DIFFERENTIABLE TRANSFORMATION GROUPS ON HOMOTOPY SPHERES

D. Montgomery and C. T. Yang

1. INTRODUCTION

The purpose of the present paper is to use techniques and results of differentiable topology in the study of differentiable transformation groups. Although our objective is the same as that in recent works of Conner and Floyd (see, for example, [4]), we do not use bordism theory, a fundamental tool employed by Conner and Floyd, but rely on pasting techniques instead. Recent papers by W. C. Hsiang and W. Y. Hsiang on differentiable actions have some points of contact with our work.

Denote by S^n the unit n-sphere in euclidean $(n+1)$-space, and regard S^{n-1} as a subspace of S^n by identifying every $(x_1, \cdots, x_n) \in S^{n-1}$ with $(x_1, \cdots, x_n, 0) \in S^n$. Let G be a compact Lie group acting as a transformation group on S^n, and let

$$F = \{x \in S^n \mid Gx = x\};$$

that is, let F be the set of the fixed points of G in S^n. If the action is linear, it is obvious that F is diffeomorphic to S^r for some integer r $(-1 \leq r \leq n)$. (It is understood that S^{-1} denotes the null set.) Moreover, there is a diffeomorphism of S^n onto itself that maps F onto S^r.

Suppose that the action of G on S^n is only differentiable. Then F is a (differentiable) submanifold of S^n. However, F may not be homeomorphic to a sphere, because F may not even be an integral cohomology sphere [2]. In this paper we shall study differentiable actions of G on S^n in which F is an integral cohomology sphere. They are more general than linear actions, but the case just mentioned is excluded.

Let G be a nontrivial compact Lie group and n(G) the integer such that G is isomorphic to a subgroup of the orthogonal group O(n(G)) but not to any subgroup of the orthogonal group O(n(G) - 1). It is easily seen that if G acts effectively and differentiably on S^n, the fixed point set F is of dimension at most n - n(G) (Proposition 1). As a modification of a theorem of Montgomery and Samelson [7], we shall show that for any nontrivial compact Lie group G, there are infinitely many effectively differentiable actions of G on $S^{n(G)+3}$ of which the fixed point sets are integral cohomology 3-spheres with fundamental groups not isomorphic to one another (Theorem 1). Hence the fixed point set of an effective differentiable action of any nontrivial compact Lie group on S^n may not be homeomorphic to a sphere, even when it is an integral cohomology sphere attaining the highest possible dimension.

We next consider the case when G is the circle group SO(2). As we said in the preceding paragraph, it is possible to have an effective differentiable action of G on S^5 such that the fixed point set F is an integral cohomology 3-sphere that is not simply connected. (Notice that n(SO(2)) = 2, so that n(SO(2)) + 3 = 5.) Therefore such an action is not equivalent to a linear action. In this particular case, it is

Received February 12, 1966.
The second author was supported in part by the U. S. Army Research Office.

crucial whether F is simply connected. In fact, we can show that if the circle group acts differentiably on S^n $(n > 6)$ in such a way that the fixed point set F is simply connected and (n - 2)-dimensional, then the action is differentiably equivalent to a linear action, so that F is diffeomorphic to S^{n-2} (Proposition 3).

Suppose that there is an effective differentiable action of the circle group on S^n with a simply connected fixed point set F. We have said that if $n > 6$ and F attains the highest possible dimension, then F is diffeomorphic to S^{n-2}. On the other hand, we shall see that if F attains the next highest possible dimension, that is, if $\dim F = n - 4$, then F may not be diffeomorphic to S^{n-4}. In fact, we construct a differentiable action of the circle group on S^{11} whose fixed point set is an exotic 7-sphere (Theorem 2).

Finally, we study free differentiable actions of the circle group on S^n. Our main result: Let the circle group G act freely and differentiably on a homotopy sphere Σ^n such that there is a sequence of invariant homotopy spheres

$$\Sigma^n \supset \Sigma^{n-2} \supset \cdots \supset \Sigma^7 \supset \Sigma^5 .$$

Then $\Sigma^5, \Sigma^7, \cdots, \Sigma^{n-2}$ are diffeomorphic to $S^5, S^7, \cdots, S^{n-2}$, respectively. Moreover, for any differentiable action of the circle group on S^5, there are, up to a differentiable equivalence, only finitely many ways to extend it to such an action on Σ^n with $\Sigma^5, \cdots, \Sigma^{n-2}$ invariant.

2. ACTIONS OF AN ARBITRARY, NONTRIVIAL, COMPACT LIE GROUP

In this section, we study effective differentiable actions of a nontrivial compact Lie group on a homotopy sphere of which the fixed point set is an integral cohomology sphere of the highest possible dimension.

PROPOSITION 1. *Let* M^n *be a connected, differentiable* n-*manifold, and let* G *be a compact Lie group acting as an effective differentiable transformation group on* M^n *having a nonempty fixed point set* F. *Then* F *is of dimension at most* $n - n(G)$, *where* $n(G)$ *is the smallest integer such that* G *is isomorphic to a subgroup of the orthogonal group* $O(n(G))$.

Proof. If p is a fixed point, there exists an invariant open neighborhood Q of p such that the action of G on Q is differentiably equivalent to an orthogonal action of G on euclidean n-space (see, for example, [8]). Therefore we may assume that Q is an open n-disk on which there exists a coordinate system with respect to which G acts orthogonally. Under this assumption, $Q \cap F$ is a linear subdisk of Q, and its orthogonal complement at p, denoted by Q', is an invariant linear subdisk of Q. Since M^n is connected, Q' contains a principal orbit. Therefore G acts effectively on Q', so that G is isomorphic to a subgroup of O(dim Q'). By the definition of n(G), $\dim Q' \geq n(G)$. Hence the dimension of F at p is $n - \dim Q' \leq n - n(G)$.

Remark. We do not know whether the proposition holds when F is empty. In other words, we can not determine whether there is an effective differentiable action of a compact Lie group G on a differentiable manifold M^n $(\dim n \leq n(G) - 2)$ without fixed point.

By an *integral cohomology* n-*sphere* we mean a compact differentiable n-manifold that has the integral cohomology group of the n-sphere S^n. By a *homotopy* n-*sphere* we mean a compact differentiable n-manifold with the homotopy type of S^n. Clearly, a homotopy n-sphere is an integral cohomology n-sphere, but an integral cohomology n-sphere need not be a homotopy n-sphere.

PROPOSITION 2. *Let* Y *be a contractible, compact, differentiable* $(m + 1)$-*manifold bounded by an integral cohomology* m-*sphere* B $(m > 0)$. *Then there exists a differentiable action of the orthogonal group* O(n) *on a homotopy* $(m + n)$-*sphere* Σ *such that*

(i) *the fixed point set* F *is diffeomorphic to* B,

(ii) *all orbits in* $\Sigma - F$ *are of the type* $O(n)/O(n-1)$,

(iii) *the orbit space* $\Sigma/O(n)$ *is diffeomorphic to* Y, *and*

(iv) *there is a differentiable cross-section* $s: \Sigma/O(n) \to \Sigma$.

Proof. By the collaring theorem, there is a diffeomorphism ϕ of $B \times [0, 1)$ onto a neighborhood of B in Y such that $\phi(x, 0) = x$ for each $x \in B$. Let D^n be the unit n-disk in euclidean n-space, S^{n-1} the boundary of D^n, $E = D^n - S^{n-1}$, and $E_0 = E - \{0\}$. Then the map

$$\psi : B \times E_0 \to \phi(B \times (0, 1)) \times S^{n-1},$$

defined by

$$\psi(x, y) = (\phi(x, \|y\|), y/\|y\|),$$

is a diffeomorphism. Therefore we can obtain a differentiable $(m + n)$-manifold Σ from the disjoint sum $M = B \times E + (Y - B) \times S^{n-1}$ by identifying each $(x, y) \in B \times E_0$ with $\psi(x, y)$, where the differentiable structure on Σ is the one such that the projection $p : M \to \Sigma$ is differentiable. Notice that when $n = 1$, Σ is the double of Y on B.

Let $G = O(n)$ act on D^n in the standard way. Then there is a differentiable action of G on M such that $g(x, y) = (x, gy)$ for each $g \in G$ and each $(x, y) \in M$. Since $g\psi = \psi g$ holds for all $g \in G$, it follows that the action of G on M induces a differentiable action of G on Σ commuting with the projection p.

From the construction of the action of G on Σ, it is clear that the fixed point set F is $p(B \times \{0\})$, which is diffeomorphic to B, and that all orbits in $\Sigma - F$ are of the type $O(n)/O(n-1)$. Moreover, the map $\xi : M \to Y$, defined by

$$\xi(x, y) = \begin{cases} \phi(x, \|y\|) & \text{if } (x, y) \in B \times E, \\ x & \text{if } (x, y) \in (Y - B) \times S^{n-1}, \end{cases}$$

induces a diffeomorphism ξ^* of Σ/G onto Y with $\xi = \xi^* p$. Let b be a point of S^{n-1}, and let $\lambda : Y \to \Sigma$ be defined by

$$\lambda(x) = \begin{cases} p(x, 0) & \text{if } (x, 0) \in B \times E, \\ p(x, b) & \text{if } (x, b) \in (Y - B) \times S^{n-1}. \end{cases}$$

It is easily seen that $s = \lambda\xi^*$ is a differentiable cross-section. Hence (i) to (iv) are satisfied, and consequently the last part of our assertion holds. For the first part of our assertion, it remains to be shown that Σ is a homotopy $(m + n)$-sphere, or equivalently, that Σ is a simply connected integral cohomology $(m + n)$-sphere. Because it is easy to compute the fundamental group and the integral cohomology group of Σ, we omit the details.

THEOREM 1. *For each nontrivial compact Lie group* G, *there exist infinitely many effective differentiable actions on the* $(n(G) + 3)$-*sphere (with the standard* *differentiable structure) for which the fixed point sets are integral cohomology spheres whose fundamental groups are not isomorphic to one another; here* $n(G)$ *denotes the smallest integer such that* G *is isomorphic to a subgroup of* $O(n(G))$.

Proof. Let Y be the contractible, compact, differentiable 4-manifold bounded by a multiply connected integral cohomology 3-sphere B as given in [6]. Using induction on r, we shall now construct, for each positive integer r, a compact differentiable 4-manifold Y_r of boundary B_r. We set $Y_1 = Y$. If Y_{r-1} has been constructed for some r $(r > 1)$, let H^4 be the upper half of the open unit 4-disk in euclidean 4-space, p a point of B_{r-1}, and q a point of B. Then there is a diffeomorphism ϕ of H^4 onto an open neighborhood of p in Y_{r-1} and a diffeomorphism ψ of H_4 onto an open neighborhood of q in Y such that $\phi(0) = p$ and $\psi(0) = q$. We define Y_r as the connected sum $Y_{r-1} \# Y$ obtained from the disjoint sum $(Y_{r-1} - \{p\}) + (Y - \{q\})$ by identifying $\phi(x)$ with $\psi((1 - \|x\|)x)$ for all $x \in H^4 - \{0\}$. It is easily seen that Y_r is unique up to a diffeomorphism, that it is simply connected, and that it is integrally cohomologically trivial. Therefore Y_r is contractible. It is also easily seen that B_r is the connected sum $B_{r-1} \# B$ in the usual sense, so that the fundamental group of B_r is the free product of r copies of the fundamental group of B. Hence, for $r \neq s$, the fundamental groups of B_r and B_s are not isomorphic.

By Proposition 2, there exist for each positive integer r a homotopy $(n(G) + 3)$-sphere Σ_r and an effective differentiable action of $O(n(G))$ on Σ_r such that (i) the fixed point set F_r is diffeomorphic to B_r, (ii) all orbits in $\Sigma_r - F_r$ are of the type $O(n(G))/O(n(G) - 1)$, (iii) the orbit space $\Sigma_r/O(n(G))$ is diffeomorphic to Y_r, and (iv) there is a differentiable cross-section $s_r : \Sigma_r/O(n(G)) \to \Sigma_r$. By (i), $F_1, F_2, \cdots$ are integral cohomology 3-spheres whose fundamental groups are not isomorphic to one another. Since $Y_r = Y_{r-1} \# Y$, it follows from (i) to (iv) (or from the construction of Σ_{r-1} and Σ_r) that

$$\Sigma_r = \Sigma_{r-1} \# \Sigma,$$

where the connected sum is obtained by means of an equivariant diffeomorphism of a neighborhood of a fixed point in Σ_{r-1} onto a neighborhood of a fixed point in Σ $(= \Sigma_1)$.

Assume first that G is a finite group of order 2, say $G = O(1)$. Since the double of Y on B is diffeomorphic to the 4-sphere S^4 (see [6]), each Σ_r is diffeomorphic to S^4. Hence our assertion holds for $G = O(1)$.

Assume next that G contains more than two elements. Then $n(G) > 1$, so that $n(G) + 3 \geq 5$. Therefore the group Θ of diffeomorphism classes of homotopy spheres of dimension $n(G) + 3$ is finite [5], and hence, for some positive integer r_0, Σ_r is diffeomorphic to the $(n(G) + 3)$-sphere with the standard differentiable structure if r is a multiple of r_0. Let G act on Σ_r as a subgroup of $O(n(G))$. The action is clearly differentiable and effective. Since G is not isomorphic to any subgroup of $O(n(G) - 1)$, the fixed point set of G coincides with that of $O(n(G))$. Hence, by letting $r = r_0, 2r_0, \cdots$, we again obtain desired actions of G.

Remarks. (1) Theorem 1 is a slight improvement on the following result of Montgomery and Samelson [7]: If G is a nontrivial compact Lie group, there exists an integer $n \geq \max\{n(G), 10\} + 4$ such that there are infinitely many differentiable actions of G on S^n whose fixed point sets are integral cohomology 4-spheres with fundamental groups not isomorphic to one another.

(2) If the action of G is required to be orientation-preserving and $n(G)$ is replaced by the smallest integer $n'(G)$ such that G is isomorphic to a subgroup of $SO(n'(G))$, then Propositions 1 and 2 and Theorem 1 remain valid.

(3) If G is a nontrivial compact Lie group that acts as an effective differentiable transformation group on S^n and whose fixed point set is a multiply connected integral cohomology sphere, then $n \geq n(G) + 3$. Hence the dimension of the sphere in Theorem 1 is the lowest possible.

(4) Let G be a compact Lie group acting effectively and differentiably on the $(n(G) + k)$-sphere and having a k-dimensional fixed point set F. In Theorem 1, we see that for $k = 3$, F need not be homeomorphic to the k-sphere. However, we do not know whether this is still true for $k = 0, 1, 2$.

The following is a special case of Theorem 1.

COROLLARY. *There exists a differentiable action of the circle group on the* 5-*sphere such that the fixed point set is a multiply connected integral cohomology* 3-*sphere.*

3. ACTIONS OF THE CIRCLE GROUP

We shall now study differentiable actions of the circle group SO(2) on a homotopy n-sphere Σ^n. It is well-known that in this case the fixed point set F is an integral cohomology sphere of even codimension (see, for example, [1]). The corollary in the last section shows that if $\dim F = n - 2$, that is, if F has the highest possible dimension under an effective action, then F need not be simply connected. The following proposition shows that whether F is simply connected is really crucial here.

PROPOSITION 3. *Let* G *be the circle group* SO(2), *and let it act as a differential transformation group on a homotopy sphere* Σ^n *such that the fixed set* F *is* $(n - 2)$-*dimensional* $(n > 6)$ *and simply connected. Then* Σ^n *is diffeomorphic to the* n-*sphere* S^n, F *is diffeomorphic to the* $(n - 2)$-*sphere, and the action of* G *on* S^n *is differentiably equivalent to a linear action.*

Proof. Since Σ^n is a homotopy n-sphere, it is an integral cohomology n-sphere, so that F is an integral cohomology $(n - 2)$-sphere. From this result and the hypothesis that F is simply connected, we infer that F is a homotopy $(n - 2)$-sphere.

It is easily seen that all orbits in $\Sigma^n - F$ are of the same type; hence we may assume that G acts freely on $\Sigma^n - F$. Therefore the orbit space Σ^n/G is a compact, differentiable $(n - 1)$-manifold with boundary F. It is easy to show that Σ^n/G is simply connected and that its integral cohomology group is trivial. Therefore Σ^n/G is homotopically trivial, and hence it is contractible. Since $n > 6$, it follows from Smale's theorem [9] that Σ^n/G is diffeomorphic to the closed unit $(n - 1)$-disk D^{n-1} in euclidean $(n - 1)$-space. Hence F is diffeomorphic to the $(n - 2)$-sphere.

Let $f^* : D^{n-1} \to \Sigma^n/G$ be a diffeomorphism. Then there is a differentiable map $f : D^{n-1} \to \Sigma^n$ such that $f^* = \pi f$, where $\pi : \Sigma^n \to \Sigma^n/G$ is the projection. Let D^k denote the closed unit k-disk in euclidean k-space. Then we may regard

$$\{(x, y) \in D^{n-2} \times D^2 \mid \|x\|^2 + \|y\|^2 = 1\}$$

as S^n, and an orthogonal action of G on S^n is given by

$$g(x, y) = (x, gy)$$

for all $g \in G$ and $(x, y) \in S^n$.

Let b be any preassigned point of the boundary of D^2. It is easily seen that $h : S^n \to \Sigma^n$, defined by

$$gf(x) = h(x, g(1 - \|x\|^2)^{1/2} b)$$

for all $x \in D^{n-1}$ and $g \in G$, is an equivariant diffeomorphism. Hence our assertion is proved.

Remark. Proposition 3 also holds for $n \leq 3$. For the cases $n = 4, 5, 6$, Proposition 3 is dependent on the unsolved cases of the Poincaré conjecture.

Suppose $G = SO(2)$ acts differentiably on the n-sphere S^n and has a simply connected fixed point set F. We shall show that if $\dim F = n - 4$, that is, if F has the second highest dimension possible under an effective action, then F is not necessarily diffeomorphic to the $(n - 4)$-sphere. We need the following preliminary result.

In the following proposition we speak of a differentiable structure on Σ^n/G, and this requires definition. Away from the fixed point set the action of G on G^n gives a fibering, and in this region, differentiable coordinates in the base space are given as usual. Let x be a point of the fixed point set F, and let x^* and F^* correspond in the base space. In Σ^n, a neighborhood of x is locally a product of a 4-cell and an $(n - 4)$-cell in F. Down below, a neighborhood of x^* is locally a product of a 3-cell and an $(n - 4)$-cell. This latter product is to be used to give local coordinates around x^*. It is not important that the natural map from above to below is not differentiable.

PROPOSITION 4. *Let* Σ^{n-1} *and* Σ^{n-4} *be homotopy spheres of dimension* $n - 1$ *and* $n - 4$, *respectively* $(n \geq 7)$, *and let* f *be an imbedding of* Σ^{n-4} *into* Σ^{n-1}. *Then there exists a differentiable action of the circle group* $G = SO(2)$ *on a homotopy* n-*sphere* Σ^n *such that*

(i) *the fixed point set* F *is* $(n - 4)$-*dimensional,*

(ii) G *acts freely on* $\Sigma^n - F$, *and*

(iii) *there is a diffeomorphism of the orbit space* Σ^n/G *onto* Σ^{n-1} *mapping* F *onto* $f(\Sigma^{n-4})$.

Proof. Let there be a Riemannian metric on Σ^{n-1}, and let

$$\lambda : E \to f(\Sigma^{n-4})$$

be the normal bundle of $f(\Sigma^{n-4})$ in Σ^{n-1}. Then, for some $\delta > 0$, the exponential function

$$\exp : E \to \Sigma^{n-1}$$

maps $K = \{v \in E \mid \|v\| \leq \delta\}$ diffeomorphically onto a closed tubular neighborhood of $f(\Sigma^{n-4})$.

Let D^{2k+2} be the closed unit $(2k + 2)$-disk in unitary $(k + 1)$-space, that is, in the set of points $(z_0, \cdots, z_k)$ satisfying

$$\|(z_0, \cdots, z_k)\|^2 = z_0 \bar{z}_0 + \cdots + z_k \bar{z}_k \leq 1,$$

where $z_0, \cdots, z_k$ are complex numbers; and let S^{2k+1} be the boundary of D^{2k+2}. Regard G as the unit circle in the complex field, and let G act on D^{2k+2} so that

$$g(z_0, \cdots, z_k) = (gz_0, \cdots, gz_k)$$

for each $g \in G$ and each $(z_0, \cdots, z_k) \in D^{2k+2}$. Then G leaves the center $(0, \cdots, 0)$ of D^{2k+2} fixed and acts freely on $D^{2k+2} - \{(0, \cdots, 0)\}$. Moreover, S^{2k+1}/G is the complex projective k-space; therefore

$$\pi_r(S^{2k+1}/G) \cong 0 \qquad \text{for } 0 \leq r \leq 2k \text{ and } r \neq 2,$$

and for $k > 0$, $\pi_2(S^{2k+1}/G)$ is infinite cyclic and has a generator represented by the 2-sphere S^3/G, with S^3 given by $z_0 \bar{z}_0 + z_1 \bar{z}_1 = 1$.

Let ∂K be the boundary of K, and let $a \in f(\Sigma^{n-4})$. Then $\lambda^{-1}(a) \cap \partial K$ is a 2-sphere, so that we have a diffeomorphism $h : \lambda^{-1}(a) \cap \partial K \to S^3/G$. When $2k > 2(n-2)$, h can be extended to an imbedding

$$h_1 : \partial K \to S^{2k+1}/G,$$

because we can first extend h to a map of ∂K into S^{2k+1}/G by means of obstruction theory and then approximate the map by an imbedding.

Let $\pi : S^{2k+1} \to S^{2k+1}/G$ be the projection. Then

$$M' = \{(v, y) \in K \times D^{2k+2} \mid \text{either } \|v\| = \|y\| = 0 \text{ or else}$$
$$\|v\| = \|y\| \neq 0 \text{ and } h_1(v/\|v\|) = \pi(y/\|y\|)\}$$

is a compact differentiable n-manifold with boundary, and there exists a differentiable action of G on M' such that

$$g(v, y) = (v, gy)$$

for each $g \in G$ and each $(v, y) \in M'$. It is easily seen that the fixed point set F of G in M' is the set $\{(v, y) \in M' \mid \|v\| = \|y\| = 0\}$ and that G acts freely on $M' - F$. Let $X = \Sigma^{n-1} - \exp(\operatorname{Int} K)$. By obstruction theory, the map

$$h_1(\exp)^{-1} : \exp(\partial K) \to S^{2k+1}/G$$

can be extended to a differentiable map

$$h_2 : X \to S^{2k+1}/G.$$

Let

$$M'' = \{(x, y) \in X \times S^{2k+1} \mid h_2(x) = \pi(y)\}.$$

Then M'' is a compact differentiable n-manifold with boundary, and there exists a free differentiable action of G on M'' such that for

$$g(x, y) = (x, gy)$$

for each $g \in G$ and each $(x, y) \in M''$. It is easily seen that there exists a natural diffeomorphism of the boundary of M' onto the boundary of M'' mapping every (v, y) into $(\exp v, y)$.

Denote by Σ^n the closed differentiable n-manifold obtained from the disjoint sum $M' + M''$ by identifying every point (v, y) of the boundary of M' with $(\exp v, y)$ of M''. A direct computation shows that the fundamental group of Σ^n vanishes and that Σ^n has the integral cohomology group of the n-sphere. Hence Σ^n is a homotopy n-sphere. Since the identification map from the boundary of M' to the boundary of M" commutes with the action of G, it follows that there exists a differentiable action of G on Σ^n whose restrictions to M' and to M" are those given in the last paragraph. Let

$$\rho : M' + M'' \to \Sigma^{n-1}$$

be defined by

$$\rho(v, y) = \exp v \quad \text{for } (v, y) \in M',$$

$$\rho(x, y) = x \quad \text{for } (x, y) \in M''.$$

Clearly ρ induces a desired diffeomorphism of Σ^n/G onto Σ^{n-1}. Hence Proposition 4 is proved.

THEOREM 2. *There exists a differentiable action of the circle group* $G = SO(2)$ *on the* 11-*sphere (with the standard differentiable structure) such that the fixed point set* F *is an exotic* 7-*sphere and* G *acts freely on the complement of* F.

Proof. Since the group Θ_7 of diffeomorphism classes of homotopy 7-spheres is of order 28 [5], there exists a homotopy 7-sphere Σ^7 representing an element of order 7. It is known [5] that Σ^7 bounds a parallelizable compact differentiable 8-manifold. It follows from a theorem of Hirsch [9] that $\Sigma^7 \times D^3$ is diffeomorphic to $S^7 \times D^3$, where D^3 is the closed unit 3-disk in euclidean 3-space and S^7 is the unit 7-sphere in euclidean 8-space. Therefore we have an imbedding

$$f : \Sigma^7 \to S^{10},$$

because there is a standard imbedding of $S^7 \times D^3$ into the unit 10-sphere S^{10} in euclidean 11-space.

By Proposition 4, there exists a differentiable action of the circle group G on a homotopy 11-sphere Σ^{11} such that the fixed point set F is 7-dimensional, G acts freely on Σ^{11} - F, and some diffeomorphism of Σ^{11}/G onto S^{10} maps F onto $f(\Sigma^7)$. As in the proof of Theorem 1, we can construct, for each positive integer r, a homotopy 11-sphere Σ_r^{11} as the connected sum of r copies of Σ^{11} at fixed points, in such a way that there exists a differentiable action of G on Σ_r^{11} whose fixed point set F_r is diffeomorphic to the connected sum of r copies of Σ^7, and whose orbit space Σ_r^{11}/G is diffeomorphic to the connected sum of r copies of Σ^{11}/G (and hence is diffeomorphic to S^{10}). Since the group Θ_{11} of diffeomorphism classes of homotopy 11-spheres is of order 992 [5], there exists a factor r of 992 such that Σ_r^{11} is diffeomorphic to the unit 11-sphere S^{11} in euclidean 12-space. However, r is not divisible by 7, so that F_r is an exotic 7-sphere. Hence Theorem 2 is proved.

Remarks. (1) In the proof of Theorem 2, Σ^7 is imbedded into S^{10} so that the orbit space Σ_r^{11}/G is diffeomorphic to S^{10}. If S^{10} is replaced by a homotopy 10-sphere representing an element of Θ_{10} of order 3 (see [5]), then for any factor r of 992, Σ_r^{11}/G is an exotic 10-sphere.

(2) The action of the circle group on the 11-sphere given in Theorem 2 is not differentiably equivalent to a linear action. However, if one fixed point is removed from the 11-sphere, then the action is differentiably equivalent to a linear action on euclidean 11-space (see [3]).

(3) Since we do not know any exotic k-sphere for $k < 7$, we do not know whether Theorem 2 holds for spheres of lower dimension. However, it is not difficult to prove the theorem for spheres of higher dimension. For example, the same argument shows that there is a differentiable action of the circle group on the 13-sphere (with the standard differentiable structure) having an exotic 9-sphere as its fixed point set. This clarifies a point discussed in [2].

4. FREE ACTIONS OF THE CIRCLE GROUP

PROPOSITION 5. *Let* D *be a contractible, compact, differentiable* (n - 1)-*manifold with boundary* Y, *and let the circle group* G *act freely and differentiably on* Y. *Then there exists a homotopy* n-*sphere* Σ^n *such that there is a free differentiable action of* G *on* Σ^n *and an imbedding* $f : D \to \Sigma^n$ *satisfying the following conditions:*

a) *For each* $g \in G$ *and each* $y \in Y$, $f(gy) = gf(y)$.

b) *Some diffeomorphism of* $G \times f(D - Y)$ *onto* $\Sigma^n - f(Y)$ *maps every* $(g, y) \in G \times f(D - Y)$ *into* gy, *where* y *goes to* gy *under the action of* G *on* Σ^n.

Proof. Let E be the open unit disk in the complex field. Then

$$X' = E \times Y$$

is an open differentiable n-manifold. By the collaring theorem, there is an imbedding

$$h : [0, 1) \times Y \to D$$

such that $h(0, y) = y$ for each $y \in Y$. Let

$$X'' = G \times (D - Y).$$

Then X" is also an open differentiable n-manifold. Moreover, there exists a diffeomorphism λ from the open subset $(E - \{0\}) \times Y$ of X' to the open subset $G \times h((0, 1) \times Y)$ of X" defined by

$$\lambda(z, y) = (z/\|z\|, h(\|z\|, (z/\|z\|)^{-1}y)),$$

where $z \in E - \{0\}$, $y \in Y$, and G is regarded as the unit circle in the complex field. Therefore we obtain a closed differentiable n-manifold Σ^n from the disjoint sum $X' + X''$ by identifying (z, y) with $\lambda(z, y)$ for all $(z, y) \in (E - \{0\}) \times Y$. It is easy to show that Σ^n is simply connected and has the integral cohomology group of the n-sphere. Hence Σ^n is a homotopy n-sphere.

Let G act on $X' + X''$ in such a way that

$$g(z, y) = (gz, gy) \quad \text{for } g \in G \text{ and } (z, y) \in X',$$

$$g(g', u) = (gg', u) \quad \text{for } g \in G \text{ and } (g', u) \in X''.$$

Obviously the action is free and differentiable. Since the action commutes with the diffeomorphism λ, it induces a free differentiable action of G on Σ^n. Hence it remains to be shown that there is an imbedding $f : D \to \Sigma^n$ satisfying a) and b).

Define $f : D \to \Sigma^n$ as follows. Let $\rho' : X' \to \Sigma^n$ and $\rho'' : X'' \to \Sigma^n$ be the natural inclusions. Whenever $u \in D - Y$, we let

$$f(u) = \rho''(1, u).$$

Whenever $u = h(t, y)$ with $(t, y) \in [0, 1) \times Y$, we let

$$f(u) = \rho'(t, y).$$

Notice that if $u = h(t, y)$ with $(t, y) \in (0, 1) \times Y$, then

$$\lambda(t, y) = (1, h(t, y)) = (1, u),$$

so that $\rho'(t, y) = \rho''(1, u)$. Hence f is well-defined. Now it is easy to see that f satisfies a) and b). This completes the proof.

The following is a converse of Proposition 5.

THEOREM 3. *Let the circle group* G *act freely and differentiably on a homotopy* n-*sphere* Σ^n, *and let* Σ^{n-2} *be an invariant submanifold of* Σ^n *that is a homotopy* (n - 2)-*sphere. Then there exists a contractible compact differentiable* (n - 1)-*manifold* D *with boundary* Y *and an imbedding* $f : D \to \Sigma^n$ *such that* a) *and* b) *of Proposition* 5 *are satisfied and* $f(Y) = \Sigma^{n-2}$.

Proof. By hypothesis, Σ^n is an integral cohomology n-sphere admitting a free action of the circle group. We infer that n is odd. It is well-known that if $n = 1, 3$, then each free differentiable action of the circle group on a homotopy n-sphere is differentiably equivalent to an orthogonal action. Therefore our conclusion holds for these cases, and we shall now assume that $n > 3$.

There exists an invariant Riemannian metric on Σ^n, and we assume that such a metric has been chosen.

For $y \in \Sigma^{n-2}$, denote by $U(y, \delta)$ the set of points $x \in \Sigma^n$ such that x and y are joined by a geodesic, of length at most δ and orthogonal to Σ^{n-2} at y. Choose a number $\delta > 0$ such that each $U(y, \delta)$ is a closed 2-cell and such that there exists a fibering map

$$\pi : U \to \Sigma^{n-2}$$

with $\pi^{-1}(y) = U(y, \delta)$ for all $y \in \Sigma^{n-2}$. Let B be the boundary of U. Both B and U are invariant, and

$$\pi : B \to \Sigma^{n-2}$$

gives a circle bundle. Whenever A is an invariant subset of Σ^n, let A^* be the orbit space A/G and p the natural projection

$$p : A \to A^*.$$

The fibering $\pi : B \to \Sigma^{n-2}$ is trivial, because all circle bundles over a homotopy n-sphere $(n \geq 3)$ are trivial. Hence B is diffeomorphic to $\Sigma^{n-2} \times G$. Now $p : B \to B^*$ is a principal fiber map in which the fibers are circles.

The remarks following next must be considered in connection with the action of G on Σ^n, since otherwise they would not be true. A basic fact is that an orbit of G in $\Sigma^n - \Sigma^{n-2}$ has linking number ± 1 with Σ^{n-2}. By examining the spectral sequence (or otherwise) we can see that B^* is an integral cohomology $(n-2)$-sphere. The homotopy sequence shows also that B^* is simply connected, so that B^* is a homotopy $(n-2)$-sphere.

There is a cross-section for the fibering $\pi : B \to \Sigma^{n-2}$ given by

$$s : \Sigma^{n-2} \to B.$$

Let

$$r : \Sigma^{n-2} \times R \to B$$

be the covering map such that whenever $(y, t) \in \Sigma^{n-2} \times R$, then $r(y, t) \in \pi^{-1}(y)$ and $r(y, 0) = s(y)$, and such that if $t < t' < t + 1$, the geodesic joining y and $r(y, t)$ and the geodesic joining y and $r(y, t')$ intersect at an angle $2\pi(t' - t)$. For each $g = \exp(2\pi\phi i) \in G$ and each $x = r(y, t) \in B$, let a new action of G be given by

$$g \circ x = r(y, t + \phi).$$

Since the metric is invariant, it follows that

$$g'(g \circ x) = g \circ (g' x)$$

for $g, g' \in G$ and $x \in B$.

We shall now show that there exists a differentiable map

$$f : \Sigma^{n-2} \times R \to R$$

such that

(a) $f(y, 0) = 0$ for all $y \in \Sigma^{n-2}$ and

(b) $r(y, f(y, t)) = (\exp(-2\pi ti))s((\exp 2\pi ti)y)$ for all $(y, t) \in \Sigma^{n-2} \times R$.

Since $r(y, f(y, 1)) = r(y, f(y, 0))$, it will follow that $f(y, 1)$ is an integer d that is independent of y. In order to prove our assertion, we consider the space $(\Sigma^{n-2} \times R) \times R$ and in it the points (y, t, t') such that

$$r(y, t') = (\exp(-2\pi ti))\, s((\exp 2\pi ti)y).$$

Because these points form a covering space of $\Sigma^{n-2} \times R$, the space $\Sigma^{n-2} \times R$ is simply connected; this shows the existence of $f(y, t)$.

We shall now see that $d = \pm 1$. Using $r'(y, t) = r(y, -t)$ for $(y, t) \in \Sigma^{n-2} \times R$, if necessary, we may assume that $d = -1$.

Let $\pi^* : B^* \to \Sigma^{n-2*}$ be induced by π. Since $p\pi = \pi^* p$, it follows that $\pi^* ps = p$, and therefore the diagram

$$\begin{array}{ccc} \Sigma^{n-2} & \underset{s}{\overset{\pi}{\rightleftarrows}} & B \\ \downarrow p & & \downarrow p \\ \Sigma^{n-2*} & \underset{\pi^*}{\longleftarrow} & B^* \end{array}$$

is commutative. When $\exp 2\pi ti \in G$ and $y \in \Sigma^{n-2}$,

$$(ps)(\exp 2\pi ti)y = p((\exp 2\pi ti)r(y, f(y, t)) = p(\exp 2\pi f(y, t)i) \circ (\exp 2\pi ti)s(y) \ .$$

We infer that for $y^* \in \Sigma^{n-2^*}$,

$$ps : p^{-1}(y^*) \to \pi^{*-1}(y^*)$$

is of degree d.

Let c and c' be the fundamental cycles on $p^{-1}(y^*)$ and $\pi^{*-1}(y^*)$. Then there exists a 2-chain b in Σ^{n-2} and a 2-chair b' in B^* such that

$$\partial b = c, \qquad \partial b' = c'.$$

Then pb and $\pi^* b'$ are 2-cycles in Σ^{n-2^*} each of which belongs to a generator of $H_2(\Sigma^{n-2^*})$. By a remark above,

$$\partial psb \sim dc' = d\partial b', \quad \partial(psb - db') = 0, \quad \text{and } psb - db' = \partial A \quad (A \text{ a 3-chain in } B^*).$$

Therefore

$$pb - \pi^* db' = \pi^*(psb - db') = \partial \pi^* A,$$

that is, the 2-cycles pb and $\pi^* db'$ are homologous, and hence $d = \pm 1$. As said earlier, we assume that $d = -1$.

From this assumption, it follows that for $(y, t) \in \Sigma^{n-2} \times R$,

$$f(y, 1+t) = -1 + f(y, t),$$

and $f(y, t) + t$, as a function of t, is a periodic function of period 1. Let

$$t(y) = \int_0^1 (f(y, t) + t)\,dt, \qquad s'(y) = r(y, t(y)).$$

Then $s' : \Sigma^{n-2} \to B$ is a cross-section to $\pi : B \to \Sigma^{n-2}$.

Whenever $g = \exp 2\pi\phi i \in G$ and $(y, t) \in \Sigma^{n-2} \times R$, then

$$\begin{aligned} \exp(2\pi f(y, t+\phi)i) \circ s(y) &= r(y, f(y, t+\phi)) = g^{-1}(\exp(-2\pi ti))s((\exp 2\pi ti)(gy)) \\ &= g^{-1} r(gy, f(gy, t)) = \exp(2\pi f(gy, t)i) \circ g^{-1} s(gy) \\ &= \exp 2\pi(f(gy, t) + f(y, \phi))i \circ s(y) . \end{aligned}$$

Hence

$$f(y, t+\phi) = f(gy, t) + f(y, \phi),$$

so that

$$t(gy) = \int_0^1 (f(gy, t) + t)\,dt = \int_0^1 (f(y, t+\phi) - f(y, \phi) + t)\,dt = t(y) - f(y, \phi) - \phi .$$

It follows that

$$s'(gy) = r(gy, t(gy)) = \exp 2\pi t(gy)i \circ s(gy) = g(\exp 2\pi(t(gy) + f(y, \phi))i \circ s(y))$$
$$= g(\exp 2\pi(t(y) - \phi)i \circ s(y)) = g(g^{-1} \circ s'(y)).$$

Let

$$h : \Sigma^{n-2} \times [0, 1] \to \Sigma^n$$

be the map such that

(a) $h(y, 1) = y$ for $y \in \Sigma^{n-2}$ and

(b) h maps $y \times [0, 1]$ proportionally onto the geodesic joining $s'(y)$ and y for all $y \in \Sigma^{n-2}$.

Then $h(\Sigma^{n-2} \times [0, 1))$ intersects each orbit in $Gh(\Sigma^{n-2} \times [0, 1))$ at exactly one point, so that $h(\Sigma^{n-2} \times [0, 1))$ is a cross-section of

$$p : Gh(\Sigma^{n-2} \times [0, 1)) \to Gh(\Sigma^{n-2} \times [0, 1))^*.$$

As before, it can be shown that

$$E_1 = (\Sigma^n - Gh(\Sigma^{n-2} \times [0, 1]))^*$$

is homologically trivial, and in fact, that E_1 is a homotopy $(n + 1)$-ball.

Let E be obtained from the disjoint union $(\Sigma^n - \Sigma^{n-2})^* + (\Sigma^{n-2} \times [0, 1])$ by identification of each $(y, t) \in \Sigma^{n-2} \times (0, 1)$ with $ph(y, t)$. Then $E_1 \subset E$, and $E - E_1$ is diffeomorphic to $\Sigma^{n-2} \times (0, 1]$. Hence E is a homotopy $(n + 1)$-ball.

The space $(\Sigma^n - \Sigma^{n-2})^*$ is contractible, and therefore there exists a cross-section

$$f' : (\Sigma^n - \Sigma^{n-2})^* \to \Sigma^n - \Sigma^{n-2}$$

of $p : \Sigma^n - \Sigma^{n-2} \to (\Sigma^n - \Sigma^{n-2})^*$. We may assume that $f'(ph(y, t)) = h(y, t)$ for each $(y, t) \in \Sigma^{n-2} \times (0, 1)$. Hence we may combine f' and h to obtain the desired map

$$f : E \to \Sigma^n.$$

This completes the proof of Theorem 3.

THEOREM 4. *Let the circle group* G *act freely and differentiably on a homotopy sphere* Σ^n *such that there exists a sequence of invariant homotopy spheres*

$$\Sigma^n \supset \Sigma^{n-2} \supset \cdots \supset \Sigma^7 \supset \Sigma^5.$$

Then $\Sigma^5, \Sigma^7, \cdots, \Sigma^{n-2}$ *are diffeomorphic to* $S^5, S^7, \cdots, S^{n-2}$, *respectively. Moreover, for any free differentiable action of the circle group on* Σ^5, *there are, up to a differentiable equivalence, only finitely many ways to extend it to such an action on* Σ^n *with* $\Sigma^5, \Sigma^7, \cdots, \Sigma^{n-2}$ *invariant.*

By Theorem 3, Σ^{n-2} bounds an $(n - 1)$-disk D^{n-1} in Σ^n, and hence Σ^{n-2} is diffeomorphic to S^{n-2}. Similarly $\Sigma^{n-4}, \cdots, \Sigma^5$ are diffeomorphic to standard spheres of appropriate dimensions.

The disk D^{n-1} is essentially obtained as follows:

$$D^{n-1} = h(\Sigma^{n-2} \times [0, 1]) \cup_\alpha E_1,$$

where

$$\alpha : \partial E_1 \to h(\Sigma^{n-2} \times \{0\})$$

is a diffeomorphism. Notice that topologically ∂E_1 and $h(\Sigma^{n-2} \times \{0\})$ are (n - 2)-spheres. The union D^{n-1} has exactly one differentiable structure, so that we obtain a unique structure obtained for D^{n-1}. However, the action on Σ^n (as well as the differentiable structure of Σ^n) depends not merely on the differentiable structure of D^{n-1}, but also on α. The action on Σ^n and its structure depends on the space Σ^n/G, which is diffeomorphic to D^{n-1} with the boundary Σ^{n-2} collapsed into its orbit space under the action of G given on Σ^{n-2}. This depends on α, and its variations are given by

$$\Sigma^n/G \# \Sigma^{n-1},$$

where Σ^{n-1} is a homotopy (n - 1)-sphere. Since the number of diffeomorphism classes of homotopy (n - 1)-spheres is finite, the theorem follows.

REFERENCES

1. A. Borel *et al.*, *Seminar on Transformation Groups*, Annals of Mathematics Studies no. 46, 1960. Princeton University Press, Princeton, N. J.
2. G. E. Bredon, *Examples of differentiable group actions*, Topology 3 (1965), 115-122.
3. E. H. Connell, D. Montgomery, and C. T. Yang, *Compact groups in* E^n, Ann. of Math. (2) 80 (1964), 94-103.
4. P. E. Conner and E. E. Floyd, *Differentiable Periodic Maps*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F. Bd. 33. Reihe: Moderne Topologie. New York, Academic Press, 1964.
5. M. A. Kervaire and J.W. Milnor, *Groups of homotopy spheres, I*, Ann. of Math. (2) 77 (1963), 504-537.
6. B. C. Mazur, *A note on some contractible 4-manifolds*, Ann. of Math. (2) 73 (1961), 221-228.
7. D. Montgomery and H. Samelson, *Examples for differentiable group actions on spheres*, Proc. Nat. Acad. Sci. U.S.A. 47 (1961), 1202-1205.
8. D. Montgomery and L. Zippin, *Topological Transformation Groups*, Interscience Tracts in Pure and Applied Mathematics, vol. 1. New York, Interscience Publishers, 1955.
9. S. Smale, *On the structure of manifolds*, Amer. J. Math. 84 (1962), 387-399.

Institute for Advanced Study
Princeton, N. J.

University of Pennsylvania
Philadelphia, Pa.