THE FUNDAMENTAL EQUATIONS OF A SUBMERSION

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1. INTRODUCTION

Let M and B be Riemannian manifolds. A *Riemannian submersion* π: M $\rightarrow$ B is a mapping of M onto B satisfying the following axioms, S1 and S2:

S1. π has maximal rank;

that is, each derivative map π_* of π is onto; hence, for each $b \in B$, $\pi^{-1}(b)$ is a submanifold of M of dimension dim M - dim B. We use the language of fiber bundles, although π certainly need not be the projection of a bundle. In particular, the submanifolds $\pi^{-1}(b)$ are called *fibers*, and a vector field on M is *vertical* if it is always tangent to fibers, *horizontal* if always orthogonal to fibers; we use corresponding terminology for individual tangent vectors. The second axiom may now be stated in the following form.

S2. π_* preserves lengths of horizontal vectors.

Submersions occur widely in geometry (for example, as projection mappings of suitable Riemannian coset spaces). In classical geometry, a surface of revolution or a family of (so-called) parallel surfaces in R^3 each leads in an obvious way to a submersion. Further examples are given in Section 5, where in particular we compute (relative to a natural Riemannian structure) the sectional curvature of the frame bundle of a Riemannian manifold.

If we consider a submersion as the generalization of an isometry M $\rightarrow$ B to the case where dim M $\geq$ dim B, then the notion bears comparison with the generalization to dim M $\leq$ dim B, that is, with an isometric immersion. The character of an immersion is described by a single tensor, the second fundamental form. For a submersion we shall define two such tensors, one of which is the second fundamental form of all the fibers. Our purpose is to find the analogues, for a submersion, of the Gauss and Codazzi equations of an immersion, and thus, in particular, to find the relations linking the Riemannian curvatures of M, B, and the fibers $\pi^{-1}(b)$.

Certain aspects of submersions have been investigated, for example, by Hermann [1], and in greater generality ("bundle-like metrics") by Reinhart [4] and Hermann [2]. Our curvature results were suggested by the special case used by Kobayashi [3]. In preparing this paper we have benefited from conversations with A. Gray, who, in particular, suggested using the term "submersion" in this context.

2. THE FUNDAMENTAL TENSORS T AND A

For a submersion π: M $\rightarrow$ B, let $\mathscr{H}$ and $\mathscr{V}$ denote the projections of the tangent spaces of M onto the subspaces of horizontal and vertical vectors, respectively. (The same letters will serve for the horizontal and vertical distributions of Chevalley

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type on M.) The letters U, V, W will always denote vertical vector fields, and X, Y, Z horizontal vector fields.

The second fundamental form of all fibers $\pi^{-1}(b)$ gives rise to a tensor field T on M, defined for arbitrary vector fields E and F by

$$T_E F = \mathscr{H}\nabla_{\mathscr{V}E}(\mathscr{V}F) + \mathscr{V}\nabla_{\mathscr{V}E}(\mathscr{H}F),$$

where ∇ is the covariant derivative of M. (It is easy to check that T is in fact a tensor—of type (1, 2).) We shall make frequent use of the following three properties of T:

1. At each point, T_E is a skew-symmetric linear operator on the tangent space of M, and it reverses the horizontal and vertical subspaces.

2. T is vertical; that is, $T_E = T_{\mathscr{V}E}$.

3. For vertical vector fields, T has the symmetry property $T_V W = T_W V$.

This last fact, well-known for second fundamental forms, follows immediately from the integrability of the vertical distribution.

To define the other tensor A mentioned in the Introduction, we simply dualize the definition of T by reversing $\mathscr{H}$ and $\mathscr{V}$. Thus

$$A_E F = \mathscr{V}\nabla_{\mathscr{H}E}(\mathscr{H}F) + \mathscr{H}\nabla_{\mathscr{H}E}(\mathscr{V}F).$$

Again A is a (1, 2)-tensor, and it has the following properties:

1'. At each point, A_E is a skew-symmetric linear operator on the tangent space of M, and it reverses the horizontal and vertical subspaces.

2'. A is horizontal; that is, $A_E = A_{\mathscr{H}E}$.

3'. For horizontal vector fields, A has the alternation property $A_X Y = -A_Y X$.

(The last property will be established in the proof of Lemma 2.)

In computations with tensor equations, we are free to make special choices of vector fields. For the purpose of relating the geometry of M to that of B, the natural choice is as follows: We define a vector field X on M to be *basic* provided X is horizontal and π-related to a vector field X_* on B. Every vector field X_* on B has a unique horizontal lift X to M, and X is basic. Thus $X \leftrightarrow X_*$ is a one-to-one correspondence between basic vector fields on M and arbitrary vector fields on B. This correspondence preserves brackets, inner products, and covariant derivatives to the following extent:

LEMMA 1. *If* X *and* Y *are basic vector fields on* M, *then*

1. $\langle X, Y\rangle = \langle X_*, Y_*\rangle \circ \pi$,

2. $\mathscr{H}[X, Y]$ *is the basic vector field corresponding to* $[X_*, Y_*]$,

3. $\mathscr{H}\nabla_X Y$ *is the basic vector field corresponding to* $\nabla^*_{X_*}(Y_*)$.

Proof. The first assertion follows from axiom S1, the second from the identity $\pi_*[X, Y] = [X_*, Y_*]$. We shall prove (3), using the standard formula for Riemannian covariant derivative in terms of inner products and brackets. In fact, for a (basic) vector field Z,

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle$$
$$- \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle .$$

But, for example, $X\langle Y, Z\rangle = X\{\langle Y_*, Z_*\rangle \circ \pi\} = X_*\langle Y_*, Z_*\rangle \circ \pi$. With a similar formula for brackets, this shows that the right side of the equation above is just $2\langle \nabla^*_{X_*} Y_*, Z_*\rangle \circ \pi$. It follows that $\nabla_X Y$ is π-related to $\nabla^*_{X_*} Y_*$, hence $\mathscr{H}\nabla_X Y$ is basic and corresponds to this covariant derivative.

We now show that A is essentially the integrability tensor of the horizontal distribution $\mathscr{H}$ on M.

LEMMA 2. *If* X *and* Y *are horizontal vector fields, then* $A_X Y = \frac{1}{2}\mathscr{V}[X, Y]$.

Proof. Since $[X, Y] = \nabla_X Y - \nabla_Y X$, we have the relation $\mathscr{V}[X, Y] = A_X Y - A_Y X$. Thus it suffices to prove the alternation property 3', or, equivalently, to show that $A_X X = 0$. We may assume that X is basic, hence that $0 = V\langle X, X\rangle = 2\langle \nabla_V X, X\rangle$ for any vertical vector field V. But $[V, X] = \nabla_V X - \nabla_X V$ is vertical (since V is π-related to the zero vector field), hence

$$\langle \nabla_V X, X\rangle = \langle \nabla_X V, X\rangle = -\langle V, \nabla_X X\rangle = -\langle V, A_X X\rangle .$$

Since $A_X X$ is vertical, the result follows.

Geometrical features of the fibers will be distinguished by a caret (^). For the covariant derivative, for example, $\hat{\nabla}_V W = \mathscr{V}\nabla_V W$, where V and W are vertical vector fields. We can now summarize the relation of T and A to the various covariant derivatives as follows:

LEMMA 3. *Let* X *and* Y *be horizontal vector fields, and* V *and* W *vertical vector fields. Then*

1. $\nabla_V W = T_V W + \hat{\nabla}_V W$,
2. $\nabla_V X = \mathscr{H}\nabla_V X + T_V X$,
3. $\nabla_X V = A_X V + \mathscr{V}\nabla_X V$,
4. $\nabla_X Y = \mathscr{H}\nabla_X Y + A_X Y$.

Furthermore, if X *is basic,* $\mathscr{H}\nabla_V X = A_X V$.

(The final assertion follows as above from the fact that [V, X] is vertical when X is basic.) In the right-hand members of the four equations, we have written the horizontal terms first; note that only four of the eight terms are tensorial.

3. COVARIANT DERIVATIVES OF T AND A

The covariant derivatives ∇T and ∇A will appear in the fundamental equations derived in the next section. First we want to determine the extent to which these derivatives are "algebraic," that is, depend only on T and A without intervention of other covariant derivatives or of curvature.

LEMMA 4. *If* X *and* Y *are horizontal and* V *and* W *are vertical, then*

$$(\nabla_V A)_W = -A_{T_V W}, \quad (\nabla_X T)_Y = -T_{A_X Y},$$

$$(\nabla_X A)_W = -A_{A_X W}, \qquad (\nabla_V T)_Y = -T_{T_V Y}.$$

Proof. For example, if E is an arbitrary vector field, then

$$(\nabla_V A)_W E = \nabla_V(A_W E) - A_{\nabla_V W}(E) - A_W(\nabla_V E).$$

But since A is horizontal, the two outer terms vanish, and in the remaining term, $\nabla_V W$ can be replaced by $\mathscr{H}\nabla_V W = T_V W$.

Because T_E and A_E are, at each point, skew-symmetric linear operators, it follows that $(\nabla_F T)_E$ and $(\nabla_F A)_E$ are also skew-symmetric and linear. However, the following lemma shows that the latter operators do not, in general, retain the property of reversing horizontal and vertical vectors.

LEMMA 5. *If* X *is a horizontal vector field and* U, V, W *are vertical, then* $\langle (\nabla_U A)_X V, W \rangle = \langle T_U V, A_X W \rangle - \langle T_U W, A_X V \rangle$.

The proof is a slightly more complicated version of the preceding one. As long as the properties 3 and 3' from Section 2 are not used, T and A are dual under reversal of "horizontal" and "vertical." (In particular, the preceding lemma gives rise to three similar formulas.) These alternation and symmetry properties are generally lost under covariant differentiation; the following lemma deals with a useful exceptional case.

LEMMA 6. *If* X *and* Y *are horizontal, and* V *and* W *vertical, then*

1. $\langle (\nabla_E A)_X Y, V \rangle$ *is alternate in* X *and* Y,
2. $\langle (\nabla_E T)_V W, X \rangle$ *is symmetric in* V *and* W.

Proof. Expand the covariant derivatives, and use properties of T and A.

We are now reduced to just four types of covariant derivatives of T and A, namely

$$\langle (\nabla_Z A)_X Y, V \rangle, \quad \langle (\nabla_W A)_X Y, V \rangle, \quad \langle (\nabla_Z T)_V W, X \rangle, \quad \langle (\nabla_W T)_U V, X \rangle,$$

with the usual notation for horizontal and vertical vector fields. As far as I know, none of these is algebraic. For the first type, we have the following identity:

LEMMA 7. *If* V *is vertical, and* $\mathfrak{S}$ *denotes the cyclic sum of over the horizontal vector fields* X, Y, Z, *then*

$$\mathfrak{S}\langle (\nabla_Z A)_X Y, V \rangle = \mathfrak{S}\langle A_X Y, T_V Z \rangle.$$

Proof. Since this is a tensor equation, we can assume that X, Y, Z are basic, and even that all three brackets $[X, Y], \cdots$ are vertical (since this can be achieved at a single point). Thus, by Lemma 2, $\frac{1}{2}[X, Y] = A_X Y$, so that

$$\tfrac{1}{2}\langle [[X, Y], Z], V \rangle = \langle [A_X Y, Z], V \rangle = \langle \nabla_{A_X Y}(Z), V \rangle - \langle \nabla_Z (A_X Y), V \rangle,$$

where V is vertical. But

$$\langle \nabla_{A_X Y}(Z), V \rangle = \langle T_{A_X Y}(Z), V \rangle = -\langle Z, T_{A_X Y}(V) \rangle$$

$$= -\langle Z, T_V(A_X Y) \rangle = \langle T_V Z, A_X Y \rangle .$$

Hence, by the Jacobi identity, $\mathfrak{S}\langle \nabla_Z(A_X Y), V \rangle = \mathfrak{S}\langle T_V Z, A_X Y \rangle$. Thus it remains only to show that $\mathfrak{S}\langle \nabla_Z(A_X Y), V \rangle = \mathfrak{S}\langle (\nabla_Z A)_X Y, V \rangle$. But

$$(*) \qquad \langle \nabla_Z(A_X Y), V \rangle - \langle (\nabla_Z A)_X Y, V \rangle = \langle A_{\nabla_Z X}(Y), V \rangle + \langle A_X(\nabla_Z Y), V \rangle .$$

Basic properties of A show that the first term on the right equals

$$-\langle A_Y(\mathscr{H}\nabla_Z X), V \rangle ,$$

and since we assume that $[X, Z] = 0$, this becomes $-\langle A_Y(\mathscr{H}\nabla_X Z), V \rangle$, from which the projection $\mathscr{H}$ may now be deleted. Thus, when $\mathfrak{S}$ is applied to equation (*), we obtain the required result.

4. FUNDAMENTAL EQUATIONS

For a submersion $\pi: M \to B$ we now derive the equations analogous to the Gauss and Codazzi equations of an immersion. These are found by applying previous results to the second structural equation of M. Five equations result, which we number as $\{n\}$, where $n = 0, 1, 2, 3, 4$ is the number of horizontal vectors in the curvature tensor $\langle R_{E_1 E_2}(E_3), E_4 \rangle$ —the others being vertical. The order in which horizontal and vertical vectors are interspersed is not essential; because of the symmetries of the curvature tensor, the five equations will completely determine the curvature of M.

The first two equations relate the geometry of M to that of the fibers $\pi^{-1}(b)$; they are simply the Gauss and Codazzi equations of the fibers. If v_1, v_2, v_3, v_4 are vertical vectors at a point $m \in M$, let $\langle \hat{R}_{v_1 v_2}(v_3), v_4 \rangle$ be the curvature tensor of the fiber $\pi^{-1}(\pi(m))$ at m.

THEOREM 1. *If* U, V, W, F *are vertical vector fields and* X *is horizontal, then*

$$\{0\} \qquad \langle R_{UV} W, F \rangle = \langle \hat{R}_{UV} W, F \rangle - \langle T_U W, T_V F \rangle + \langle T_V W, T_U F \rangle ,$$

$$\{1\} \qquad \langle R_{UV} W, X \rangle = \langle (\nabla_V T)_U W, X \rangle - \langle (\nabla_U T)_V W, X \rangle .$$

The proof is the same as for a single submanifold.

Now we turn to the pair of equations relating the geometry of M and B; these are in a sense the duals of the Gauss and Codazzi equations above. The horizontal lift of the curvature tensor R* of B will also be denoted by R*; explicitly, if h_1, h_2, h_3, h_4 are horizontal tangent vectors to M, we set

$$\langle R^*_{h_1 h_2}(h_3), h_4 \rangle = \langle R^*_{h_{1*} h_{2*}}(h_{3*}), h_{4*} \rangle ,$$

where $h_{i*} = \pi_*(h_i)$.

THEOREM 2. *If* X, Y, Z, H *are horizontal vector fields and* V *is vertical, then*

$$\{4\} \qquad \langle R_{XY}Z, H\rangle = \langle R^*_{XY}Z, H\rangle - 2\langle A_XY, A_ZH\rangle + \langle A_YZ, A_XH\rangle + \langle A_ZX, A_YH\rangle,$$

$$\{3\} \qquad \langle R_{XY}Z, V\rangle = \langle(\nabla_Z A)_X Y, V\rangle + \langle A_XY, T_VZ\rangle - \langle A_YZ, T_VX\rangle - \langle A_ZX, T_VY\rangle.$$

Proof. Since $\{4\}$ and $\{3\}$ are tensor equations, we can assume that X, Y, Z are basic vector fields whose brackets are vertical. Making repeated use of Lemma 3, we compute each term of

$$(1) \qquad R_{XY}Z = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z.$$

Because of the bracket condition above, Lemma 1 implies that $[X, Y] = 2A_X Y$. Thus we deduce from Lemma 3 that

$$(2) \qquad \nabla_{[X,Y]}Z = 2A_Z(A_XY) + 2T_{A_XY}(Z).$$

In view of Lemma 1, we write the basic vector field $\mathscr{H}\nabla_Y Z$ as $\nabla^*_Y Z$. Thus $\nabla_Y Z = \nabla^*_Y Z + A_Y Z$, and another application of Lemma 3 yields the equation

$$(3) \qquad \nabla_X\nabla_Y Z = \nabla^*_X\nabla^*_Y Z + A_X(\nabla^*_Y Z) + A_XA_Y Z + \mathscr{V}\nabla_X(A_Y Z).$$

Combining the three equations, we obtain the relations

$$(4) \qquad \mathscr{H}R_{XY}Z = -[\nabla^*_X, \nabla^*_Y]Z + 2A_Z A_X Y - A_XA_Y Z + A_YA_X Z,$$

$$(5) \qquad \mathscr{V}R_{XY}Z = 2T_{A_XY}(Z) - \mathscr{V}\nabla_X(A_Y Z) + \mathscr{V}\nabla_Y(A_X Z) - A_X(\nabla^*_Y Z) + A_Y(\nabla^*_X Z).$$

By Lemma 1 and the definition of R^* (preceding this theorem) the second structural equation of B lifts to M in the form

$$R^*_{XY}Z = \nabla^*_{\mathscr{H}[X,Y]}Z - [\nabla^*_X, \nabla^*_Y]Z.$$

We have arranged that $\mathscr{H}[X, Y] = 0$; thus the first term on the right in (4) is simply $R^*_{XY}Z$. Taking inner products, in this equation, with the horizontal vector field H, we obtain the dual Gauss equation $\{4\}$.

Taking inner products in (5) with the vertical vector field V, we find that

$$(6) \qquad \langle R_{XY}Z, V\rangle = 2\langle T_{A_XY}(Z), V\rangle - \langle\nabla_X(A_Y Z), V\rangle + \langle\nabla_Y(A_X Z), V\rangle - \langle A_X(\nabla_Y Z), V\rangle + \langle A_Y(\nabla_X Z), V\rangle.$$

But in the proof of Lemma 7 we saw that

$$(7) \qquad \langle T_{A_XY}(Z), V\rangle = \langle T_V Z, A_X Y\rangle.$$

Furthermore,

$$(8) \qquad \langle\nabla_Y(A_X Z), V\rangle - \langle\nabla_X(A_Y Z), V\rangle = \langle(\nabla_Y A)_X Z, V\rangle - \langle(\nabla_X A)_Y Z, V\rangle + \langle A_X(\nabla_Y Z), V\rangle - \langle A_Y(\nabla_X Z), V\rangle.$$

(Two terms cancel, since [X, Y] is vertical.) Substituting (7) and (8) in (6), we find that

$$\langle R_{XY}Z, V\rangle = \langle(\nabla_Y A)_X Z, V\rangle - \langle(\nabla_X A)_Y Z, V\rangle + 2\langle A_X Y, T_V Z\rangle .$$

In view of Lemmas 6 and 7, this equation is equivalent to the dual Codazzi equation $\{3\}$ as stated above. *A priori* the latter seems preferable, since it contains only a single covariant derivative.

We now state the last of the five equations.

THEOREM 3. *If* X *and* Y *are horizontal vector fields, and* V *and* W *are vertical, then*

$$\{2\} \qquad \begin{aligned}\langle R_{XV}Y, W\rangle = {} & \langle(\nabla_X T)_V W, Y\rangle + \langle(\nabla_V A)_X Y, W\rangle \\ & - \langle T_V X, T_W Y\rangle + \langle A_X V, A_Y W\rangle .\end{aligned}$$

We omit the proof, which follows the same lines as that of the preceding theorem.

The symmetries of curvature lead not only to new formulations of these equations, but also to identities involving the covariant derivatives of T and A. For example, application of $\{2\}$ to the symmetry $\langle R_{XV}Y, W\rangle = \langle R_{YW}X, V\rangle$ gives such an identity; and this, combined with the cyclic symmetry of curvature, leads to an alternative formulation of $\{2\}$:

$$\{2'\} \qquad \begin{aligned}\langle R_{VW}X, Y\rangle = {} & \langle(\nabla_V A)_X Y, W\rangle - \langle(\nabla_W A)_X Y, V\rangle + \langle A_X V, A_Y W\rangle \\ & - \langle A_X W, A_Y V\rangle - \langle T_V X, T_W Y\rangle + \langle T_W X, T_V Y\rangle .\end{aligned}$$

In the case of sectional curvature, these results become quite simple. If a and b are tangent vectors (tacitly assumed to be independent), the tangent plane they span is denoted by P_{ab}.

COROLLARY 1. *Let* π: M $\to$ B *be a submersion, and let* K, K_*, *and* $\hat{K}$ *be the sectional curvatures of* M, B, *and the fibers. If* x *and* y *are horizontal vectors at a point of* M, *and* v *and* w *are vertical, then*

1. $K(P_{vw}) = \hat{K}(P_{vw}) - \dfrac{\langle T_v v, T_w w\rangle - \|T_v w\|^2}{\|v \wedge w\|^2}$,
2. $K(P_{xv})\,\|x\|^2\,\|v\|^2 = \langle(\nabla_x T)_v v, x\rangle + \|A_x v\|^2 - \|T_v x\|^2$,
3. $K(P_{xy}) = K_*(P_{x_* y_*}) - \dfrac{3\,\|A_x y\|^2}{\|x \wedge y\|^2}$, *where* $x_* = \pi_*(x)$.

The first equation here is one formulation of the Gauss equation for the fibers; we shall see in the next section that the third often gives a quite efficient way of computing the sectional curvature of B.

We might just as well have worked with pseudo-Riemannian manifolds; but note that in the Riemannian case, equation (3) shows that submersions are curvature-increasing (more precisely, nondecreasing) on horizontal tangent planes.

5. EXAMPLES

We apply the results above to three familiar situations; for brevity, we compute only sectional curvatures.

Complex projective space. Let N be the outward unit normal on the unit sphere $S^{2n+1} \subset R^{2n+2} = C^{n+1}$. If J is the natural almost complex structure on C^{n+1}, then the integral curves of the tangent vector field JN are great circles in S^{2n+1} that are the fibers of a bundle mapping π: $S^{2n+1} \to CP^n$ onto complex projective space. The usual Riemannian structure on CP^n is characterized by the fact that π is a submersion. Since the fibers are geodesics, the tensor T vanishes. The tensor A can be described as follows: If X and Y are horizontal vector fields on S^{2n+1} (that is, orthogonal to JN), then

$$A_X Y = \langle X, JY \rangle JN, \qquad A_X(JN) = JX.$$

In fact, $A_X(JN) = \mathscr{H}\nabla_X(JN) = \mathscr{H}J\nabla_X N = \mathscr{H}JX = JX$. (The distinction between covariant derivative of S^{2n+1} and of C^{n+1} is irrelevant here.) Hence

$$\langle A_X Y, JN \rangle = -\langle Y, A_X(JN) \rangle = -\langle Y, JX \rangle = \langle X, JY \rangle.$$

If X is a basic vector field on S^{2n+1}, then so is JX. This determines the usual almost complex structure on CP^n. It now follows from equation (3) of Corollary 1 that the sectional curvature of a plane P_{xy} tangent to CP^n is

$$1 + 3\langle x, Jy \rangle^2 / \|x \wedge y\|^2 .$$

A Riemannian homogeneous space G/K. Suppose for simplicity that G is a Lie group furnished with a two-sided invariant Riemannian structure. If K is a closed subgroup, then the usual Riemannian structure on G/K is characterized by the fact that the natural mapping π: G → G/K is a submersion. The fibers (left cosets of G mod K) are totally geodesic; hence T = 0.

Now suppose that X and Y are left-invariant horizontal vector fields on G, in other words, that X and Y are in $\mathfrak{K}^\perp$, where $\mathfrak{K} \subset \mathfrak{G}$ are the Lie algebras of $K \subset G$. By Lemma 1, $A_X Y = \frac{1}{2}\mathscr{V}[X, Y] \in \mathfrak{K}$, and it is known that

$$K(P) = \frac{1}{4}\|[X, Y]\|^2 / \|X \wedge Y\|^2$$

for planes P spanned by X and Y. Then for the planes $P_* = \pi_*(P)$ tangent to G/K, Corollary 1 shows that sectional curvature is given by the well-known formula

$$K_*(P_*)\|X \wedge Y\|^2 = \frac{1}{4}\|[X, Y]\|^2 + \frac{3}{4}\|\mathscr{V}[X, Y]\|^2 = \frac{1}{4}\|\mathscr{H}[X, Y]\|^2 + \|\mathscr{V}[X, Y]\|^2 .$$

The frame bundle F(B) *of a Riemannian manifold.* F(B) is a principal bundle over B with structural group O(n); we identify the elements of the Lie algebra o(n) with skew-symmetric matrices, and use the inner product,

$$\langle a, b \rangle = -\operatorname{trace} ab = \sum_{i,j} a_{ij} b_{ij} .$$

Then there exists a natural Riemannian structure on F(B) such that the projection π: F(B) $\rightarrow$ B is a submersion. To define it, let ω be the Riemannian connection form on F(B) —values in o(n) —and let $\mathscr{H}$ = kernel ω be the Riemannian connection on F(B). If v is a vertical vector and $\omega(h) = 0$, define

$$\|v\| = \|\omega(v)\|, \qquad \langle v, h \rangle = 0, \qquad \|h\| = \|\pi_*(h)\| .$$

Evidently π is then a submersion with $\mathscr{H}$ as its horizontal distribution.

A straightforward computation shows that the fibers are totally geodesic; therefore again T = 0. To compute A, recall that if X and Y are horizontal vector fields on F(B), then (with one set of conventions) $\omega([X, Y]) = -\Omega(X, Y)$, where Ω is the curvature form of B on F(B). Hence, by Lemma 2, if x and y are horizontal vectors, $A_x y$ *is the vertical vector such that* $\omega(A_x y) = -\frac{1}{2}\Omega(x, y)$. If x, y, and a vertical vector v are all tangent to F(B) at $f = (f_1, \cdots, f_n)$, we compute $A_x v$ as follows. First,

$$\langle A_x v, y \rangle = -\langle v, A_x y \rangle = -\langle \omega(v), \omega(A_x y) \rangle = \frac{1}{2}\langle \omega(v), \Omega(x, y) \rangle$$

$$= \frac{1}{2}\sum_{i,j} \omega_{ij}(v)\Omega_{ij}(x, y) = -\frac{1}{2}\sum_{i,j} \omega_{ij}(v) \langle R^*_{f_i f_j}(x_*), y_* \rangle ,$$

where, as usual, $x_* = \pi_*(x)$ and R^* is the curvature tensor of B. Since R^*_{st} is alternate and bilinear in s and t, the curvature operator R_α is well-defined for any 2-vector α in $\wedge^2 B_{\pi(f)}$. The vertical vector v determines such a 2-vector, namely $\alpha(v) = \sum \omega_{ij}(v) f_i \wedge f_j$. Thus the equations above imply that $A_x v$ *is the horizontal vector at* f *such that* $\pi_*(A_x v) = -\frac{1}{2}R^*_{\alpha(v)}(\pi_* x)$.

We now use Corollary 1 to determine the sectional curvature K of the frame bundle F(B). In the notation of that corollary, we find that

$$K(P_{vw}) = \frac{1}{4}\|[\omega(v), \omega(w)]\|^2/\|v \wedge w\|^2 ,$$

$$K(P_{xv}) = \frac{1}{4}\|R^*_{\alpha(v)}(x_*)\|^2/\|x\|^2\,\|v\|^2 ,$$

$$K(P_{xy}) = K_*(P_{x_* y_*}) - \frac{3}{4}\|R^*_{x_* y_*}\|^2/\|x \wedge y\|^2 .$$

In the first of these, we are using the curvature formula referred to in the preceding example, since O(n) is isometric to each (totally geodesic) fiber. In the last formula, the norm of the curvature operator is that of its matrix $-\Omega(x, y)$.

6. INFLUENCE OF THE FUNDAMENTAL TENSORS

We shall now prove two results; the first shows that the tensors T and A control a submersion π in the sense in which the second fundamental form determines an immersion.

THEOREM 4. *Let* π *and* $\bar{\pi}$ *be submersions of a connected Riemannian manifold* M *onto* B. *If* π *and* $\bar{\pi}$ *have the same tensors* T *and* A, *and if their derivative maps agree at one point of* M, *then* $\pi = \bar{\pi}$.

Proof. Let D be the set of points m in M for which $\pi_{*m} = \bar{\pi}_{*m}$ (hence also $\pi(m) = \bar{\pi}(m)$). Evidently, D is a closed set in M, and since M is connected, it suffices to prove that if p is in D, then $\pi = \bar{\pi}$ on some neighborhood of p in M. We shall carry out this proof by showing that π is completely determined on such a neighborhood by T, A, and π_{*p}. Note that the latter gives the horizontal and vertical subspaces $\mathscr{H}_p$ and $\mathscr{V}_p$ of M_p.

A curve α in a fiber of π: M $\to$ B is a geodesic of the fiber if and only if $\alpha'' = T_{\alpha'}(\alpha')$, where α' is the velocity of α, and where α'' is its M-acceleration. Conversely, consider the set of all curves α in M that solve the initial value problem

$$\alpha'' = T_{\alpha'}(\alpha'), \qquad \alpha'(0) \in \mathscr{V}_p.$$

For each prescribed value of $\alpha'(0)$, the solution is unique; thus these curves are the geodesics in $\pi^{-1}(\pi(p))$ starting at p. Consequently, their images fill a convex neighborhood F of p in this fiber.

Next let X be a vector field defined only on F, but with values tangent to M. If X is basic, then by Lemma 3,

$$\nabla_V X = A_X V + T_V X, \qquad X(p) \in \mathscr{H}_p$$

for each vertical vector field V tangent to F. Conversely, the vector fields X satisfying this condition are exactly the basic vector fields on F; this follows if we apply Lemma 3 to show that $V\langle X, X\rangle = V\langle X, Y\rangle = 0$ for each basic vector field Y.

Now let N(F) be the normal bundle of $F \subset M$, and let E: N(F) $\to$ M be the exponential map. Since E is regular at the zero vector 0_p, the image of E is a neighborhood $\tilde{F}$ of p in M. To show that T, A, and π_{*p} completely determine π on this neighborhood, we suppose that m is a point of $\tilde{F}$, so that it may be written as $m = \gamma_x(1)$, where x is normal to F (hence horizontal) and γ_x is the geodesic with initial velocity x. The arguments above have shown that T, A, and π_{*p} determine F and the basic vector fields on F; but if X is the basic extension of x, then $\pi_*(x) = \pi_*(X(p))$. Finally, we use a fact established in [1] and [2]: if β is a geodesic of M such that β' is horizontal at one point, then β' is everywhere horizontal and $\pi \circ \beta$ is a geodesic of B. It follows that

$$\pi(m) = \pi_*(\gamma_x(1)) = \gamma_{\pi_*(x)}(1) = \exp_{\pi(p)}(\pi_*(X(p))),$$

and the proof is complete.

The simplest type of submersion is the projection of a Riemannian product manifold on one of its factors. We shall say that a submersion π: M $\to$ B is *trivial* provided it differs from such a projection only by an isometry of M. Equivalently, π: M $\to$ B is trivial provided there exist a Riemannian manifold F and a submersion ϕ: M $\to$ F *dual to* π in the sense that the horizontal distribution of ϕ is the vertical distribution of π (hence *vice versa*). Our goal now is a necessary and sufficient condition for a submersion to be trivial. Vanishing of both T and A is certainly necessary; however, this local condition is not sufficient, as one can see, for example, from the projection of a flat Möbius band onto its central circle. To find a suitable global condition, we recall some results due to Hermann [1]. If π: M $\rightarrow$ B is a submersion and M is complete, then B and the fibers are also complete. In this case, if γ: [a, b] $\rightarrow$ B is a geodesic segment (possibly broken) in B, then for each point m lying over γ(a) there exists a unique horizontal lift γ^m: [a, b] $\rightarrow$ M of γ such that $\gamma^m(a) = m$—and γ^m is also geodesic. The function

$$\phi_\gamma \colon \pi^{-1}(\gamma(a)) \rightarrow \pi^{-1}(\gamma(b))$$

such that $\phi_\gamma(m) = \gamma^m(b)$ is a diffeomorphism. Furthermore, *if* $T = 0$, *then* ϕ_γ *is an isometry.* Now fix a base point o in B. The set of all the diffeomorphisms ϕ_γ: $\pi^{-1}(o) \rightarrow \pi^{-1}(o)$, where γ is a geodesic loop at o, forms a group G that we call *the group of the submersion.*

THEOREM 5. *Let* π: M $\rightarrow$ B *be a submersion of a complete Riemannian manifold* M. *Then* π *is trivial if and only if the tensor* T *and the group* G *of the submersion both vanish.*

Proof. The necessity of the conditions is obvious; to prove the sufficiency, let $F = \pi^{-1}(o)$ ($o \in B$). By the definition of the diffeomorphisms ϕ_γ, the vanishing of G means that for each $m \in M$, every horizontal geodesic from m to F ends at the same point of F, say $\phi(m)$. It is easy to verify that ϕ is differentiable; we assert that ϕ: M $\rightarrow$ F is a submersion dual to π. In fact, this is a consequence of the following two remarks:

(1) The restriction of ϕ to any fiber $\pi^{-1}(b)$ is a mapping ϕ_γ of the type defined above; furthermore, since $T = 0$, this restriction is an isometry onto F.

(2) If h is a π-horizontal vector, then by an earlier remark the geodesic γ_h with initial velocity h remains horizontal. Thus from the definition of ϕ we can deduce that the curve $\phi \circ \gamma_h$ is constant, hence $\phi_*(h) = 0$.

REFERENCES

1. R. Hermann, *A sufficient condition that a mapping of Riemannian manifolds be a fiber bundle,* Proc. Amer. Math. Soc. 11 (1960), 236-242.

2. ———, *On the differential geometry of foliations*, Ann. of Math. (2) 72 (1960), 445-457.

3. S. Kobayashi, *Topology of positively pinched Kaehler manifolds*, Tôhoku Math. J. (2) 15 (1963), 121-139.

4. B. L. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. (2) 69 (1959), 119-132.

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