# PRIMES IN ARITHMETIC PROGRESSIONS

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For any positive integer q and any integer a relatively prime to q, let

$$\psi(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n),$$

where $\Lambda(n)$ is the arithmetical function which is log p if n is a power of a prime p and 0 otherwise. The prime number theorem for arithmetic progressions, in the form which Walfisz deduced from Siegel's result on L-functions, states that if $q \le (\log x)^N$ for each fixed N, then

$$\psi(x; q, a) = \frac{x}{\phi(q)} + O(x e^{-C\sqrt{\log x}}) \tag{1}$$

for some positive constant C depending on N.

In an important recent paper [1], Bombieri has investigated the behaviour of the error term in this theorem as q varies, up to $x^{1/2}(\log x)^{-B}$ for some fixed positive B. He defined

$$E(x, q) = \max_a |\psi(x; q, a) - x/\phi(q)|, \qquad E^*(x, q) = \max_{y \le x} E(y, q),$$

and he proved in his Theorem 4 that for any fixed positive A there exists a positive constant B such that, if $X \le x^{1/2}(\log x)^{-B}$, then

$$\sum_{q \le X} E^*(x, q) \ll \frac{x}{(\log x)^A}.$$

(We use Vinogradov's notation $\ll$ to indicate an inequality with an unspecified constant factor.) Bombieri's proof was based on a general theorem of the 'large sieve' type (his Theorem 3), but it employed also a whole range of methods and techniques from analytic number theory. In particular, the crucial step was the proof of a density theorem on the zeros of L-functions.

The object of the present paper is to establish another result on the average of the error term

$$\psi(x; q, a) - x/\phi(q);$$

this result can be deduced very simply from a general large sieve theorem similar to Bombieri's Theorem 3, which we have proved elsewhere [2, Corollary 2 to Theorem 4].

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THEOREM. *For each fixed positive* A,

$$(2) \qquad \sum_{q \le X} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} (\psi(x; q, a) - x/\phi(q))^2 \ll \frac{x^2}{(\log x)^{A-5}}$$

*provided*

$$X \le x(\log x)^{-A}.$$

It will be seen that the upper bound for q is much larger than in Bombieri's theorem and that it approaches the limit beyond which the error term loses its significance. On the other hand, it does not seem to us that the above theorem implies Bombieri's result.

The theorem can equally well be stated for the error term

$$\pi(x; q, a) - (\mathrm{li}\, x)/\phi(q),$$

and there is no material difference in the proof.

The large sieve result referred to above is as follows.

*For any character* $\chi$ *to the modulus* q, *let*

$$\tau(\chi) = \sum_{m=1}^{q} \chi(m) e(m/q) \qquad [e(\theta) = e^{2\pi i \theta}],$$

*and let the* $a_n$ *be any real or complex numbers. Then, for any* X,

$$\sum_{q \le X} \frac{1}{q} \min\left(1, \frac{x}{2Xq}\right) \sum_{\chi} |\tau(\chi)|^2 \left| \sum_{n \le x} \chi(n) a_n \right|^2 \le 3.1(x+X) \sum_{n \le x} d(n) |a_n|^2 .$$

It is well known that if $\chi$ is a primitive character (mod q), then $|\tau(\chi)|^2 = q$. Hence the preceding result implies that for $X \le x$

$$(3) \qquad \sum_{q \le X} \min\left(1, \frac{x}{Xq}\right) \sum_{\chi}^{*} \left| \sum_{n \le x} \chi(n) a_n \right|^2 \ll x \sum_{n \le x} d(n) |a_n|^2,$$

where $\sum_{\chi}^{*}$ denotes a sum over all the primitive characters to the modulus q.

We now prove an identity that relates the sum on the left of (2), with a slight modification, to a sum involving characters. It is well known that

$$\psi(x; q, a) = \frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a) \sum_{n \le x} \Lambda(n) \chi(n) = \frac{1}{\phi(q)} \psi(x, \chi_0) + \frac{1}{\phi(q)} \sum_{\chi \ne \chi_0} \bar{\chi}(a) \psi(x, \chi),$$

where $\chi_0$ denotes the principal character (mod q) and

$$\psi(x, \chi) = \sum_{n \le x} \chi(n) \Lambda(n).$$

Since

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} \overline{\chi}(a)\chi'(a) = \begin{cases} \phi(q) & \text{if } \chi = \chi', \\ 0 & \text{otherwise,} \end{cases}$$

we deduce that

$$(4) \qquad \sum_{\substack{a=1\\(a,q)=1}}^{q} (\psi(x;\, q,\, a) - \psi(x,\, \chi_0)/\phi(q))^2 = \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\psi(x,\, \chi)|^2 .$$

This is the identity in question.

We have, for $q \leq x$,

$$\psi(x,\, \chi_0) = \sum_{\substack{n \leq x\\(n,q)=1}} \Lambda(n) = \sum_{n \leq x} \Lambda(n) - \sum_{\substack{p^m \leq x\\ p|q}} \log p$$

$$= \psi(x) + O(\log^2 x) = x + O(x e^{-C'\sqrt{\log x}})$$

by the classical prime number theorem, where $C'$ is some positive (absolute) constant. Hence the sum on the left of (2) does not exceed

$$2 \sum_{q \leq X} \sum_{\substack{a=1\\(a,q)=1}}^{q} (\psi(x;\, q,\, a) - \psi(x,\, \chi_0)/\phi(q))^2 + O\left( x^2 e^{-2C'\sqrt{\log x}} \sum_{q \leq X} 1/\phi(q) \right) .$$

The O-term is negligible in comparison with the right-hand side of (2). Thus in view of (4) it suffices to prove that, for $X \leq x(\log x)^{-A}$,

$$(5) \qquad \sum_{q \leq X} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\psi(x,\, \chi)|^2 \ll x^2 (\log x)^{-A+5} .$$

Each character occurring here is induced (to the modulus $q$) by some primitive character $\chi_1$ to a modulus $q_1$ ($q_1 > 1$) that divides $q$, and each primitive character $\chi_1$ induces at most one character $\chi$ for each $q$. We have the inequalities

$$|\psi(x,\, \chi) - \psi(x,\, \chi_1)| \leq \sum_{\substack{n \leq x\\(n,q) > 1}} \Lambda(n) \ll (\log x)^2 .$$

Hence

$$|\psi(x,\, \chi)|^2 \ll |\psi(x,\, \chi_1)|^2 + (\log x)^4 ,$$

and the contribution of the term $(\log x)^4$ to the sum in (5) is negligible. So it suffices to prove that

$$\sum_{q \leq X} \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} |\psi(x,\, \chi_1)|^2 \ll x^2 (\log x)^{-A+5} .$$

For given $q_1$, the sum over $q$ is subject to the conditions $q_1 | q$ and $q \leq X$. Under these conditions, the sum is

$$\sum_q \frac{1}{\phi(q)} = \sum_{r \leq X/q_1} \frac{1}{\phi(q_1 r)} \leq \frac{1}{\phi(q_1)} \sum_{r \leq X/q_1} \frac{1}{\phi(r)} \ll \frac{\log x}{\phi(q_1)}.$$

Hence it suffices to prove that

$$\sum_{1 < q_1 \leq X} \frac{1}{\phi(q_1)} \sum_{\chi_1} |\psi(x, \chi_1)|^2 \ll x^2 (\log x)^{-A+4},$$

where the summation is over primitive characters $\chi_1$ to the modulus $q_1$. Reverting to the symbol $q$, and recalling that

$$\frac{q}{\phi(q)} \ll \log \log X \ll \log x,$$

we see that it will suffice to prove that

$$\sum_{1 < q \leq X} \frac{1}{q} \sum_{\chi}{}^{*} |\psi(x, \chi)|^2 \ll x^2 (\log x)^{-A+3}, \tag{6}$$

for $X \leq x(\log x)^{-A}$, where $\sum_{\chi}^{*}$ denotes a summation over primitive characters to the modulus $q$.

We dissect the sum on the left of (6) into three parts $S_1$, $S_2$, $S_3$, corresponding to the following ranges for $q$:

$$S_1\colon\ 1 < q \leq (\log x)^N,$$

$$S_2\colon\ (\log x)^N < q \leq x^{1/2},$$

$$S_3\colon\ x^{1/2} < q \leq x(\log x)^{-A}.$$

For $S_1$ we use the Siegel-Walfisz theorem [3, Satz 7.2 combined with Satz 8.2] in the form

$$|\psi(x, \chi)| \ll x e^{-C_1 \sqrt{\log x}},$$

where $C_1$ is a positive constant depending on N. This gives

$$S_1 \ll (\log x)^N x^2 e^{-2C_1 \sqrt{\log x}}.$$

For $S_2$ we use (3) with $X = x^{1/2}$, and note that $\min\left(1, \frac{x^{1/2}}{q}\right) = 1$. With $a_n = \Lambda(n)$, (3) gives

$$\sum_{q \leq x^{1/2}} \sum_{\chi}{}^{*} |\psi(x, \chi)|^2 \ll x \sum_{n \leq x} d(n)\, \Lambda^2(n) \ll x^2 (\log x)^3.$$

Hence

$$S_2 \ll (\log x)^{-N} \sum_{q \leq x^{1/2}} \sum_{\chi}{}^{*} |\psi(x, \chi)|^2 \ll x^2 (\log x)^{-N+3}.$$

Finally, for $S_3$, we use (3) with $X = x(\log x)^{-A}$. For $q > x^{1/2}$,

$$\min\left(1, \frac{x}{Xq}\right) = \min\left(1, \frac{(\log x)^A}{q}\right) = \frac{(\log x)^A}{q}.$$

Hence (3) implies that

$$\sum_{x^{1/2} < q \leq x(\log x)^{-A}} \frac{(\log x)^A}{q} \sum_{\chi}{}^{*} |\psi(x, \chi)|^2 \ll x \sum_{n \leq x} d(n)\Lambda^2(n) \ll x^2 (\log x)^3.$$

It follows that $S_3 \ll x^2 (\log x)^{-A+3}$.

Taking $N = A$ and combining the results for $S_1$, $S_2$, $S_3$, we obtain (6), and this establishes the theorem.

*Postscript* (added June 1, 1966). An inspection of recent literature on the large sieve has brought to our attention the fact that M. B. Barban has recorded a result (*Analogues of the divisor problem of Titchmarsh*, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom. 18 (1963) no. 4, 5-13) of the same kind as our main theorem, but slightly weaker in the sense that the exponent of $\log x$ on the right of (2) is smaller than ours by a constant factor.

## REFERENCES

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