INVOLUTIONS FIXING PROJECTIVE SPACES

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The object of this paper is to prove the following result.

THEOREM. *Suppose* (T, M^n) *is a differentiable involution on a closed manifold* M^n $(n > 2r)$, *and its fixed point set is real projective space* $RP(2r)$. *Then* $n = 4r$, *and* (T, M^n) *is cobordant to the involution of* $RP(2r) \times RP(2r)$ *that sends* (x, y) *into* (y, x).

This result was suggested by Conner and Floyd [2, Section 27]. In particular, Conner and Floyd proved that $n = 4r$, and that if ξ: $E \to RP(2r)$ denotes the normal bundle of $RP(2r)$ in M^n, then the Stiefel-Whitney class of ξ is $(1 + d)^m$, where both m and the binomial coefficient $\binom{m}{2r}$ are odd, and where d is the nonzero class of $H^1(RP(2r); Z_2)$.

Proof of the theorem. Let $RP(\xi)$ be the total space of the $RP(2r - 1)$-bundle associated with ξ, and let p: $RP(\xi) \to RP(2r)$ be the projection. Borel and Hirzebruch [1] have shown that $H^*(RP(\xi); Z_2)$ is the free module over $H^*(RP(2r); Z_2)$, *via* p^*, on the classes $1, c, \cdots, c^{2r-1}$, where c is the characteristic class of the double cover of $RP(\xi)$ by the sphere bundle of ξ. Multiplication in $H^*(RP(\xi); Z_2)$ is given by the formula

$$0 = \sum_{0}^{2r} c^{2r-i} p^*(w_i(\xi)) = \sum_{0}^{2r} \binom{m}{i} c^{2r-i} \alpha^i$$

$$= c^{2r} + c^{2r-1}\alpha + \text{terms of higher degree in } \alpha$$

(since m is odd), where $\alpha = p^*(d)$. The Stiefel-Whitney class of $RP(\xi)$ is

$$w = (1 + \alpha)^{2r+1} \left\{ \sum_{0}^{2r} \binom{m}{i} (1 + c)^{2r-i} \alpha^i \right\}.$$

(See [2, Theorem 23.3].)

By Theorem 28.1 of [2], the antipodal involution on the sphere bundle of ξ bounds a free involution, or equivalently, all of the generalized Stiefel-Whitney numbers $c^i w_\omega[RP(\xi)]$ of $RP(\xi)$ are zero (here w_ω denotes any product $w_{i_1} \cdots w_{i_s}$ of Stiefel-Whitney classes).

Since m and $\binom{m}{2r}$ are odd, $m \geq 2r + 1$. If $m = 2r + 1$, then the bundle ξ and the normal bundle of $RP(2r)$ in $RP(2r) \times RP(2r)$, which is the tangent bundle τ of $RP(2r)$, have the same Stiefel-Whitney class. Thus the bundles $(\xi, RP(2r))$ and

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$(\tau, RP(2r))$ are cobordant, so that the normal bundles of the fixed point sets of the two involutions are cobordant. By Theorem 28.1 of [2], the involutions are then cobordant.

Thus, one may assume that $m > 2r + 1$. Since $\alpha^i = 0$ for $i > 2r$,

$$w = \frac{(1+\alpha)^{2r+1}}{(1+c)^{m-2r}}\left\{\sum_{i=0}^{2r}\binom{m}{i}(1+c)^{m-i}\alpha^i\right\} = \frac{(1+\alpha)^{2r+1}}{(1+c)^{m-2r}}\left\{\sum_{i=0}^{m}\binom{m}{i}(1+c)^{m-i}\alpha^i\right\}$$

$$= \frac{(1+\alpha)^{2r+1}}{(1+c)^{m-2r}}(1+c+\alpha)^m = \frac{(1+\alpha)^{2r+1}(1+c+\alpha)^{2r+1}}{(1+c)}\left\{\frac{1+c+\alpha}{1+c}\right\}^{m-(2r+1)}$$

$$= \frac{(1+c+\alpha(c+\alpha))^{2r+1}}{(1+c)}\left\{\frac{1+c+\alpha}{1+c}\right\}^{m-(2r+1)}$$

$$= \{1+\alpha(c+\alpha)+c\alpha(c+\alpha)+c^2\alpha(c+\alpha)+\cdots\}\cdot\{1+c^2+\alpha^2(c+\alpha)^2\}^r$$

$$\cdot\{1+\alpha+c\alpha+c^2\alpha+\cdots\}^{m-(2r+1)}.$$

Because m is odd, $m - (2r+1)$ is even. Let $m - (2r+1) = 2^s(1+2v)$, with $v \geq 0$, $s \geq 1$.

If $s = 1$, then $w_2(RP(\xi)) = \alpha(c+\alpha) + rc^2 + \alpha^2$ or $c\alpha = w_2(RP(\xi)) + rc^2$. Then

$$c^{2r}\alpha^{2r-1} = c(c\alpha)^{2r-1} = c(w_2 + rc^2)^{2r-1},$$

which is zero, since it gives a generalized Stiefel-Whitney number when evaluated on the fundamental class of $RP(\xi)$. On the other hand,

$$c^{2r} = c^{2r-1}\alpha + \text{terms of higher degree in } \alpha,$$

so that

$$c^{2r}\alpha^{2r-1} = c^{2r-1}\alpha^{2r},$$

which is the nonzero class of $H^{4r-1}(RP(\xi); Z_2)$.

Thus $s > 1$, and $w_2(RP(\xi)) = \alpha(c+\alpha) + rc^2$ or $\alpha(c+\alpha) = w_2 + rc$

If $2^s \leq 2r$, then

$$w(RP(\xi)) = \frac{(1+c+w_2+rc^2)^{2r+1}}{(1+c)}\{1+\alpha^{2^s}+\cdots\},$$

so that

$$w_{2^s}(RP(\xi)) = \alpha^{2^s} + P(w_2, c),$$

where $P(w_2, c)$ denotes a polynomial in w_2 and c. Since $2r \geq 2^s$, we let $2r = k2^s + u$ $(k \geq 1,\ 0 \leq u < 2^s)$. Then

$$[\alpha(c+\alpha)]^u\alpha^{k2^s}c^{2r-1-u} = \alpha^u c^u \alpha^{k2^s}c^{2r-1-u} = \alpha^{2r}c^{2r-1},$$

because all other terms of $[\alpha(c+\alpha)]^u\alpha^{k2^s}$ contain a higher power of α and are therefore zero. Thus

$$c^{2r-1}\alpha^{2r} = (w_2 + rc^2)^u(w_{2^s} + P(w_2, c))^k c^{2r-1-u}$$

is zero, since it gives a generalized Stiefel-Whitney number. Since this is still the nonzero class, we have established a contradiction.

Thus $2^s > 2r$, and

$$w(\xi) = (1 + d)^m = (1 + d)^{2r+1}(1 + d)^{m-(2r+1)} = (1 + d)^{2r+1}(1 + d^{2^s})^{2v+1};$$

but $d^{2^s} = 0$, since $2^s > 2r$. Hence $w(\xi) = (1 + d)^{2r+1}$, and as we noted for the case $m = 2r + 1$, this establishes the theorem.

REFERENCES

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