A MILDLY WILD TWO-CELL

Ralph Tindell

1. INTRODUCTION

The results in this paper grew from an attempt to answer the following question of R. H. Fox: Does there exist in 3-space or in 4-space a wild 2-cell with an interior point p such that every 2-cell subset that has p on its boundary is tame? [5, Problem 21.] Doyle [4] has shown that no such cell exists in 3-space. In Section 5, we give an affirmative answer for 4-space, along with a discussion of mildly wild n-cells in (n + 2)-space. (An n-cell C^n in E^{n+2} is said to be *mildly wild* if it is wild and one of its interior points p has the property that each n-cell subset of C^n having p on its boundary is tame.) In Section 3 we prove a general theorem on ε-taming. In Section 4 we prove an ε-taming theorem about almost piecewise linear imbeddings; it is the main tool in the construction of the mildly wild 2-cell; we also show, in Section 4, that each almost polyhedral 2-sphere in 4-space is the union of two flat cells.

2. DEFINITIONS

We assume familiarity with the material contained in Chapters 1 and 3 of [18], and we adhere to the notation given there. By a *simplex* we mean a closed rectilinear simplex, and by a *complex* we mean a closed rectilinear simplical complex (which may be assumed to be a subcomplex of a rectilinear division of some Euclidean space E^q). $K \downarrow L$ means that K *collapses* to L (see Chapter 3 of [7]). We shall abbreviate "piecewise linear" (or piecewise linearly) to pwl. If M is a manifold, we shall denote its *interior* by int M and its *boundary* by ∂M; we shall write $\mathrm{Int}_X A$ for the interior of A as a subset of the topological space X, and $\bar{A}$ for the closure of A.

If a space C is homeomorphic (respectively, pwl homeomorphic) to a k-simplex, we say that C is a k-*cell* (respectively, k-*ball*). An n, m *cell pair* (n, m *ball pair*) is a pair (C^n, C^m) of cells (balls) with $C^m \subset C^n$ and $C^m \cap \partial C^n = \partial C^m$; an n, m *semi-cell pair* (n, m *semi-ball pair*) is a pair (C^n, C^m) of cells (balls) with $C^m \subset C^n$ and such that

$$C^m \cap \partial C^n = \partial C^m \cap \partial C^n = C^{m-1}$$

is an (m - 1)-cell (an (m - 1)-ball). The *standard* n, m *ball pair* (Σ, τ) and the *standard* n, m *semi-ball pair* (Σ, σ) are defined as follows: let σ' be an (m - 1)-simplex in E^{m-1}, and let $u = (0, \cdots, 0, -1)$ and $v = (0, \cdots, 0, 1)$ belong to E^m; then σ is the m-simplex $u * \sigma$,

$$\tau = \sigma \cup (v * \sigma'),$$

Received May 12, 1966.

The author was supported by the National Science Foundation Grant GP 4006; the contents of this paper form a part of the author's doctoral thesis written at Florida State University under the direction of Professor James J. Andrews.

and Σ is the (n - m)-fold suspension of τ, that is, the join $\tau * S^{n-m}$ of τ with the (n - m)-sphere. The *standard sphere pair* is the pair $(\partial\Sigma, \partial\tau)$. A cell pair (ball pair) $C^{n,m}$ is said to be *flat (unknotted)* if there exists a homeomorphism (pwl homeomorphism) g of C^n onto Σ such that $g(C^m) = \tau$; we similarly define flat (unknotted) semi-pairs. A sphere pair $S^{n,m}$ is said to be *flat (unknotted)* if it is homeomorphic (pwl homeomorphic) to the standard pair $(\partial\Sigma, \partial\tau)$. If B^m is an m-ball in an n-sphere S^n, the pair (S^n, B^m) is said to be *unknotted* if it is pwl homeomorphic to $(\partial\Sigma, \partial\Sigma \cap \partial\sigma)$. A ball pair $C^{n,m}$ is said to be *locally unknotted* if for some triangulation J, L of C^n, C^m and each vertex a of L the pair (lk(a, J), lk(a, L)) is unknotted. (This will then be true for each triangulation of (C^n, C^m).)

Throughout the remainder of this paper, M will denote a combinatorial m-manifold with compact boundary, N will denote a combinatorial n-manifold, and f an imbedding (not necessarily pwl) of M into int N. f is said to be *locally* pwl at $p \in M$ if for some subpolyhedron P of M with $p \in \mathrm{Int}_M P$, the restriction $f \mid P$ of f to P is pwl. The imbedding (locally pwl imbedding) f is said to be *locally flat (locally unknotted)* at $p \in M$ if there is a closed neighborhood U of f(p) in N such that $(U, U \cap f(M))$ is a flat n, m cell pair (unknotted n, m ball pair) when $p \in$ int M, or a flat n, m semi-cell pair (unknotted n, m semi-ball pair) when $p \in \partial M$. Such a neighborhood U is called a *canonical neighborhood,* and an unknotting homeomorphism of $(U, U \cap f(M))$ is called a *canonical homeomorphism.*

A natural question arises when f is pwl (and n - m = 2): are the notions of local flatness and local unknottedness equivalent? In [15], we gave a partial answer to this question, and in particular we showed that the answer is affirmative when m = 2, n = 4.

Let A be a closed subset of N, and ε a positive number. An ε*-push of* N, A is an isotopy $\{h_t\}$ of N onto itself such that h_0 is the identity, $d(x, h_t(x)) < \varepsilon$ for all $x \in N$ and $t \in [0, 1]$, and $h_t(x) = x$ whenever $d(x, A) \geq \varepsilon$ (d denotes distance). A pwl ε*-push* of N, A is an ε-push that is a pwl isotopy (in the sense of [8]). We shall call a homeomorphism of N onto itself an ε-push if it is the final stage of an ε-push as defined above. Our definition of an ε-push is the same as that given originally by Gluck [7].

We shall have occasion to use relative regular neighborhoods as defined by Hudson and Zeeman [9]: If X, Y are polyhedra in a combinatorial manifold M^m, a polyhedron Z in M^m is said to be a *regular neighborhood of* X mod Y in M if

1. Z is an m-manifold,

2. $Z \downarrow \overline{X - Y}$,

3. $X - Y \subset \mathrm{Int}_M Z$ and $Z \cap Y = \partial Z \cap Y = \overline{X - Y} \cap Y$.

In [9], it is proved that such a relative regular neighborhood exists if X is "link collapsible" on Y, a condition that is satisfied if X is a manifold and $X \cap Y = \partial X \cap Y$. It is also shown in [9] that if $X \cap \partial M$ is link collapsible on $Y \cap \partial M$, then $\overline{Z \cap \partial M - Y}$ is a regular neighborhood of $X \cap \partial M$ mod $Y \cap \partial M$ in ∂M. A detailed discussion of these topics is given in [9], though the reader is cautioned that the uniqueness theorems given there are false (see [14]; for a corrected version of the uniqueness theorems see [10]).

3. AN ε-TAMING THEOREM

THEOREM 3.1. *Let* $f: M^m \to \text{int } N^n$ *be an imbedding that is locally* pwl *on* int M *and locally flat on* ∂M^n. *Then for each* $\varepsilon > 0$, *there exists an* ε*-push* h *of* N, $f(\partial M)$ *such that* hf *is* pwl *and* $hf(M) \subset f(M)$.

Proof. Since f is locally flat at each point $p \in \partial M$, there exist a canonical neighborhood U_p of f(p) of diameter less than ε and a canonical homeomorphism g_p of $(U_p, U_p \cap f(M))$ onto the standard n, m semi-ball pair (Σ, σ). Let $Q_p = \text{st}(p, J_p^{(r)})$, where J_p is a subdivision of the original triangulation J of M and contains p as a vertex, and where $J_p^{(r)}$ is a barycentric r^{th} derived J of mesh small enough so that $Q_p \subset \text{Int}_M(f^{-1}(U_p))$. Then $g_p f$ maps Q_i into int Σ and onto a subcell of σ; moreover,

$$g_p f(Q_p \cap \partial M) = g_p f(Q_p) \cap \sigma'$$

is an (n - 1)-cell in the interior of σ'. Recall that Σ is the (n - m)-fold suspension of $\sigma \cup (\sigma' * v)$; let $\Sigma' \subset \Sigma$ be the (n - m)-fold suspension of

$$g_p f(Q_p) \cup (g_p f(Q_p) \cap \sigma') * v,$$

using the same suspension points as in Σ. Clearly, $(\Sigma', g_p f(Q_p))$ is homeomorphic to (Σ, σ); thus $V_p = g_p^{-1}(\Sigma')$ is a canonical neighborhood of f(p), and since $V_p \subset U_p$, V_p has diameter less than ε. From the construction, we see that $V_p \cap f(M) = f(Q_p)$. We now define Q_p' as $\text{st}(p, J_p^{(r+1)}) \subset \text{int}_M Q_p$; the sets $\text{int}_M Q_p'$ cover the compact set ∂M, so that there is a finite subcover $Q_{p_1}', \cdots, Q_{p_k}'$. For simplicity, we write Q_i' and V_i for Q_{p_i}' and V_{p_i}.

Let η_i be a pwl homeomorphism of $(Q_i, Q_i \cap \partial M)$ onto (σ, σ'), and let g_i be a homeomorphism of V_i onto Σ that extends

$$\eta_i f^{-1}: f(Q_i) = V_i \cap f(M) \to \sigma.$$

Since η_i^{-1} is uniformly continuous, there exists a $\delta_i' > 0$ such that if h is a δ_i'-push of Σ, σ', then

$$(*) \qquad d(x, \lambda_i(x)) = d(x, \eta_i^{-1} h' \eta_i(x)) = d(\eta_i^{-1}\eta_i(x), \eta_i^{-1} h' \eta_i(x)) < q/k,$$

where $q = \min\{d(Q_i', \overline{M - Q_i})\}$. Since g_i^{-1} is uniformly continuous, there exists a $\delta_i'' > 0$ such that if h' is a δ_i''-push of Σ, σ', then

$$(**) \qquad h_i = g_i^{-1} h' g_i \text{ is an } \varepsilon/k\text{-push of } V_i, V_i \cap f(M).$$

Let $\delta = \min\{\delta_i', \delta_i''\}$, and let h' be a pwl δ-push of Σ, σ' such that

(1) $h'(x) = x$ for all $x \in \partial\Sigma$,

(2) $h'(\sigma) \subset \sigma$, and

(3) $h'(\text{int } \sigma') \subset \text{int } \sigma$;

such a push is easily constructed.

Define $h_i: N \to N$ by

$$h_i(x) = \begin{cases} g_i^{-1} h' g_i(x) & (x \in V_i), \\ x & (x \in \overline{N - V_i}); \end{cases}$$

since by (**) each h_i is an ε/k-push of N, f(∂M), $h = h_k \cdots h_1$ is an ε-push of N, f(∂M). To see that h satisfies the conclusions of the theorem, we define the pwl homeomorphism λ_i of M onto itself by

$$\lambda_i(x) = \begin{cases} \eta_i^{-1} h' \eta_i(x) & (x \in Q_i), \\ x & (x \in \overline{M - Q_i}), \end{cases}$$

and we let λ be the pwl homeomorphism $\lambda_k \cdots \lambda_1$ mapping M into itself. Suppose $x \in \text{int}\, M$; then by invariance of domain, $\lambda(x) \in \text{int}\, M$. If $x \in \partial M$, then $x \in Q_i'$ for some i, since the Q_i' cover M. By (*),

$$d(x, \lambda_{i-1} \cdots \lambda_1(x)) < (i-1)q/k < q \leq d(Q_i', \overline{M - Q_i});$$

hence $\lambda_{i-1} \cdots \lambda_1(x) \in \text{int}_M\, Q_i$; but properties (2) and (3) of h' imply that $\lambda_i(\text{int}_M\, Q_i) \subset \text{int}\, M$ and thus $\lambda_i(\lambda_{i-1} \cdots \lambda_1(x)) \in \text{int}\, M$, so that

$$\lambda(x) = \lambda_k \cdots \lambda_{i+1}(\lambda_i \cdots \lambda_1(x))$$

is in int M. We have thus shown that $\lambda(M) \subset \text{int}\, M$.

Suppose $y \in f(M)$; then

$$\begin{aligned} h(y) &= h_m \cdots h_1(y) = (g_m^{-1} h' g_m)(g_{m-1}^{-1} h' g_{m-1}) \cdots (g_1^{-1} h' g_1)(y) \\ &= (f\eta_m^{-1} h' \eta_m f^{-1})(f\eta_{m-1}^{-1} h' \eta_{m-1} f^{-1}) \cdots (f\eta_1^{-1} h' \eta_1 f^{-1})(y) \\ &= f\lambda_m \lambda_{m-1} \cdots \lambda_1 f^{-1}(y) = f\lambda f^{-1}(y). \end{aligned}$$

Hence $hf(x) = f\lambda(x)$ for all $x \in M$. But $\lambda(M) \subset \text{int}\, M$ and f is pwl on subsets of int M, so that $f\lambda = hf$ is pwl; since $\lambda(M) \subset M$, $f\lambda(M) \subset f(M)$. The proof of the theorem is thus complete.

COROLLARY 3.2. *Let* $p \in \partial M$, *let the imbedding* $f: M^m \to \text{int}\, N^n$ *be locally flat at* p, *and suppose that for some neighborhood* V *of* p *in* M, f *is locally* pwl *at each point of* V - ∂M. *Then for each* $\varepsilon > 0$, *there exists an* ε-*push* h *of* N, f(p) *such that* hf *is locally* pwl *at* p *and* $hf(M) \subset f(M)$.

4. ALMOST PIECEWISE LINEAR IMBEDDINGS

LEMMA 4.1. *Let* B^n *be a locally unknotted ball in the manifold* M^{n+2}, *and let* $B^n \cap \partial M = \partial B^n \cap \partial M$ *be either empty or an* (n - 1)-*ball* B^{n-1}. *Let* N^{n+2} *be a second derived neighborhood of* B^n mod $\overline{\partial B^n - \partial M}$ *in* M. *Then* (N^{n+2}, B^n) *is an unknotted ball pair and* $N^{n+2} \cap \partial M$ *is either empty* (*when* $B^n \cap \partial M = \emptyset$), *or else it is an* (n + 1)-*ball* N^{n+1} *with* (N^{n+1}, B^{n-1}) *an unknotted ball pair.*

A direct proof of this lemma is given in [16]; it is straightforward, though tedious. A variation of Lemma 4.1 is given in [10].

THEOREM 4.2. *Let* f *be an imbedding into* E^n $(n \geq 4)$ *of* $\triangle^m$ *that is locally* pwl *(and locally unknotted) except at a single boundary point. Then* f *is a flat imbedding.*

Proof. Let $(B^{n-1}, \triangle^{m-1})$ be the standard $n-1$, $m-1$ ball pair in E^{n-1}, with $\triangle^{m-1}$ a simplex, and let $a = (a_1, \cdots, a_n)$ be the point in E^n such that $a_i = 0$ $(i = 1, 2, \cdots, n-1)$ and $a_n = 1$. Let

$$B = a * B^{m-1} \quad \text{and} \quad \triangle = a * \triangle^{m-1}.$$

Then define $\triangle_i$ (respectively, B_i) to be the set of points $(x_1, \cdots, x_n)$ of $\triangle$ (respectively, of B) satisfying the condition $(i-1)/i \leq x_n \leq i/(i+1)$. Then $(B, \triangle)$ is an unknotted n, m ball pair, and $(B - a, \triangle - a)$ is the union of the unknotted ball pairs $(B_i, \triangle_i)$ $(i = 1, 2, \cdots)$. Also, $B_i \cap B_j = \emptyset$ if $j \neq i-1, i, i+1$, and

$$(B_i, \triangle_i) \cap (B_{i+1}, \triangle_{i+1}) = (\partial B_i \cap \partial B_{i+1}, \partial\triangle_i \cap \partial\triangle_{i+1}) = (A_i, \alpha_i)$$

is an unknotted $n-1$, $m-1$ ball pair.

We may assume that f maps from $\triangle$ into E^n and that it is locally pwl (and locally unknotted) off the vertex a.

CLAIM. *There exist* n*-balls* N_i $(i = 1, 2, \cdots)$ *such that*

1. $N_i \cap f(\triangle) = f(\triangle_i)$ *and* $(N_i, f(\triangle_i))$ *is an unknotted ball pair,*
2. $(N_i, f(\triangle_i)) \cap (N_{i+1}, f(\triangle_{i+1})) = (\partial N_i \cap \partial N_{i+1}, \partial f(\triangle_i) \cap \partial f(\triangle_{i+1})) = (A_i^*, f(\alpha_i))$,
3. $N_i \cap N_j = \emptyset$ *if* $j \neq i-1, i, i+1$,
4. $\lim_{j\to\infty}(\text{diam } N_j) = 0$.

Proof of the claim. Let $\triangle_j'$ be the m-ball $\bigcup_{i=1}^{j} \triangle_i$; by hypothesis, $f \mid \triangle_j'$ is pwl and locally unknotted; therefore, by Theorem 8 of [8], $f \mid \triangle_j'$ may be extended to a pwl homeomorphism F_{j-1} of E^n onto itself; moreover, by Theorem 4 of [8], we may assume that F_{j-1} is the identity outside some compact set and is thus uniformly continuous. Note that we have arranged our subscripts so that $F_j = f$ on $\triangle_{j+1}'$.

Let M_1 be a second derived neighborhood of $\triangle_1 \bmod \partial\triangle_1$ in E^n. Then $(M_1, \triangle_1)$ is an unknotted ball pair by a variation of Lemma 4.1; moreover, we may assume that $M_1 \cap \triangle_2 = \triangle_1 \cap \triangle_2 = \alpha_1$, and since $\triangle_1$ and $F_1^{-1}f(\overline{\triangle - \triangle_1})$ are disjoint closed sets and F_1 is uniformly continuous, we may assume that

$$M_1 \cap F_1^{-1}f(\overline{\triangle - \triangle_1}) = \emptyset \quad \text{and} \quad \text{diam } F_1(M_1) < \text{diam } f(\triangle_1) + 1.$$

(We need merely take M_1 as a second derived neighborhood in a subdivision of sufficiently fine mesh.) Let $N_1 = F_1(M_1)$; then, by construction,

$$N_1 \cap f(\triangle) = f(\triangle_1) = F_1(\triangle_1),$$

and the conditions of the claim are met for $k \leq 1$.

Suppose we have constructed $N_1, \cdots, N_k$ satisfying the conditions of the claim, condition 4 being replaced by

4'. diam $N_i \leq$ diam $f(\triangle_i) + 1/i$.

We define $N_{k+1} = F_{k+1}(M_{k+1})$, where M_{k+1} is a second derived neighborhood of $\triangle_{k+1} \bmod \overline{\partial\triangle_{k+1} - \partial M}$ in $\overline{E^n - M}$. Here

$$M = F_{k+1}^{-1}(N_k') = F_{k+1}^{-1}\left(\bigcup_{j=1}^{k} N_j\right).$$

Inductively we can show that $N_k' = \bigcup_{j=1}^{k} N_j$ is an n-ball, so that $\overline{E^n - M}$ is a manifold. By our induction hypothesis,

$$\Delta_{k+1} \cap \partial(\overline{E^n - M}) = F_k^{-1}(f(\alpha_k)) = \alpha_k,$$

(α_k is an $(m - 1)$-ball). By Lemma 4.1, (M_{k+1}, Δ_{k+1}) is an unknotted ball pair intersecting M in an unknotted face $(\partial M_{k+1} \cap \partial(\overline{E^n - M}), \alpha_k)$. Moreover, as in the case $k = 1$, we may assume that

(i) $\operatorname{diam} F_{k+1}(M_{k+1}) < \operatorname{diam} f(\Delta_{k+1}) + 1/(k + 1)$ and

(ii) $M_k \cap F_{k+1}^{-1} f(\Delta) = \Delta_{k+1}$.

Since $N_{k+1} = F_{k+1}(M_{k+1})$, we have satisfied the conclusion of the claim. (Note that $\lim_{j\to\infty} (\operatorname{diam} N_j) = \lim_{j\to\infty} (\operatorname{diam} f(\Delta_j) + 1/j) = \lim_{j\to\infty} (\operatorname{diam} f(\Delta_j)) = 0$.)

Proof of the theorem. Since the $n - 1$, $m - 1$ ball pairs (A_i, α_i) and $(A_i^*, f(\alpha_i))$ are unknotted, $f \mid \alpha_i$ may be extended to a homeomorphism h_i^* of A_i onto A_i^* (see Chapter 4 of [18]); then we may extend h_{i-1}^* and h_i^* over the annulus pair

$$(\overline{\partial B_i - A_{i-1} - A_i}, \overline{\partial \Delta_i - \alpha_{i-1} - \alpha_i})$$

to a homeomorphism h_i' of ∂B_i onto ∂N_i such that $h_i' \mid \partial\Delta_i = f \mid \partial\Delta_i$ (see [16]). We may then extend h_i' to a homeomorphism h_i of B_i onto N_i that agrees with f on Δ_i.

The conditions of the claim imply that

$$N = \left(\bigcup_{j=1}^{\infty} N_j\right) \cup f(a)$$

is a cell and that $(N, f(\Delta))$ is a cell pair; it is actually a flat cell pair, for the homeomorphism h defined by

$$h(x) = \begin{cases} h_i(x) & \text{if } x \in B_i, \\ f(a) & \text{if } x = a \end{cases}$$

carries the flat pair (B, Δ) onto $(N, f(\Delta))$. Moreover, N is locally polyhedral except at $f(a)$, and hence it is locally flat except possibly at a. Since, by Cantrell's theorem [1], N is flat, the homeomorphism $h: B \to N$ may be extended to all of E^n; since $h^{-1} f(\Delta) = \Delta$, our proof implies that $f(\Delta)$ is flat, and the proof is complete.

We note that if $n - m \geq 3$, the hypothesis of local unknottedness in Theorem 4.2 is superfluous, by [17], that it may also be removed in the case $n - m = 1$, and that in the special case $n = 4$, the condition $m = 2$ may be replaced by local flatness, by [15]. Finally, we remark that one can choose the flattening homeomorphism h to be locally pwl off a. These comments also apply to the next theorem.

THEOREM 4.3. *Let* $f: M^m \to \operatorname{int} N^n$ $(n \geq 4)$ *be an imbedding that is locally* pwl *(and locally unknotted) except at a countable subset* S *of* ∂M. *Then for each* $\varepsilon > 0$, *there exists an* ε*-push* h *of* N_1, f(S) *such that* $hf(M) \subset f(M)$ *and* hf *is* pwl *and locally unknotted.*

This theorem follows from Corollary 3.2 and Theorem 4.2 by the methods of [2].

We remark that Černavskiĭ [3] and Lacher [11] have independently proved a generalized form of Theorem 4.3. They showed that an imbedding cannot fail to be locally flat at a countable subset of ∂M; however, they do not obtain an ε-taming theorem, except when $m < 2(n - 1)/3$ [3].

Also, Charles Seebeck [12] has recently extended Theorem 4.3 by removing the hypothesis $S \subset \partial M$ (and, of course, the conclusion $hf(M) \subset f(M)$) in the case $n - m \geq 3$.

COROLLARY 4.4. *Let* S^2 *be a* 2*-sphere in* E^4 *that is locally polyhedral except at a countable subset* B; *then there exist flat* 2*-cells* D_1 *and* D_2 *such that*

$$\partial D_1 = \partial D_2 = D_1 \cap D_2 \quad \textit{and} \quad S^2 = D_1 \cup D_2.$$

Proof. Let f be an imbedding of the standard 2-sphere $\partial\Delta^3$ that is locally pwl off $f^{-1}(B)$; let α be an arc in $\partial\Delta^3$ that is locally polyhedral except at its subset $f^{-1}(B)$. Then $f \mid \partial\Delta^3 - \alpha$ is a pwl imbedding of a combinatorial 2-plane into E^4; by [14], the set of points K' at which it is locally knotted is a subpolyhedron of dimension at most 0; hence K' is countable. Likewise, the set K" of points of $\alpha - f^{-1}(B)$ at which f is locally knotted is countable; let $K = K' \cup K'' \cup f^{-1}(B)$, and let γ be a 1-sphere in $\partial\Delta^3$ that is locally polyhedral except at its subset K. Let C_1, C_2 be the closed complementary domains of γ in $\partial\Delta^3$. Then, by Theorem 4.3, $f \mid C_i$ is locally flat and hence flat; set $D_i = f(C_i)$, and the proof is complete.

COROLLARY 4.5. *If* D^2 *is a* 2*-cell that is locally polyhedral off a countable subset of its interior, then* $D^2 = D_1 \cup D_2$, *where each* D_i *is flat and* $D_1 \cap D_2 = \partial D_1 \cap \partial D_2$ *is an arc.*

5. A MILDLY WILD TWO-CELL

CONJECTURE 1_n. *There exists a mildly wild* n*-cell in* E^{n+2}.

CONJECTURE 2_n. *Every* n*-cell in* E^n *is tame in* E^{n+2}.

We shall show that for each $n \geq 2$, Conjecture 1_n is equivalent to Conjecture 2_n. Conjecture 1_1 is true by [5], and Conjecture 2_1 is trivial. Also, since the classical Schoenflies Theorem implies Conjecture 2_2, we shall have proved Conjecture 1_2 (Corollary (5.2)).

THEOREM 5.1. *If every* n*-cell in* E^n *is tame in* E^{n+2}, *then there exists a mildly wild* n*-cell in* E^{n+2}.

Proof. Let $D^n = D$ be a wild n-cell in E^{n+2} that is locally pwl and locally unknotted except at a single interior point a (we have constructed such examples in [13]). Suppose C is an n-cell subset of D with $a \in C$; then there exists an arc α in D with endpoints a, b such that $\alpha \cap D = b$, $\alpha \cap C = a$, and such that α is polyhedral except at a. Now D - a is a pwl punctured ball and can be triangulated by an (infinite) complex L in such a way that $st(v, L) \cap C = \emptyset$ for each vertex v of $\alpha - a$ and the diameters of simplexes of L near a become arbitrarily small. Then

$$N' = N(\alpha - a, L'') = \bigcup \{\sigma \in L'' : \sigma \cap \alpha - a = \emptyset\}$$

is an ascending union of cells, as in the claim of Theorem 4.2, and $N' \cup a = N$ is a cell; moreover, $\overline{D - N}$ is a cell that is locally pwl except at a, and $\overline{D - N}$ contains C. $\overline{D - N}$ is locally unknotted at interior vertices and hence is locally unknotted wherever it is locally pwl—in other words, except at a (see Corollary 3.5 of [15]). Theorem 4.2 ways that $\overline{D - N}$ is flat; let $g: E^{n+2} \to E^{n+2}$ be a homeomorphism with $g(\overline{D - N}) = \Delta^n \subset E^n$. Then $g(C) \subset \Delta^n \subset E^n$; hence $g(C)$ is tame in E^{n+2}. Since g is a homeomorphism of E^{n+2} onto itself, it follows that C is tame.

COROLLARY 5.2. *There is a mildly wild two-cell in* E^4.

LEMMA 5.4. *Let* T *be a compact subset of* E^n *such that* $\text{Fr } T = T - \text{Int}_{E^n} T$ *is a manifold. Then corresponding to every point* $p \in \text{Fr } T$, *there exists an* n*-cell* F *in* E^n *such that* $T \subset F$, $T \cap \partial F = p$, *and* F *is flat in* E^n *for* $n \geq 4$ *and flat in* E^{n+1} *in the case* $n = 3$.

Proof. Since T is compact, there is a combinatorial ball F' such that $T \subset \text{int } F$. Since Fr T is a manifold, $p \in \text{Fr } T$ is accessible by an almost polyhedral arc; that is, there exists an arc α in F' with endpoints p, q such that $\alpha \cap T = p$, $\alpha \cap \partial F' = q$, and α is polyhedral off p. We thicken α to a tapering cell N, as in the proof of Theorem 5.1, and we let $F = \overline{F' - N}$. Then F is a flat cell (see the proof of Theorem 5.1), and $T \subset F$ and $T \cap \partial F = p$, for $n \geq 4$. If $n = 3$, we can still have F satisfying the last two claims; but since Theorem 4.2 is not applicable in this case, we may claim only that F is flat in E^4.

There exists a set T in E^3, whose boundary is a 2-sphere, such that for some point $p \in \text{Fr } T$ no flat 3-cell F (in E^3) contains $T - p$ in its interior and p on its boundary. However, T is not itself a 3-manifold. It seems probable that if T were a 3-manifold, we could find such an F. To show this, one need only consider the case where T is a 3-cell:

CONJECTURE 3. *If* T *is a* 3*-cell in* E^3 *and* $p \in \partial T$, *then there exists a flat* 3*-cell* $F \supset T$ *such that* $T \cap \partial F = p$.

THEOREM 5.4. *If there exists a mildly wild* n*-cell in* E^{n+2}, *then every* n*-cell in* E^n *is tame in* E^{n+2}.

Proof. Suppose D is a mildly wild n-cell in E^{n+2} with distinguished interior point p, and let C be an n-cell in E^n. Let F be a flat n-cell in E^n (all we need is that F is flat in E^{n+2}), with $C \subset F$ and $C \cap \partial F = q$. Let D' be a subcell of D with $p \in \partial D'$, and let $h: E^{n+2} \to E^{n+2}$ be a homeomorphism such that h(D') is a combinatorial n-cell. Let σ be an n-simplex of h(D') with $h(p) \in \sigma$, and let $g: E^{n+2} \to E^{n+2}$ be a homeomorphism such that $g(F) = \sigma$ and $g(q) = h(p)$ (such a homeomorphism exists, because σ and F are flat n-cells). Then $h^{-1} g: E^{n+2} \to E^{n+2}$ has the properties that

$$h^{-1} g(C) \subset D \quad \text{and} \quad p = h^{-1} g(q) \in h^{-1} g(\partial C).$$

Since D is mildly wild, $h^{-1} g(C)$ must be tame in E^{n+2}, and hence C is tame in E^{n+2}. The proof is thus complete.

REFERENCES

1. J. C. Cantrell, *Almost locally flat embeddings of* S^{n-1} *in* S^n, Bull. Amer. Math. Soc. 69 (1963), 716-718.
2. J. C. Cantrell and C. H. Edwards, Jr., *Almost locally polyhedral curves in Euclidean* n-*space*, Trans. Amer. Math. Soc. 107 (1963), 451-457.
3. A. V. Černavskiĭ, *Isotopies in Euclidean spaces*, Russian Math. Surveys 19, Nr. 6 (1964), 63-65.
4. P. H. Doyle, *On the embedding of complexes in* 3-*space*, Illinois J. Math. 8 (1964), 615-620.
5. R. H. Fox, *Some problems in knot theory*, Topology of 3-Manifolds and Related Topics, 168-176, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.
6. R. H. Fox and E. Artin, *Some wild cells and spheres in three-dimensional space*, Ann. of Math. (2) 49 (1948), 979-990.
7. H. Gluck, *Embeddings in the trivial range*, Ann. of Math. (2) 81 (1965), 195-210.
8. V. K. A. M. Gugenheim, *Piecewise linear isotopy and embedding of elements and spheres, I*, Proc. London Math. Soc. (3) 3 (1953), 29-53.
9. J. F. P. Hudson and E. C. Zeeman, *On regular neighbourhoods*, Proc. London Math. Soc. (3) 14 (1964), 719-745.
10. L. S. Husch, *On relative regular neighborhoods* (submitted for publication).
11. R. C. Lacher, *Some conditions for manifolds to be tame*, Ph.D. Thesis, University of Georgia (1966).
12. Charles Seebeck, *ε-taming in codimension three*, Michigan Math. J. (to appear).
13. Ralph Tindell, *Some wild imbeddings in codimension two*, Proc. Amer. Math. Soc. 17 (1966), 711-716.
14. ———, *A counterexample on relative regular neighborhoods*, Bull. Amer. Math. Soc. 72 (1966), 892-893.
15. ———, *The knotting set of a piecewise linear imbedding* (submitted for publication).
16. ———, *Unknotting manifolds in* S^n (submitted for publication).
17. E. C. Zeeman, *Unknotting combinatorial balls*, Ann. of Math. (2) 78 (1963), 501-526.
18. ———, Seminar in Combinatorial Topology (mimeographed notes) Inst. Hautes Etudes Sci., Paris, 1963.

Florida State University