QUASI-ISOMORPHISM OF PRIMARY GROUPS

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The primary groups G and H are said to be quasi-isomorphic if there exist a subgroup G^* of G and a subgroup H^* of H such that G/G^* and H/H^* are bounded and $G^* \cong H^*$. Beaumont and Pierce have recently given necessary and sufficient conditions on the Ulm invariants in order that G and H be quasi-isomorphic in case G and H are countable [1] and in case G and H are direct sums of cyclic groups [2]. The conditions of Beaumont and Pierce are (equivalent to) the following:

(i) there exists an integer $k \geq 0$ such that for all integers $n \geq 0$ and $r \geq 0$

$$\sum_{j=n+k}^{n+k+r} \mathscr{F}_G(j) \leq \sum_{j=n}^{n+2k+r} \mathscr{F}_H(j), \qquad \sum_{j=n+k}^{n+k+r} \mathscr{F}_H(j) \leq \sum_{j=n}^{n+2k+r} \mathscr{F}_G(j),$$

where $\mathscr{F}_G$ is the Ulm function defined on G.

(ii) $p^\omega G \cong p^\omega H$.

It was shown in [1] that conditions (i) and (ii) are always necessary in order that G and H be quasi-isomorphic for arbitrary primary groups G and H. The proof of this is indeed simple. However, the proof in [1] of the sufficiency of (i) and (ii) for the countable case is rather laborious, and the proof in [2] of the sufficiency of (i) for direct sums of cyclic groups is based on the countable case. Much simpler proofs are given below; in fact, the results of [1] and [2] are immediate consequences of Lemma 1. Moreover, we are able to extend these results to direct sums of countable groups and beyond.

We write $G \doteq H$ to mean that G and H are quasi-isomorphic.

THEOREM. *Suppose that* G *and* H *are primary groups such that* $G/p^\omega G$ *and* $H/p^\omega H$ *are direct sums of cyclic groups. Then conditions* (i) *and* (ii) *are necessary and sufficient in order that* $G \doteq H$.

As we have mentioned, the necessity of (i) and (ii) is readily established in [1]. Thus we are concerned only with the sufficiency.

LEMMA 1. *Suppose that* G *and* H *are direct sums of cyclic groups and that condition* (i) *holds. If* $\mathscr{F}_G(0) = |G|\,|H|\,\aleph_0 = \mathscr{F}_H(0)$, *then there exists an isomorphism from* $G[p]$ *onto* $H[p]$ *that alters heights (computed in* G *and* H*) no more than* k.

Proof. For each nonnegative integer n, let A_n be a closed initial segment of ordinal numbers such that the segment has cardinality $\mathscr{F}_G(n)$. Similarly, let B_n be a closed initial segment of ordinals such that the segment has cardinality $\mathscr{F}_H(n)$. Define

$$A = [(n, x)\colon 0 \leq n < \omega \text{ and } x \in A_n], \qquad B = [(n, x)\colon 0 \leq n < \omega \text{ and } x \in B_n],$$

and consider the lexicographical order on A and B. Call the first component n of the element (n, x) in A or B the *index* of the element. For each nonnegative

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integer n, let $A^{(n)}$ and $B^{(n)}$, respectively, denote the set of all elements of A and B having index less than or equal to n. Notice that

$$|G| \leq \mathscr{F}_H(0) \quad \text{and} \quad |H| \leq \mathscr{F}_G(0),$$

since $\mathscr{F}_G(0) = |G|\,|H|\,\aleph_0 = \mathscr{F}_H(0)$.

We show that there is an injection of A into B that does not alter indices more than k. Since

$$|A^{(k)}| \leq |A| \leq \mathscr{F}_H(0) = |B_0| = |B^{(0)}|,$$

there is an injection π of $A^{(k)}$ into $B^{(0)}$. Define $\beta_0 = \operatorname{Sup}[\pi(A^{(k)})]$, computed in B. Suppose, for the nonnegative integer m, that π has been extended to an injection of $A^{(k+m)}$ into $B^{(2k+m)}$ in such a way that indices are not altered more than k. Among all such extensions of π choose one, π^*, such that $\beta_1 = \operatorname{Sup}[\pi^*(A^{(k+1)})]$ is as small as possible — if $m \geq 1$. Indeed, among all such extensions of π, choose π^* such that $(\beta_1, \beta_2, \cdots, \beta_m)$ is minimal with respect to the lexicographical order, where $\beta_i = \operatorname{Sup}[\pi^*(A^{(k+i)})]$. For each nonnegative integer j, let δ_j be the first element of B having index at least $j+1$; if $\mathscr{F}_H(j+1) \neq 0$, then $\delta_j = (j+1, 0)$. Since $\pi(A^{(k)}) \subseteq B^{(0)}$, $\beta_0 < \delta_0$. Choose n to be the greatest positive integer not exceeding $m+1$ such that $\beta_{n-1} < \delta_{n-1}$. If $n = m+1$, then $\pi^*(A^{(k+m)}) \subseteq B^{(m)}$. By hypothesis,

$$|A_{k+m+1}| \leq \sum_{m+1}^{2k+m+1} |B_j|;$$

hence π^* can be extended to an injection of $A^{(k+m+1)}$ into B that does not alter indices more than k. Now consider the case $n \leq m$; here $1 \leq n \leq m$. Since $\pi^*(A^{(k+n-1)}) \subseteq B^{(n-1)}$ and since $\pi^*(A^{(k+n)})$ is not contained in $B^{(n)}$, $|A_{k+n}| > |B_n|$. It follows, in turn, that

$$|A_{k+n}| + |A_{k+n+1}| > |B_n| + |B_{n+1}| \quad \text{if } n+1 < m,$$

and inductively, that

$$\sum_{n}^{j} |A_{k+i}| > \sum_{n}^{j} |B_i| \quad \text{for } n \leq j \leq m;$$

otherwise we would obtain a contradiction on the choice of n. Note, for example, that we may assume that all the elements of B having index exactly j are contained in $\pi^*(A^{(k+j)})$ for $n \leq j \leq m$. Now $\sum_n^m |A_{k+i}| > \sum_n^m |B_i|$; but setting $r = m+1-n$, we have the relations

$$\sum_{n}^{m+1} |A_{k+i}| = \sum_{k+n}^{k+n+r} \mathscr{F}_G(i) \leq \sum_{n}^{2k+n+r} \mathscr{F}_H(i) = \sum_{n}^{2k+m+1} |B_i|.$$

Thus again it is obvious that π^* can be extended to an injection of $A^{(k+m+1)}$ into B in such a way that indices are not altered more than k. In order to complete the induction, we choose (in this as well as in the preceding case) an extension having the smallest possible least upper bound in B on its image set. This completes the proof that there is an injection of A into B that alters indices no more than k. Similarly, there is such an injection of B into A. An analysis of the standard proof of the Bernstein theorem yields the proof that there is an equivalence (a one-to-one and onto mapping) between A and B that alters indices no more than k. But such an equivalence corresponds in a natural way to an isomorphism between G[p] and H[p] that alters heights no more than k, and the lemma is proved.

An important consequence of Lemma 1 is the following corollary.

COROLLARY 1 (Beaumont and Pierce [2]). *If the primary groups* G *and* H *are direct sums of cyclic groups satisfying condition* (i), *then* $G \doteq H$.

Proof. There is no loss of generality in assuming that

$$\mathscr{F}_G(0) = |G|\,|H|\,\aleph_0 = \mathscr{F}_H(0),$$

because neither quasi-isomorphism nor condition (i) is affected by addition to G and H of a p-bounded summand. Let π be an isomorphism from G[p] onto H[p] that does not alter heights more than k. Define $S = [(x, \pi(x)): x \in G[p]]$, and let K be a pure subgroup of G + H such that K[p] = S [4]. Then K is a subdirect sum $G^* \dot{+} H^*$ with trivial kernels, where $G^* \subseteq G$ and $H^* \subseteq H$. We know that $G^* \cong H^*$, and it is an elementary exercise to show that $p^k G \subseteq G^*$ and $p^k H \subseteq H^*$, since π alters heights no more than k. Thus $G \doteq H$.

COROLLARY 2. *If the primary groups* G *and* H *are direct sums of countable groups and satisfy conditions* (i) *and* (ii), *then* $G \doteq H$.

Proof. $G/p^\omega G$ and $H/p^\omega H$ are direct sums of cyclic groups satisfying condition (i). Thus $G/p^\omega G \doteq H/p^\omega H$. Let $G^*/p^\omega G \cong H^*/p^\omega H$, where

$$G/G^* \cong (G/p^\omega G)/(G^*/p^\omega G) \quad \text{and} \quad H/H^* \cong (H/p^\omega H)/(H^*/p^\omega H)$$

are bounded. In order to show that $G \doteq H$, it suffices to show that $G^* \cong H^*$. However, $p^\omega G^* = p^\omega G$ and $p^\omega H^* = p^\omega H$. It follows from [6] or [8] that G^* and H^* are direct sums of countable groups, and they have the same Ulm invariants. Thus G^* and H^* are isomorphic ([3], [7]).

Remark. If G and H are countable, then $G^* \cong H^*$ by the classical uniqueness theorem of Ulm.

Proof of the theorem. Suppose that G and H are primary groups such that $G/p^\omega G$ and $H/p^\omega H$ are direct sums of cyclic groups. Hill and Megibben have shown that $G \cong H$ if $G/p^\omega G \cong H/p^\omega H$ and $p^\omega G \cong p^\omega H$ [5]. This result yields the isomorphism $G^* \cong H^*$ (in Corollary 2) under the weaker hypothesis of the theorem.

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