b. Let Y_1' denote the component of $D_1'' - D_2'$ that contains $\text{Bd } D_1'' - (K_1' \cup K_2')$. Each component of $D_1'' - \text{Cl } Y_1'$ is an open disk that has one of the points a and b on its boundary. Now we can use a modification of Bing's procedure [4, p. 297] to move each component of $\text{Int } D_2' \cap \text{Cl } Y_1'$ slightly to one side of D_2' and to replace each component of $D_1'' - \text{Cl } Y_1'$ with an open disk slightly to one side of D_2'. We obtain a disk D_1''' such that

$$D_1''' \cap D_2' = K_1' \cup K_2', \quad H_1' \cup H_2' \subset \text{Bd}(D_1''' \cup D_2'), \quad D_1''' \cup D_2' \subset U.$$

Thus we have shown that $H_1' \cup H_2'$ can be collared from U.

THEOREM 5. *If*

(i) S *is a* 2-*sphere in* S^3,

(ii) *each component of* $S^3 - S$ *is an open* 3-*cell, and*

(iii) S *can be pierced on each of its arcs with a tame disk,*

then S *is tame.*

Proof. The hypothesis implies that every arc on S is tame. Thus it follows from a theorem proved by Doyle and Hocking [11] that S does not have an isolated wild point. We combine this with Theorem 4 to conclude that S is locally tame at each of its points. Thus S is tame [1], [17], [8].

THEOREM 6. *If*

(i) S *is a* 2-*sphere in* S^3,

(ii) *for each* $\varepsilon > 0$, S *can be approximated in the component* U *of* S^3 - S *with a map that moves no point more than a distance* ε, *and*

(iii) *each tame arc on* S *can be collared from* U,

then S *is tame from* U.

Proof. As observed by Hempel [15], it follows from the hypothesis and the Sphere Theorem [18] that U is an open 3-cell. Thus by Theorem 4, there exists a point p such that S is locally tame from U at each point of S - p. That S is tame from U follows from [16, Theorem 1 and Corollary 2].

COROLLARY 3. *If*

(i) S *is a* 2-*sphere in* S^3,

(ii) *for each* $\varepsilon > 0$, S *can be approximated in each component of* S^3 - S *with a map that moves no point more than a distance* ε, *and*

(iii) S *can be pierced with a disk on each tame arc on* S,

then S *is tame.*

5. 2-MANIFOLDS THAT ARE ALMOST TAME

Some of the theorems in Sections 3 and 4 can be adapted to a 2-manifold that separates a 3-manifold. Of course, we need to formulate a suitable condition to replace the requirement that a complementary domain of the 2-sphere S in S^3 be an open 3-cell.

In all of the theorems stated in this section, M denotes a compact connected 2-manifold that separates a connected 3-manifold M^3, and U denotes a component of M^3 - M. Suppose there exists a sequence $M_1, M_2, \cdots$ of polyhedral 2-manifolds converging to M such that for some $y \in U$ each M_j separates y from M in M^3 and

(12) for some $\gamma > 0$ and for each positive integer j, every unknotted simple closed curve in M_j of diameter less than γ is the boundary of a disk in M_j.

We notice that these requirements are satisfied if M is a 2-sphere in S^3 such that the component U of S^3 - M is an open 3-cell.

THEOREM 7. *If* M *can be locally peripherally collared from* U, *then there exists a point* p *such that* M *is locally tame from* U *at each point of* M - p.

Proof. We obtain a proof of this theorem by changing slightly the proof of Theorem 1. We replace S^3, S, S_1, and S_2 with M^3, M, M_1, and M_2, respectively, and unless otherwise specified, we follow the same procedure and use the same notation as in the proof of Theorem 1.

Let p_1 and p_2 be two points of M, and let K be a disk in M such that $p_1 \cup p_2 \subset \text{Int } K$. We require that the disks D_1 and D_2 be subsets of Int K, and we assume, for convenience, that M_1 separates $J_1 \cup J_2 \cup a_1 a_2$ from M in M^3 and that $M_1 \subset N(M, \delta)$. In view of (12) and the fact that each simple closed curve in

$A_1 \cup A_2$ is unknotted, we assume further that each simple closed curve in each $A_i \cap M_j$ is the boundary of a disk in M_j. In the proof of Theorem 1, we used the fact that each component of $S^3 - S_1$ is simply connected, to obtain the contradiction that it is impossible for J' to link J_1. Since we do not know that $M^3 - M_1$ is simply connected, we construct J' so close to K that it is impossible for J' to link J_1.

We choose a positive number σ such that $\sigma < \delta$ and

(13) each simple closed curve in $N(K, \sigma)$ can be shrunk to a point in the component of $M^3 - M_1$ that contains K.

Furthermore, we require that σ be so small that, for each arc $p_i x$ $(i = 1, 2)$ that lies in $N(M, \sigma)$ and has diameter at least 3δ, there exist an arc $p_i b$ in $p_i x$ and a point c in $K - (D_1 \cup D_2)$ such that

(14) $$p_i b \subset N(K, \sigma)$$

and

(15) the rectilinear arc bc has length less than σ and does not intersect $A_1 \cup D_1 \cup A_2 \cup D_2$.

We identify A_1' and A_2' as in the proof of Theorem 1, so that $A_1' \cup A_2' \subset N(K, \sigma)$. For convenience, we assume that $M_2 \subset N(M, \sigma)$. We obtain the disk D' in M_2 and consider the case where $\text{Bd}\, D' \subset A_1'$. We adjust D' to a disk D in $D' \cup (N(M, \sigma) \cap U)$. If we suppose that $\text{Diam}\, D \geq 6\delta$, we can follow the procedure in the proof of Theorem 1 to construct an arc $p_1 b$ and a rectilinear arc bc satisfying requirements (14) and (15) above so that $b \in D$ and the arc $p_1 b \cup bc$ does not intersect $A_1 \cup (D_1 - p_1)$. Let cp_1 be an arc in $N(K, \sigma)$ such that

$$cp_1 - (c \cup p_1) \subset M^3 - (M \cup U),$$

and let J' denote the simple closed curve $p_1 b \cup bc \cup cp_1$. As in the proof of Theorem 1, we have constructed J' so that it links J_1. It follows from (13) that J' can be shrunk to a point in the component of $M^3 - M_1$ that contains K. Thus, since M_1 separates J_1 from K in M^3, we have the contradiction that J' does not link J_1. As in the proof of Theorem 1, we have shown that M can be locally spanned from U at one of the points p_1 and p_2. Hence the conclusion of Theorem 7 follows from [9, Theorem 10].

THEOREM 8. *If each tame arc in* M *can be collared from* U, *then there exists a point* p *such that* M *is locally tame from* U *at each point of* $M - p$.

Proof. Following Bing's work on the existence of tame Sierpiński curves in a 2-manifold in a 3-manifold [7, p. 513], we observe that the lemma we used in the proof of Theorem 4 can be proved for a 2-manifold in a 3-manifold. Thus Theorem 8 follows directly from Theorem 7 and this adjusted lemma.

THEOREM 9. *If* M *can be pierced on each of its arcs with a tame disk, then* M *is tame.*

Proof. We apply Theorem 8 in the manner in which we used Theorem 4 in the proof of Theorem 5.

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The University of Utah