EXAMPLES OF GENERALIZED-MANIFOLD APPROACHES TO TOPOLOGICAL MANIFOLDS

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Dedicated to R. L. Wilder on his seventieth birthday.

INTRODUCTION

The theory of generalized manifolds and the study of topological manifolds have no doubt influenced each other in methods and motivations. Of course, the theory of topological manifolds has its own approaches, and the extent to which we can apply the theory of generalized manifolds is necessarily limited. At times, however, the latter theory may offer a better picture of the problem involved, and it may even play an essential role. I hope to illustrate this point with a few simple examples.

1. SEPARABLE 1- AND 2-gms

One of the most useful aspects of gms in connection with topological manifolds is the fact that 1- and 2-gms that are separable are actual manifolds [8, Chapter 9]. This is not the case for dimensions greater than 2. In fact, the monotone mapping theorem [9], [10] of Wilder opened a convenient way of constructing gms that are not manifolds, for dimensions exceeding 2. In order to show that a given space be a 2-manifold, it therefore suffices to prove that it is a 2-gm. To give an example of this approach, we recall the following famous theorem of R. L. Moore.

If G *is an upper-semicontinuous decomposition of the plane into continua that do not separate the plane, then the decomposition space is homeomorphic to the plane.*

This theorem is an immediate corollary of the monotone mapping theorem of Wilder that I have already mentioned. I point out that despite appearances to the contrary, theorems on generalized manifolds sometimes have significant implications concerning manifolds.

2. HANDLING OF BOUNDARIES

By this time, it is well known that a cartesian factor of a manifold need not be a manifold (see [1]). Suppose A and B are spaces, and suppose one wishes to show that $A \times B$ is a manifold with nonempty boundary. It is often necessary to consider separately the candidate for $\mathrm{Bd}(A \times B)$ and that for $\mathrm{Int}(A \times B)$. This means that one has to analyze $\mathrm{Int}(A \times B)$ in terms of A and B. Since A and B need not be manifolds, this may pose a difficulty. The most natural way to take care of this is to regard $A \times B$ as a gm and to use the formula $\mathrm{Bd}(A \times B) = \mathrm{Bd}\, A \times B \cup A \times \mathrm{Bd}\, B$. Here Bd is taken in the sense of gms, and this is possible because by the factorization theorem [6] of Raymond A and B are gms.

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Consider the following example. Let S be a 2-sphere embedded (not necessarily tamely) in the 3-sphere S^3. Let U be a complementary domain of S in S^3. Then $\overline{U} = U \cup S$.

Though many interesting spaces that are not manifolds are still factors of manifolds, in this particular case $\overline{U}$ cannot be a factor of a manifold unless it is already a manifold. To see this, suppose for some space B that $\overline{U} \times B$ is a manifold, and let $p \in \text{Bd}\, \overline{U} = S$ and $q \in B$. Then $(p, q) \in \text{Bd}(\overline{U} \times B)$, and since $\overline{U} \times B$ is a genuine manifold, $\text{Int}(\overline{U} \times B)$ is 1-LC (locally simply connected) at (p, q). Hence $\text{Int}\, \overline{U} = U$ is 1-LC at p. According to a result of Bing [2, corollary to Theorem 2, p. 300], this means that S is tame and $\overline{U}$ is a 3-cell. (This observation may be stated in a more general form; see Remark 1 at the end of this text.)

3. SUSPENDING HOMOLOGY SPHERES

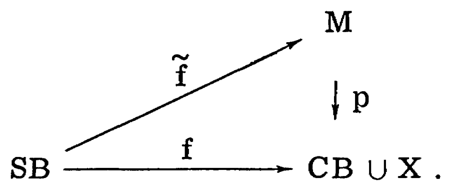
One of the easiest ways of going beyond the class of topological manifolds and yet staying in the class of gms is to take the suspension of a closed manifold that is a homology sphere. Even when one is not primarily interested in the suspension itself, this construction sometimes facilitates the proof. For example, consider the following proposition, which came to light during a conversation with M. Brown, W. Browder, and D. Epstein.

PROPOSITION. *An* n-*manifold* X *is compact and contractible if and only if it is connected and its boundary* B *is a homology sphere such that the inclusion* $B \subset X$ *is inessential.*

We prove only the sufficiency. Our proof is based on an observation by Epstein. Let CB be the cone over B, and form an n-gm $CB \cup X$ by attaching CB and X along B. We denote by SB the suspension of B, and we construct a map $f\colon SB \to CB \cup X$ as follows. Since the inclusion $B \subset X$ is inessential, we may choose f so that it maps the upper cone of SB onto CB by the identity and the lower cone of SB into X. Now both SB and $CB \cup X$ are gms. The behavior of f on the upper cone ensures that the cohomology degree of f is 1 (or -1) over the integers. More precisely, let U be an open connected subset of the open upper cone of SB, and consider the commutative diagram

$$\begin{array}{ccc} H_c^n(U) & \overset{f^*}{\underset{\simeq}{\longleftarrow}} & H_c^n(U) \\ \simeq \downarrow i^* & & \downarrow i^* \\ H^n(SB) & \overset{f^*}{\longleftarrow} & (CB \cup X)\,. \end{array}$$

The right-side i* is nontrivial. Since this is true over integer- or Z_2-coefficients, $CB \cup X$ is orientable and compact. It follows that the lower f^* is an isomorphism. Using the fact that Poincaré duality is induced by capping with the orientation class, we deduce without difficulty that $CB \cup X$ is a homology n-sphere, in other words, that after deletion of CB, X is homologically trivial. To see that $\pi_1 X = 1$, let $p\colon \widetilde{M} \to CB \cup X$ be the universal covering, and let $\tilde{f}\colon SB \to \widetilde{M}$ be a lifting of f, as is indicated in the diagram



Let q be the vertex of the upper cone of SB (we note that q is also the vertex of CB in CB ∪ X). Since deg f ≠ 0, it follows that deg $\tilde{f} \neq 0$. Hence $\tilde{M}$ is compact and $\tilde{f}$ is onto. Thus $\tilde{f}(q) = p^{-1}(q)$. Thus p is a trivial covering, and therefore $\pi_1(CB \cup X)$ is trivial. Since the inclusion $B \subset X$ is inessential, Van Kampen's theorem implies that $\pi_1 X = 1$. (The hypothesis that B be a homology sphere is not necessary. For a stronger result, see Remark 2 in Section 5.)

4. SHRINKING A MANIFOLD

The fourth and final example is of an obvious kind. Namely, we apply a known result in generalized manifolds to a problem on topological manifolds.

Consider the following question: What closed manifolds A can be embedded (not necessarily locally nicely) in a higher-dimensional manifold M in such a way that the space M/A obtained from M by shrinking A to a point is again a manifold? As P. Doyle has pointed out, we may suppose that M/A is an n-sphere.

Using a theorem due to Raymond and myself [5, Theorem 3 (2)], J. Hocking and I have been able to show that the Z_2-cohomology ring $H^*(A; Z_2)$ of A must be a truncated polynomial ring generated by a single element of degree $p > 0$. A result [7, Theorem 4.5] of Steenrod and Adams implies that p = 1, 2, 4, or 8. In particular, the n-sphere can be shrunk as indicated in the question if and only if n = 1, 2, 4, or 8, and no nontrivial product of manifolds can be so shrunk. A purely topological proof that (for example) a torus cannot be shrunk in this way might be very difficult. For the details, see [4].

5. REMARKS

I add some remarks that were not a part of my talk.

1. The example $\overline{U}$ of Section 2 can be stated in a more general form. One need only assume that a 2-manifold M is embedded in a 3-manifold as a closed subset and that M has two complementary domains. U can be either of the two complementary domains. The result is that $\overline{U}$ is a manifold if and only if $\overline{U} \times B$ is a manifold for some space B.

2. After my talk, Morton Brown expressed his belief that the homology sphere assumption in the proposition of Section 3 is not needed, and that the mere assumption that the boundary of X be nonempty and compact should suffice. I shall show that this is indeed the case. Since nothing is new for $n \leq 2$, we may assume $n \geq 3$.

Since the inclusion $B \subset X$ is inessential, the sequences

$$0 \to H^{q-1}(B) \to H^q(X, B) \to H^q(X) \to 0,$$

$$0 \to H_q(X) \to H_q(X, B) \to H_{q-1}(B) \to 0$$

are exact for all $q > 1$ over any abelian group. In particular,

$$0 \neq H^{n-1}(B; Z_2) \simeq H^n(X, B; Z_2) \simeq H_0(X; Z_2)$$

(here H_0 denotes the 0th Borel-Moore homology group [3]). Since X is connected, this implies that X is compact and B is connected.

Letting $q = n$ over Z, we see that X and B are both orientable or both non-orientable.

Case 1. X and B are orientable. The exactness of the sequence

$$0 \to H_1(X) \to H_1(X, B) \to H_0(B) \overset{\simeq}{\to} H_0(X)$$

implies that $H_1(X) \simeq H_1(X, B)$, and by Poincaré duality,

$$H^{n-1}(X, B) \simeq H^{n-1}(X) \quad \text{and} \quad H^{n-2}(B) = 0.$$

Hence $H_1(B) = 0$.

Suppose it has been proved that $H_i(B) = 0$ for all i $(1 \leq i < k < n - 1)$. Then

$$0 \to H_k(X) \to H_k(X, B) \to 0 \quad \text{and} \quad H^{n-k}(X, B) = H^{n-k}(X),$$

and therefore $H^{n-k-1}(B) = 0$ and finally $H_k(B) = 0$.

Note that in concluding the triviality of some $H_i(B)$, we use implicitly the commutative diagram

$$\begin{array}{ccc} H^i(X, B) & \to & H^i(X) \\ \downarrow & & \downarrow \\ H_{n-i}(X) & \to & H_{n-i}(X, B), \end{array}$$

where the vertical homomorphisms are Poincaré duality isomorphisms and the horizontal homomorphisms are induced by the inclusion. Thus B is a homology sphere, and by the proposition in Section 3, X is contractible.

Case 2. X and B are not orientable. Let p: $\tilde{X} \to X$ be the 2-sheeted orientable covering. $\tilde{B} = p^{-1}(B)$ is connected, since B is not orientable. Since p induces monomorphisms for homotopy groups of positive dimensions, the homomorphisms $\pi_i(\tilde{B}) \to \pi_i(\tilde{X})$ are trivial for all $i \geq 1$. Hence the inclusion $\tilde{B} \subset \tilde{X}$ is inessential, and by Case 1, $\tilde{X}$ is contractible and has the fixed-point property. On the other hand, since p is 2-sheeted, there exists a fixed-point-free involution. This contradiction rules out Case 2.

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